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Control Theory of Partial Differential Equations

Oleg Imanuvilov
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Control Theory of Partial Differential Equations

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Control Theory of Partial Differential Equations

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Preface

The present volume contains contributions by participants in the “Conference on Control Theory for Partial Differential Equations,” which was held over a two-and-a-half day period, May 30 to June 1, 2003, at Georgetown University, Washington, D.C. The conference was dedicated to the occasion of the retirement of Professor Jack Lagnese from the Mathematics Department of Georgetown University.

It seemed most appropriate to honor the productive and successful scientific career of Jack Lagnese by convening a conference that would bring together a select group of international specialists in the theory of partial differential equations and their control. Over the years, many of the invitees have enjoyed a personal and professional association with Jack. The lasting impact of Jack’s contributions to control theory of partial differential equations and applied mathematics is well documented by over 80 research articles and three books. In addition, Jack served the scientific community for many years in his capacity, at various times, as a program director in the Applied Mathematics Program within the National Science Foundation, as an editor on the boards of several journals, as editor-in-chief of the *SIAM Journal on Control and Optimization*, and as president of the SIAM Activity Group on Control and Systems Theory. He was also a consultant to The National Institute for Standards and Technology for a number of years.

Control theory for distributed parameter systems, and specifically for systems governed by partial differential equations, has been a research field of its own for more than three decades. Although having a distinctive identity and philosophy within the theory of dynamical systems, this field has also contributed to the general theory of partial differential equations. Optimal interior and boundary regularity of mixed problems, global uniqueness issues for over-determined problems and related Carleman estimates, various types of *a priori* inequalities, and stability and long-time behavior are just some examples of important developments in the theory of partial differential equations arising from control theoretic considerations. In recent years, the field has broadened considerably as more realistic models have been introduced and investigated in areas such as elasticity, thermoelasticity, and aeroelasticity; in problems involving interactions between fluids and elastic structures; and in other problems of fluid dynamics, to name but a few. These new models present fresh mathematical challenges. For example, the mathematical foundations of fundamental theoretical issues have to be developed, and conceptual insights that are useful to the designer and the practitioner need to be provided. This process leads to novel numerical challenges that must also be addressed. The papers contained in this volume provide a broad range of significant recent developments, new discoveries, and mathematical tools in the field and further point to challenging open problems.

The conference was made possible through generous financial support by the National Science Foundation and Georgetown University, whose sponsorship is greatly appreciated.

We wish to thank Marcel Dekker for agreeing to include this volume in its well-known and highly regarded series “Lecture Notes in Pure and Applied Mathematics” and for its high professional standards in handling this volume.

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Chapter 1

Asymptotic Rates of Blowup for the Minimal Energy Function for the Null Controllability of Thermoelastic Plates: The Free Case

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Abstract Continuing the analysis undertaken in References 8 and 9, we consider the null-controllability problem for thermoelastic plate partial differential equations (PDEs) models in the absence of rotational inertia, defined on a two-dimensional domain Ω, and subject to the *free* mechanical boundary conditions of second and third order. It is now known that such uncontrolled systems generate *analytic* semigroups on finite energy spaces. Consequently, the concept of *null* controllability is indeed an appropriate question for consideration. It is shown that all finite energy states can be driven to zero by means of $L^2[(0, T) \times \Omega]$ controls in either the mechanical or thermal component. However, the main intent of the paper is to quantify the singularity, as $T \downarrow 0$, of the minimal energy function relative to null controllability. In particular we shall show that in the case of one control function acting upon the system, the singularity of minimal energy is optimal; it is in fact of order $\mathcal{O}(T^{-\frac{5}{2}})$, which is the same rate of blowup as that of any finite dimensional approximation of the problem. The PDE estimates, which are obtained in the process of deriving this sharp numerology, will have a strong bearing on regularity properties of related stochastic differential equations.

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[2]The research of Irena Lasiecka was partially supported by NSF DMS-01043 and ARO DAAD19-02-1-0179.

1.1 Introduction

In this chapter we address specific questions related to the null controllability of thermoelastic plates subject to *free mechanical boundary conditions*, these being represented by shear forces and moments. These particular boundary conditions are of particular interest in the control theory of plates [22, 24, 23]. As we shall see below, the model under consideration is one which corresponds to an infinite speed of propagation; accordingly, null controllability—in arbitrarily short time—is an appropriate topic for study in regard to these plates. We will give at length a full and precise description of our thermoelastic control problem; but for the benefit of the reader and in order to motivate the specific problem under study, we will first provide a few opening remarks.

1.1.1 Motivation

There are several ways of controlling a given plate dynamic. This control can be accomplished by using: 1. internal controls, 2. boundary controls, or 3. controls localized on an open subset of Ω. In addition, one may use either one control action (be it thermal or mechanical) or simultaneous mechanical and thermal controls (i.e., controls located on *both* the mechanical and thermal components of the system). Depending on the objective to be achieved, one framework of control might be more advantageous than another. For instance, if the particular issue at hand is to guarantee the minimal support of control functions, then boundary control would be the most appropriate control situation. However, if one is concerned with the cost of control—or equivalently, with quantifying the associated "minimal energy"—then internal controls should be considered. In this connection, a question of both practical and mathematical relevance is the question of finding the optimal asymptotics that describe the singularity of the associated minimal energy, as $T \downarrow 0$. Since the work of T. Seidman in Reference 34, the optimal asymptotics are well defined and well known for *finite dimensional* control systems. In fact, these asymptotics are given by the *sharp* formula $T^{-k-\frac{1}{2}}$ where index k corresponds to the Kalman rank condition and measures the defect of controllability (see below). The above formula actually gives a *lower* bound for the singularity of the minimal energy associated with any PDE system.

Given then the existence of formula in Reference 34 for controlled finite dimensional systems, we are in a position to loosely define the "optimal" singularity for any controlled PDE. In fact, for a given infinite dimensional system, *the "optimal" rate of singularity of its associated minimal energy will be the rate of singularity enjoyed by approximating (or truncated) finite dimensional systems* (assuming of course that each finite dimensional truncation has the same Kalman rank). For example, scalar first order (in time) models will have its optimal rate of blowup of minimal energy as being $\mathcal{O}(T^{-\frac{3}{2}})$; in general, the optimal singularity for vectorial coupled structures will depend on the number of controls used with respect to number of interactions. Thus, in the case of thermal plates with one control only, the optimal singularity of any finite dimensional truncation is $T^{-\frac{5}{2}}$ (this is seen below). In the case of two controls used (both thermal and mechanical) the optimal singularity is $T^{-\frac{3}{2}}$. Whether, however, the minimal energy asymptotics actually obeys the optimal rate of singularity (predicted from finite dimensions) is an altogether different matter. Indeed, in References 34 and 36 (highly nontrivial) finite-dimensional estimates are derived and can be subsequently applied to finite-dimensional truncations of infinite-dimensional systems; however, the delicate estimates are controlled by a constant C_n, say, where n stands for the dimensionality of the respective approximation. These constants may well tend to infinity as n goes to infinity. In such an event (as seen in References 14, 6, and 40) the optimal asymptotics for the original PDE are lost. This brings us to the key question asked in this chapter: *Is it possible to achieve the optimal rate of singularity for a (fully infinite dimensional) controlled PDE model*?

The answer to the above question—in the negative—has been known for many years in the case of the heat equation with either boundary or localized controls. Indeed, the rate for boundary control of the heat equation is the exponential blowup rate $e^{\mathcal{O}(\frac{1}{T})}$; see References 35 and 37. This rate is known to be sharp [20]. A similar negative answer has been provided in the case of thermoelastic systems under the influence of boundary controls—in fact, such boundary controls likewise lead to $e^{\mathcal{O}(\frac{1}{T})}$ exponential blowup [25]. Therefore, in light of the rational rates of minimal energy blowup exhibited by finite-dimensional controlled systems (as shown in Reference 34) and of the definition given above for optimal rates of minimal energy blowup for controlled PDEs, it is manifest that thermoelastic plates under the influence of boundary or localized controls will not give rise to minimal energies that exhibit an optimal (finite dimensional) singularity. Thus, in searching for PDE control situations, which will yield up the optimal algebraic singularity enjoyed by finite dimensional truncations, the only reasonable choice left is the implementation of *internal* controls. In the specific context of our thermoelastic PDE, the relevant question then becomes: *Do the minimal energies of internally controlled (fully infinite dimensional) thermoelastic plates exhibit the optimal rate of blowup* $O(T^{-\frac{5}{2}})$ *by either mechanical or thermal control*?

The relevance of this question should not be underestimated from both a practical and mathematical point of view. Indeed, from a practical point of view one would like to know whether a given finite-dimensional approximation of the system contains critical information and moreover reflects controllability properties of the original PDE model. From a mathematical point of view, the solution to the null controllability problem is not only of interest in its own right as an issue in control theory, but this solution can also give rise to deep and significant connections between the algebraic optimal singularity of minimal energy and other fields of analysis, including stochastic analysis. In point of fact, within the field of stochastic differential equations, there is an acute need to know of those PDE control environments that will yield up optimal (and algebraic) rates of singularity of minimal energy. These particular rates are critical in finding the regularity and solvability of certain stochastic differential equations [14, 15, 19], as well as in setting conditions for the hypoellipticity of certain degenerate infinite dimensional elliptic problems [32]. It is shown in Reference 32 that Hormander's hypoellipticity condition is strongly linked to the singularity of the minimal energy function. Null controllability is also related to the analysis of regularity of the Bellman's function, which is associated with the minimal time control problem. Indeed, as eloquently described in References 14 and 15, this property bears a close relation to the regularity of some Markov semigroups, including Orstein–Uhlenbeck processes and related Kolmogorov equations. For some of these semigroups (see, e.g., Reference 15—Theorem 8.3.3) the minimal energy singularity associated with null controllability describes differentiability properties and regularizing effects of the Orstein–Uhlenbeck process. Moreover, the regularity of solutions to the Kolmogorov equation depends on the singularity of the minimal energy as $T \downarrow 0$. Also, as shown in Reference 14, optimal estimates for the norms of controls are critical in being able to prove Liouville's property for harmonic functions of Markov processes (see p. 108 in Reference 15). In sum, there is an abundance of examples from the literature that clearly illustrate that, in the context of computing optimal minimal energy asymptotics as $T \downarrow 0$, the tools of controllability can potentially enable a mathematical control theorist to transcend his or her deterministic realm so as to solve fundamental problems in other areas of analysis, including stochastic PDEs.

In addition, the procurement of optimal algebraic estimates for the minimal energy allows one to clearly explain the role of the hyperbolic-parabolic coupling within the PDE structure (in Eq. (1.1) below). In particular, it has been shown recently in Reference 25 that, owing to optimal algebraic singularities of minimal energy, it is possible to offset the singularity of minimal energy by introducing a very strong coupling within the system. Thus, in some sense, the lack of a second control in the system may be quantitatively compensated for by taking large values of the coupling parameter "α." From our remarks above, it is clear that this compensatory phenomenon will not be observed with boundary or partially supported controls, which, as we have said, lead to blowups of exponential type.

Having decribed the goal and motivation for the problem considered, we shall describe the main contribution of this chapter within the context of recent work in that area. The problem of controllability/reachability for thermal plates has attracted considerable attention in recent years with many contributions available in the literature [22, 23, 24, 1, 2, 3, 10, 18, 16, 17, 11], but we shall focus particularly on works related directly to singular behavior, as $T \downarrow 0$, of the minimal energy relative to null controllability.

The study of optimal singularity for thermoelastic plates with internal controls started in References 8, 9, and 40, where for the first time the optimal rates $T^{-\frac{5}{2}}$ were established for the "commutative" case (i.e., plates with hinged mechanical boundary conditions). The proof given in Reference 40 is based on a spectral method that exploits the commutativity in an essential way, whereas the proof given in Reference 8 is based on weighted energy estimates, thereby giving one the chance to extend this method to other noncommutative models (e.g., clamped or free mechanical boundary conditions). The "commutative" case (hinged boundary conditions) has been also treated in Reference 11, where null controllability with thermal controls of partially localized support was proved. For this commutative model under boundary control (either thermal or boundary), the exponentially blowing up and sharp asymptotics $e^{\mathcal{O}(\frac{1}{T})}$ have been shown [25]. The techniques used in these papers rely critically on spectral analysis and commutativity.

It turns out that a proper and necessarily more technical extension of the method introduced in Reference 8 will allow the consideration of *noncommutative* models. (By "noncommutative models" we mean those models wherein the domains of the respective spatial differential operators of the plate and heat dynamics do not necessarily enjoy any sort of compatibility.) In particular, the optimal singularity of the (null control) minimal energy is proved in Reference 9 for clamped plates with one control only. It should be stressed that the proof in the noncommutative case depends in an essential manner upon estimates provided by the analyticity of the underlying thermoelastic semigroup; this property of analyticity was discovered for the clamped case in References 31 and 28 and for free case in Reference 27. The most challenging case is, of course, that of the *free* mechanical boundary conditions (introduced in the context of control theory in Reference 22), in which a coupling between thermal and mechanical variables also occurs on the boundary. This additional coupling compels us to develop below a delicate string of trace estimates that measure the singularity at the boundary.

The main aim of this chapter is to provide a complete analysis of the free case. We shall show that in the case of mechanical control one still obtains the optimal singularity. Instead, in the case of thermal control the estimate is "off" by $3/4$. A question whether this estimate can be improved, thereby leading to the optimal singularity $T^{-\frac{5}{2}}$, still remains an *open question*.

1.1.2 Description of the PDE Model and Statement of the Problem

Having given our general remarks above, we now proceed to precisely describe the present problem under consideration; this work will continue and extend the analysis that has been previously undertaken in References 6, 7, 8, 9 through and 40. We will consider throughout the two-dimensional PDE system of thermoelasticity in the absence of rotational inertia. As we have already stated, it is now known that for all possible mechanical boundary conditions, the thermoelastic PDE model is associated with the generator of an *analytic* C_0-semigroup (see References 31, 28, 39, and 27). Given then that the underlying PDE dynamics are "parabolic-like," it is natural to consider the *null controllability problem* for the thermoelastic system, namely, can one find $L^2(Q)$ controls (mechanical or thermal) that steer the solution of the PDE from the initial data to the zero state? (We shall make this control theoretic notion more precise below. As usual, Q here denotes the cylinder $\Omega \times (0, T)$.) Having established $L^2(Q)$-null controllability for the PDE, and moreover assuming that the controllability time is arbitrary, we can subsequently proceed to measure the rate of blowup, as $T \downarrow 0$, for the minimal energy function that is associated to null controllability. As is well known, and as we shall see below, this work is very much tied up with obtaining the *sharp* observability inequality associated with null controllability; moreover, this analysis is rather sensitive to the mechanical boundary conditions imposed. In Reference 8—as well as in Reference 40 via a very different methodology—the problem of blowup for the minimal energy function was undertaken in the canonical case of *hinged* mechanical boundary conditions; in Reference 9, we revisit this problem for the more difficult *clamped* case. In this paper, we complete the picture by analyzing the singularity of minimal energy for the case of the thermoelastic PDE under the so-called *free* boundary conditions. In general, the analyses involved in the attainment of (null *and* exact control) observability inequalities for thermoelastic systems are profoundly sensitive to the particular set of boundary conditions are being imposed. But the free case, presently under consideration, will give rise to the most problematic scenario of all. This situation is due to the high degree of coupling between the mechanical and the thermal variables, with the coupling taking place in the PDE itself and in the free mechanical boundary conditions.

We describe the problem in detail. Let Ω be a bounded open set of $\mathbb{R}^2$, with smooth boundary Γ. For the free case, following [22, 23] the corresponding model PDE is as follows: the (mechanical) variables $[\omega(t,x), \omega_t(t,x)]$ and the (thermal) variable $\theta(t,x)$ solve, for given data $\{[\omega_0, \omega_1, \theta_0], u_1, u_2\}$, the PDE system

$$\begin{cases} \begin{cases} \omega_{tt} + \Delta^2\omega + \alpha\Delta\theta = a_1 u_1 \\ \theta_t - \Delta\theta - \alpha\Delta\omega_t = a_2 u_2 \end{cases} \quad \text{on } (0,T)\times\Omega \\ \\ \begin{cases} \Delta\omega + (1-\mu)B_1\omega + \alpha\theta = 0 \\ \dfrac{\partial\Delta\omega}{\partial\nu} + (1-\mu)\dfrac{\partial B_2\omega}{\partial\tau} - \omega + \alpha\dfrac{\partial\theta}{\partial\nu} = 0 \end{cases} \quad \text{on } (0,T)\times\Gamma \\ \\ \dfrac{\partial\theta}{\partial\nu} + \lambda\theta = 0 \ \text{ on } (0,T)\times\Gamma, \quad \text{where } \lambda > 0 \\ \\ \omega(t=0) = \omega_0;\ \omega_t(t=0) = \omega_1;\ \theta(t=0) = \theta_0 \quad \text{on } \Omega. \end{cases} \tag{1.1}$$

Here, $\alpha > 0$ is the parameter that *couples* the disparate dynamics (i.e., the heat equation vs. the Euler plate equation); the constant $\mu \in (0,1)$ is Poisson's ratio. Also, the (control) parameters a_1 and a_2 satisfy $a_1 \geq 0$, $a_2 \geq 0$ and $a_1 + a_2 > 0$ (in other words, at least one of the controls, be it thermal or mechanical, is always present.) The (free) boundary operators B_i are given by

$$\begin{aligned} B_1 w &\equiv 2\nu_1\nu_2\frac{\partial^2 w}{\partial x\partial y} - \nu_1^2\frac{\partial^2 w}{\partial y^2} - \nu_2^2\frac{\partial^2 w}{\partial x^2}; \\ B_2 w &\equiv \left(\nu_1^2 - \nu_2^2\right)\frac{\partial^2 w}{\partial x\partial y} + \nu_1\nu_2\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right). \end{aligned} \tag{1.2}$$

The PDE Eq. (1.1) is the model explicitly derived and analyzed in References 24 and 22 in the "limit case." That is to say, we are considering the two-dimensional thermoelastic system in the absence of rotational forces; the small and nonnegative, classical parameter γ is taken here to be zero. As we stated at the outset, it is now well known that the lack of rotational inertia in the model Eq. (1.1) will result in the corresponding dynamics having their evolution described by the generator of an *analytic* semigroup on the associated basic space of finite energy. In short, the present case $\gamma = 0$ corresponds to *parabolic-like* dynamics; this is in stark contrast to the case $\gamma > 0$—as analyzed in the control papers [22], [23], [3] and myriad others—for which the corresponding PDE manifests *hyperbolic-like* dynamics.

In fact, if we define

$$\mathbf{H} \equiv H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \tag{1.3}$$

then one can proceed to show by the Lumer Phillips theorem that the thermoelastic plate model can be associated with the generator of a C_0-semigroup of contractions on $\mathbf{H}$. That is to say, there exists $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, and $\{e^{\mathcal{A}t}\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ such that $[\omega, \omega_t, \theta]$ satisfies the PDE (1.1) if and only if $[\omega, \omega_t, \theta]$ satisfies the abstract ODE

$$\frac{d}{dt}\begin{bmatrix} \omega(t) \\ \omega_t(t) \\ \theta(t) \end{bmatrix} = \mathcal{A}\begin{bmatrix} \omega(t) \\ \omega_t(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ a_1 u_1(t) \\ a_2 u_2(t) \end{bmatrix}; \quad \begin{bmatrix} \omega(0) \\ \omega_t(0) \\ \theta(0) \end{bmatrix} = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix}.$$

In consequence of this relation, we have immediately from classical semigroup theory that

$$\{[\omega_0, \omega_1, \theta_0], [u_1, u_2]\} \in \mathbf{H} \times [L^2(Q)]^2 \Rightarrow [\omega, \omega_t, \theta] \in C([0, T]; \mathbf{H}). \tag{1.4}$$

Because of the underlying analyticity, which will ultimately mean that there are smoothing effects associated with the application of the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$, the null controllability problem for the controlled PDE Eq. (1.1)—with respect to internal L^2-controls—is an appropriate one to study. Moreover, one might speculate that, as in the case of the canonical heat equation [12], should the PDE Eq. (1.1) in fact be null controllable, it will be so in *arbitrary* small time (because of the underlying infinite speed of propagation). It is this speculation that motivates our working definition of null controllability for the present paper.

DEFINITION 1.1 *The PDE* (1.1) *is said to be null controllable if, for any* $T > 0$ *and arbitrary initial data* $\mathbf{x} \equiv [\omega_0, \omega_1, \theta_0] \in \mathbf{H}$, *there exists a control function* $[u_1, u_2] \in [L^2(Q)]^2$ *such that the corresponding solution* $[\omega, \omega_t, \theta] \in C([0, T]; \mathbf{H})$ *satisfies* $[\omega(T), \omega_t(T), \theta(T)] = [0, 0, 0]$.

However, the issue of null controllability, although certainly an important part of this paper, is subordinate to our main objective, which is to measure the rate of singularity of the associated *minimal energy function*.

We develop this notion of "minimal energy." Assume for the time being that the Eq. (1.1) is null controllable within the class of $[L^2(Q)]^2$-controls, in the sense of the Definition 1.1. Subsequently, one can then speak of the associated minimal norm control, relative to given initial data $\mathbf{x} \equiv [\omega_0, \omega_1, \theta_0] \in \mathbf{H}$ and given terminal time T. That is to say, we can consider the problem of finding a control $\mathbf{u}_T^0(\mathbf{x})$ that steers the solution $[\omega, \omega_t, \theta]$ of Eq. (1.1) (with $[u_1, u_2] = \mathbf{u}_T^0(\mathbf{x})$ therein) from initial data $\mathbf{x}$ to zero in arbitrary time T and minimizes the L^2 norm. In fact, by standard convex optimization arguments (see, e.g., Reference 13), given any $\mathbf{x} \in \mathbf{H}$ and fixed T, one can find a control $\mathbf{u}_T^0(\mathbf{x})$ which solves the problem

$$\left\|\mathbf{u}_T^0(\mathbf{x})\right\|_{[L^2(Q)]^2} = \min \|\mathbf{u}\|_{[L^2(Q)]^2},$$

where, above, the minimum is taken with respect to all possible null controllers $\mathbf{u} = [u_1, u_2] \in [L^2(Q)]^2$ of the PDE (1.1) (which steer initial data $\mathbf{x}$ to rest at time $t = T$). Subsequently, we can define the *minimal energy function* $\mathcal{E}_{\min}(T)$ as

$$\mathcal{E}_{\min}(T) \equiv \sup_{\|\mathbf{x}\|_{\mathbf{H}}=1} \left\|\mathbf{u}_T^0(\mathbf{x})\right\|_{[L^2(Q)]^2}. \tag{1.5}$$

Under the assumption of null controllability, as defined in Definition 1.1, we have that $\mathcal{E}_{\min}(T)$ is bounded away from zero. A natural follow-up question is "how does $\mathcal{E}_{\min}(T)$ behave as terminal time $T \downarrow 0$, or equivalently (by Eq. (1.5)), for given time T, how exactly does the quantity $\|\mathbf{u}_T^0(\mathbf{x})\|_{[L^2(Q)]^2}$ grow as $T \downarrow 0$?"

The problem of studying the rate of blowup for minimal norm controls is a classical one and has its origins from the finite dimensional setting. In fact, a very complete and satisfactory solution has been given in Reference 34 for the following controlled ODE in $\mathbb{R}^n$:

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + B\vec{u}(t), \quad \vec{y}_0 \in \mathbb{R}^n \tag{1.6}$$

where $\vec{u} \in L^2(0, T; \mathbb{R}^m)$ and A (resp., B) is an $n \times n$ (resp. $n \times m$) matrix, with $m \leq n$ (so consequently the solution $\vec{y} \in C([0, T]; \mathbb{R}^n)$. The problem in this finite dimensional milieu, like that for our controlled PDE (1.1), is to ascertain the rate of singularity for the associated minimal energy function, which is defined in the same way as in Eq. (1.5). The solution to this problem is tied up with the classical Kalman's rank condition. Namely, a beautifully simple (though highly nontrivial) formula in Reference 34—an alternative constructive proof of this formula is given in Reference 40; see also Reference 36—yields that the minimal energy function associated to the null controllability of Eq. (1.6) is $\mathcal{O}(T^{-k-\frac{1}{2}})$, where k is the Kalman's rank of the system Eq. (1.6) (that is, k is the smallest integer such that *rank* $([B, AB, \ldots, A^k B]) = n$; see Reference 41).

By a formal application of Seidman's finite dimensional result, one can get an inkling of the numerology involved in the computation of the minimal energy $\mathcal{E}_{\min}(T)$ for the PDE system Eq. (1.1). For example, let us consider the thermoelastic Eq. (1.1) but with now ω satisfying the canonical *hinged* mechanical/Dirichlet thermal boundary conditions

$$\omega|_\Gamma = \Delta\omega|_\Gamma = \theta|_\Gamma = 0 \text{ on } \Sigma. \tag{1.7}$$

In this case, it is shown in Reference 27 that when, say, thermal control only is implemented (i.e., $a_1 = 0$ in Eq. (1.1)), the thermoelastic PDE under the hinged boundary conditions Eq. (1.7) may be associated with the ordinary differential equation (ODE) (1.6), with

$$A = \Delta \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ a_2 \end{bmatrix}. \tag{1.8}$$

This ODE in three space dimensions is a direct consequence of the analysis undertaken for the canonical hinged case in Reference 26. By way of obtaining the ODE (1.6), we have formally "factored out" the Laplacian from the (rearranged) infinitesimal generator of the thermoelastic semigroup, which is given in (Section 1.2.2) of Reference 27 (see also Reference 28, p. 311). Considering now finite dimensional truncations of Δ (by making use of the spectral resolution of the Laplacian under Dirichlet boundary conditions) and applying the algorithm of Seidman to the given controllability pair $[A, B]$ in Eq. (1.8), we compute readily that the minimal energy function associated with the null controllability of the finite dimensional Eq. (1.6)—an approximation in some sense of the thermoelastic system under the hinged boundary conditions—blows up at a rate on the order of $T^{-\frac{5}{2}}$. These numerics lead to the following question: Does the minimal energy Eq. (1.5) (i.e., the minimal energy for the full-fledged *infinite dimensional* system) obey the law $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$? Of course, Seidman's formula for matrices gives no conclusive proof as to what is actually happening for the fully infinite dimensional model. In fact, it is well known that the minimal energy of a given infinite dimensional system may bear no relation to the limit of minimal energies of any given sequence of finite dimensional approximations. For example, it was shown in Reference 14 that the growth of the minimal energy function for a given infinite dimensional system may be arbitrarily large, even when Kalman's rank $k = 1$ and spectral diagonal systems are being considered. Moreover, in Reference 35 it is shown that for the case of the boundary controlled heat equation, the sharp observability inequality corresponding to the (null) minimal energy of a given heat operator's finite dimensional truncation obeys *rational* rates of singularity. On the other hand, the asymptotics of the minimal energy, which are obtained for the (infinite dimensional) heat equation, are of *exponential* type. A similar phenomenon is observed in References 40 and 6, wherein strongly damped wave equations under internal control are considered. In this situation, with the damping operator given by A^{β}, the asymptotics of minimal energy behave as $T^{-\frac{\beta}{2(1-\beta)}}$ for any $\beta > \frac{3}{4}$. Thus, when the damping operator approaches Kelvin's Voight damping, the singularity loses its algebraic character with $\frac{\beta}{2(1-\beta)} \uparrow \infty$. Instead, for $\beta \leq \frac{3}{4}$, the singularity is optimal (i.e., the same as that for finite dimensional truncations) and is equal to $T^{-\frac{3}{2}}$.

But as formal as the application of Seidman's finite dimensional algorithm may seem in the present context, there is in fact a relevance here to the thermoelastic PDE, which is approximately described by controllability pair $[A, B]$. The minimal energy function with respect to null controllability of the thermoelastic PDE, under the hinged boundary conditions Eq. (1.7), does indeed obey the singular rate $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$. This minimal energy analysis for the hinged case was shown independently in References 8 and 40 (and most recently in Reference 25 where the asymptotics with respect to the coupling α are also provided). In Reference 40, it is of prime importance that the hinged boundary conditions Eq. (1.7) be in play, for these mechanical boundary conditions allow a fortuitous spectral resolution of the underlying thermoelastic generator. With the eigenfunctions of the thermoelastic dynamics in hand, it is shown in Reference 40 via a constructive class of suboptimal steering controls that the delicate observability estimates for solutions for the spectrally truncated adjoint problem—adjoint with respect to null controllability—are preserved; as a consequence, a rational rate of singularity for the infinite dimensional null minimal energy is obtained in the limit. However, for other sets of mechanical boundary conditions, including the physically relevant clamped and (above all) free boundary conditions under consideration at present, there will be no such available spectral decomposition.

On the other hand, the methodology employed in References 8 and 9, and the present work, is "eigenfunction independent"; in particular, we blend a weighted multiplier method of Carleman's type with boundary trace estimates exhibiting singular behavior of the boundary traces. This rather special behavior is a consequence of the underlying analyticity. In principle, our work in Reference 8 to estimate the blowup of the "minimal norm control" as $T \downarrow 0$ is applicable to a variety of dynamics. (In fact, our method of proof in Reference 8 and in the present work is used in Reference 7 to estimate the minimal norm control of the abstract wave equation under Kelvin–Voight damping.) Moreover, the robustness of our method allows us in Reference 9 to analyze the rate of singularity of the minimal energy function for the null controllability of thermoelastic plates in the case of *clamped* boundary conditions. As we said above, there is no spectral decomposition or factorization of the thermoelastic generator in the case of mechanical boundary conditions other than the canonical hinged case and thus no rigorous association with the abstract ODE (1.8). Still, we show in Reference 9 that for the clamped case, the minimal energy obeys the singular rate "predicted" in Reference 34, namely, $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$. Our intent in this paper is to bring the story to a close by investigating the minimal energy function for the null controllability of thermoelastic systems under the high-order free boundary conditions that are present in Eq. (1.1).

1.1.3 Main Result

In regards to our stated problem, the main result is as follows:

THEOREM 1.1

Let terminal time $T > 0$ be arbitrary and $a_1, a_2 \geq 0$ with $a_1 + a_2 > 0$. Then, given initial data $[\omega_0, \omega_1, \theta_0] \in \mathbf{H}$, there exist control(s) $[u_1, u_2] \in [L^2(Q)]^2$ such that the corresponding solution $[\omega, \omega_t, \theta]$ of (1.1) *satisfies $[\omega(T), \omega_t(T), \theta(T)] = [0, 0, 0]$. (That is to say, the PDE model Eq.* (1.1) *is null controllable within the class of* $[L^2(Q)]^2$*—controls in arbitrary short time.) Moreover, We have the following rates of blowup for the minimal energy function:*

1. *(thermal control) If* $a_1 = 0$, *then* $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{13}{4}-\epsilon})$ *for all* $\epsilon > 0$;
2. *(mechanical control) If* $a_2 = 0$, *then* $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$;
3. *If* $a_1 > 0$ *and* $a_2 > 0$, *then* $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{3}{2}})$.

REMARK 1.1 The null controllability of thermoelastic plates with free boundary conditions and under one internal control (be it mechanical or thermal) appears to be, as far as we know, a new result in the literature. The Theorem 1.1 above, in addition to asserting the said null controllability property, provides the asymptotics for the singularity of the associated minimal energy function. These asymptotics are optimal in the case of a single mechanical control and in the case of two controls acting upon the system. In the case of a single thermal control the estimate is "off" by 3/4 with respect to the desired "finite dimensional prediction" in Reference 34. Whether this estimate can be improved upon is an open question.

Our method of proof of Theorem 1.1 is based on weighted energy estimates that are flexible enough to accomodate analytic estimates and the resulting singularity. The proof has the following main technical ingredients:

1. special weighted *nonlocal multipliers* introduced in Reference 4 and subsequently invoked in References 3, 5, 29, and 6, and elsewhere;
2. the analyticity of semigroups associated with thermoelastic PDE models in the absence of rotational forces, as demonstrated in References 31, 26, 27, and 28;
3. new singular estimates for boundary traces of solutions of Eq. (1.9), which are of their own intrinsic interest and which are needed to handle the boundary terms resulting from the weighted estimates employed.

1.2 The Necessary Observability Inequality

The proof of Theorem 1.1 is based on the derivation of the observability inequality associated with the null controllability of the PDE (1.1) with respect to thermal or mechanical control or both. This inequality is formulated in terms of the solution of the homogeneous PDE, which is "dual" or "adjoint" to that in Eq. (1.1). Namely, we shall consider solutions $[\phi, \phi_t, \vartheta]$ to the following system:

$$\begin{cases} \begin{cases} \phi_{tt} + \Delta^2\phi + \alpha\Delta\vartheta = 0 & \text{on } (0,T)\times\Omega \\ \vartheta_t - \Delta\vartheta - \alpha\Delta\phi_t = 0 & \text{on } (0,T)\times\Omega \end{cases} \\ \begin{cases} \Delta\phi + (1-\mu)B_1\phi + \alpha\vartheta = 0 \\ \dfrac{\partial\Delta\phi}{\partial\nu} + (1-\mu)\dfrac{\partial B_2\phi}{\partial\tau} - \phi + \alpha\dfrac{\partial\vartheta}{\partial\nu} = 0 \end{cases} \quad \text{on } \Sigma \\ \dfrac{\partial\vartheta}{\partial\nu} + \lambda\vartheta = 0 \ \text{ on } \Sigma, \quad \lambda > 0 \\ [\phi(0), -\phi_t(0), \vartheta(0)] = [\phi_0, \phi_1, \vartheta_0] \in \mathbf{H}. \end{cases} \tag{1.9}$$

If we define the bilinear form $a(\cdot,\cdot) : H^2(\Omega) \times H^2(\Omega) \to \mathbb{R}$ by

$$a(w,\tilde{w}) \equiv \int_\Omega [w_{xx}\tilde{w}_{xx} + w_{yy}\tilde{w}_{yy} + \mu(w_{xx}\tilde{w}_{yy} + w_{yy}\tilde{w}_{xx}) + 2(1-\mu)w_{xy}\tilde{w}_{xy}]\, d\Omega,$$

then we can state the "Green's formula," which involves this bilinear form (see Reference 22) and which is valid for functions $w, \tilde{w}$ (smooth enough):

$$\int_\Omega (\Delta^2 w)\tilde{w}\, d\Omega = a(w,\tilde{w}) + \int_\Gamma \left[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau}\right]\tilde{w}\, d\Gamma$$
$$- \int_\Gamma [\Delta w + (1-\mu)B_1 w]\,\frac{\partial \tilde{w}}{\partial \nu} d\Gamma. \tag{1.10}$$

Let $\mathcal{E}(t)$ denote the energy of the adjoint system Eq. (1.9), where

$$\mathcal{E}(t) \equiv \frac{1}{2}a(\phi(t),\phi(t))d + \frac{1}{2}\int_\Gamma |\phi(t)|^2\, d\Gamma + \frac{1}{2}\int_\Omega |\vartheta(t)|^2\, d\Omega. \tag{1.11}$$

In terms of this energy, then one can show by classical functional analytical arguments (see, e.g., References 41 and 3) that the PDE (1.1) is null controllable, in the sense of 1, if and only if the adjoint variables $[\phi, \phi_t, \vartheta]$ of Eq. (1.9) satisfy the following *continuous observability inequality*, for some constant C_T:

$$\|[\phi(T), \phi_t(T), \vartheta(T)]\|_{\mathbf{H}} \leq C_T(a_1\|\phi_t\|_{L^2(Q)} + a_2\|\vartheta\|_{L^2(Q)}). \tag{1.12}$$

Having worked to establish the sharp constant C_T in the observability inequality Eq. (1.12), one can proceed through an algorithmic procedure—using an explicit representation of the minimal norm control, by convex optimization—so as to have that for all terminal time $T > 0$,

$$\mathcal{E}_{\min}(T) = \mathcal{O}(C_T).$$

Because the details of this argument are known and have been previously spelled out (see, e.g., References 9 and 8), we defer from repeating them here.

Because of this characterization of the behavior of $\mathcal{E}_{\min}(T)$ with the constant C_T in Eq. (1.12), our work will accordingly be geared toward establishing this inequality (where, again, control parameters a_i satisfy $a_1, a_2 \geq 0$, and $a_1 + a_2 > 0$).

1.3 Some Preliminary Machinery

In this section, we explicitly define the underlying generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, which describes the thermoelastic flow. Subsequently, a proposition is derived with which to associate powers of this generator with specific Sobolev spaces. This characterization of the powers will be critical in work.

- To start, we define the linear operator $A_D : D(A_D) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned} A_D &\equiv -\Delta; \\ D(A_D) &= H^2(\Omega) \cap H_0^1(\Omega). \end{aligned} \tag{1.13}$$

- We will also need the following (Dirichlet) map $D : L^2(\Gamma) \to L^2(\Omega)$:

$$Df = g \Leftrightarrow \Delta g = 0 \text{ on } \Omega \quad \text{and} \quad g|_\Gamma = f \text{ on } \Gamma. \tag{1.14}$$

  By the classical elliptic regularity, we have that $D \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{1}{2}}(\Omega))$ for all s (see Reference 30).

- We also define the linear operator $\mathring{\mathbf{A}} : D(A_D) \subset L^2(\Omega) \to L^2(\Omega)$ by setting $\mathring{\mathbf{A}}\varpi = \Delta^2\varpi$, for $\varpi \in D(\mathring{\mathbf{A}})$, where

$$D(\mathring{\mathbf{A}}) = \Big\{ \varpi \in H^4(\Omega) : [\Delta\varpi + (1-\mu)B_1\varpi]_\Gamma = 0 \text{ and } \left[\frac{\partial\Delta\varpi}{\partial\nu} + (1-\mu)\frac{\partial B_2\varpi}{\partial\tau} - \varpi\right]_\Gamma = 0 \Big\},$$

where the boundary operators B_i are as defined in Eq. 1.3.

This operator is densely defined, positive definite, and self-adjoint. Consequently by Reference 21, one has the characterization

$$D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \approx H^2(\Omega); \text{ with moreover } \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} = a(\phi,\phi) + \int_\Gamma \phi^2 d\Gamma.$$

- Moreover, we define the elliptic operators G_i by

$$G_1 h = v \Leftrightarrow \begin{cases} \Delta^2 v = 0 \quad \text{on } \Omega \\ \begin{cases} \Delta v + (1-\mu)B_1 v = h \\ \dfrac{\partial\Delta v}{\partial\nu} + (1-\mu)\dfrac{\partial B_2 v}{\partial\tau} - v = 0 \end{cases} \text{on } \Gamma \end{cases};$$

$$G_2 h = v \Leftrightarrow \begin{cases} \Delta^2 v = 0 \quad \text{on } \Omega \\ \begin{cases} \Delta v + (1-\mu)B_1 v = 0 \\ \dfrac{\partial\Delta v}{\partial\nu} + (1-\mu)\dfrac{\partial B_2 v}{\partial\tau} - v = h \end{cases} \text{on } \Gamma \end{cases}. \tag{1.15}$$

By elliptic regularity (see, e.g., Reference 30) one has that for all real s,

$$G_1 \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{5}{2}}(\Omega)); \; G_2 \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{7}{2}}(\Omega)). \tag{1.16}$$

With these operators defined above, we have that the generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ of the thermoelastic semigroup may be given the explicit representation

$$\mathcal{A} = \begin{bmatrix} 0 & \mathbf{I} & 0 \\ -\mathring{\mathbf{A}} & 0 & \alpha(A_D(I - D\gamma_0) - \mathring{\mathbf{A}}G_1\gamma_0 + \lambda\mathring{\mathbf{A}}G_2\gamma_0) \\ 0 & -\alpha A_D(I - D\gamma_0) & -\alpha A_D(I - D\gamma_0) \end{bmatrix};$$

$$D(\mathcal{A}) = \Big\{ [\omega_0, \omega_1, \theta_0] \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega) : \mathring{\mathbf{A}}\,[\omega_0 + \alpha\,(G_1\gamma_0 - \lambda G_2\gamma_0)\,\theta_0] \in L^2(\Omega) \text{ and } \left[\frac{\partial\theta_0}{\partial\nu} + \lambda\theta_0\right]_\Gamma = 0 \Big\} \tag{1.17}$$

(here, $\gamma_0 \in \mathcal{L}(H^1(\Omega), H^{\frac{1}{2}}(\Gamma))$ is the classical Sobolev trace map; i.e., $\gamma_0 f = f|_\Gamma$ for $f \in C^\infty(\overline{\Omega})$).

As we have said, it is now known that the generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ for the thermoelastic plate, with free mechanical boundary conditions, is associated with an *analytic* C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ of contractions on $\mathbf{H}$ (see Reference 27 and references therein), with moreover $\mathcal{A}^{-1}$ being bounded on $\mathbf{H}$.

For this realization of the generator, we now proceed to show the following:

PROPOSITION 1.1
Let integer $k = 1, 2, \ldots$. Then $D(\mathcal{A}^k) \subset H^{2k+2}(\Omega) \times H^{2k}(\Omega) \times H^{2k}(\Omega)$.

PROOF OF PROPOSITION 1.1 Let first $[\omega_0, \omega_1, \theta_0] \in D(\mathcal{A})$. Then by definition, $\omega_1, \theta_0 \in H^2(\Omega)$. Moreover, from the abstract representation in Eq. (1.17), we have

$$\mathring{\mathbf{A}}\omega_0 + \alpha\mathring{\mathbf{A}}G_1\theta_0|_\Gamma - \alpha\lambda\mathring{\mathbf{A}}G_2\theta_0|_\Gamma + \alpha\Delta\theta_0 = f \in L^2(\Omega).$$

Because $\theta_0|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$, then consequently from the elliptic regularity results posted in Eq. (1.16),

$$\omega_0 = \mathring{\mathbf{A}}^{-1}f - \alpha G_1\theta_0|_\Gamma + \alpha\lambda G_2\theta_0|_\Gamma - \alpha\mathring{\mathbf{A}}^{-1}\Delta\theta_0 \in H^4(\Omega).$$

So the assertion is true for $k = 1$. ∎

Proceeding now by induction, suppose that the result holds true for integer $k - 1$, $k \geq 2$, and let $[\omega_0, \omega_1, \theta_0] \in D(\mathcal{A}^k)$. Then, because

$$\mathcal{A}\begin{bmatrix}\omega_0\\ \omega_1\\ \theta_0\end{bmatrix} \in D(\mathcal{A}^{k-1}),$$

we have

$$\begin{aligned} \omega_1 &\in H^{2k}(\Omega);\\ \mathring{\mathbf{A}}\omega_0 + \alpha\mathring{\mathbf{A}}G_1\theta_0|_\Gamma - \alpha\lambda\mathring{\mathbf{A}}G_2\theta_0|_\Gamma + \alpha\Delta\theta_0 &= f \in H^{2k-2}(\Omega);\\ A_R\theta_0 - \alpha\Delta\omega_1 &= g \in H^{2k-2}(\Omega). \end{aligned} \tag{1.18}$$

Here, $A_R : D(A_R) \subset L^2(\Omega) \to L^2(\Omega)$ is the elliptic operator defined by

$$A_R f = -\Delta f; \qquad D(A_R) = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu} + \lambda f = 0,\ \lambda > 0 \right\}. \tag{1.19}$$

Reading off the third equation in Eq. (1.18), we obtain, after using elliptic regularity,

$$\theta_0 = A_R^{-1}\left(g + D\gamma_0\theta_0 - \alpha\Delta\omega_1\right) \in H^{2k}(\Omega).$$

In turn, we can use again the result in Eq. (1.16) to have that

$$\omega_0 = \mathring{\mathbf{A}}^{-1}f - \alpha G_1\gamma_0\theta_0 + \alpha\lambda G_2\gamma_0\theta_0 - \alpha\mathring{\mathbf{A}}^{-1}\Delta\theta_0 \in H^{2k+2}(\Omega).$$

This concludes the proof of Proposition 1.1.

1.4 A Singular Trace Estimate

In this section, we exploit the underlying analyticity of the thermoelastic semigroup so as to generate pointwise (in time) estimate of boundary traces of the adjoint variables $\phi_t(t)$ and $\vartheta(t)$ of Eq. (1.9). These estimates will be of use to us in the proof of Theorem 1.1, inasmuch as they each reflect a proper "distribution" between the measurement $\mathcal{E}(t)$ of the energy and the observation term—be it ϕ_t or ϑ. The price to pay for these benefical estimates is the appearance therein of singular weights of the form $\frac{1}{t^s}$, where parameter s will depend on the order of derivatives present.

LEMMA 1.1
Let $\vec{x}(t) \equiv [\phi(t), \phi_t(t), \vartheta(t)]$ *denote the solution of the adjoint system Eq.* (1.9), *subject to the initial condition* $\vec{x}(0) = [\phi_0, -\phi_1, \vartheta_0] \in \mathbf{H}$. *Let,moreover,* D_m *be a differential operator of order* $m \geq 0$ *with respect to the interior variables. Then for integers* $k = 1, 2, \dots$, *and all* $t > 0$ *we have*

1. $\|D_m\vartheta(t)\|_{L(\Gamma)} \leq \dfrac{C_k}{t^{\frac{m}{2}+\frac{1}{4}}} \left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^k}} \|\vartheta(t)\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}$;
2. $\|D_m\phi_t(t)\|_{L(\Gamma)} \leq \dfrac{C_k}{t^{\frac{m}{2}+\frac{1}{4}}} \left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^k}} \|\phi_t(t)\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}$;
3. $\|D_1\phi_{tt}(t)\|_{L(\Gamma)} \leq \dfrac{C_k}{t^{\frac{7}{4}}} \left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}$.

PROOF OF LEMMA 1.1 By a trace interpolation result (see, e.g., Reference 38) and the iterative use of a classical PDE moment inequality, we have the following string of estimates, which is valid for any $g \in H^{2^{k+1}(m+1)}(\Omega)$:

$$\begin{aligned}
\|D_m g\|_{L(\Gamma)} &\leq C\,\|D_m g\|_{L(\Omega)}^{\frac{1}{2}} \|D_m g\|_{H^1(\Omega)}^{\frac{1}{2}} \leq C\,\|g\|_{H^m(\Omega)}^{\frac{1}{2}} \|g\|_{H^{m+1}(\Omega)}^{\frac{1}{2}} \\
&\leq C\,\|g\|_{L(\Omega)}^{\frac{1}{2}} \|g\|_{H^{2m}(\Omega)}^{\frac{1}{4}} \|g\|_{H^{2(m+1)}(\Omega)}^{\frac{1}{4}} \leq C\,\|g\|_{L(\Omega)}^{\frac{3}{4}} \|g\|_{H^{4m}(\Omega)}^{\frac{1}{8}} \|g\|_{H^{4(m+1)}(\Omega)}^{\frac{1}{8}} \\
&\leq \dots \leq C\,\|g\|_{L(\Omega)}^{1-\frac{1}{2^k}} \|g\|_{H^{2^k m}(\Omega)}^{\frac{1}{2^{k+1}}} \|g\|_{H^{2^k(m+1)}(\Omega)}^{\frac{1}{2^{k+1}}}.
\end{aligned} \tag{1.20}$$

Now by virtue of the analyticity of the thermoelastic semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ and Proposition 1.1, we have for all $t > 0$,

$$[\phi(t), \phi_t(t), \vartheta(t)] \in D(\mathcal{A}^{2^{k-1}m}) \Rightarrow [\phi_t(t), \vartheta(t)] \in [H^{2km}(\Omega)]^2. \tag{1.21}$$

Setting now $g \equiv \vartheta(t)$ (resp., $\phi_t(t)$) in Eq. (1.20), we obtain

$$\begin{aligned}
\|D_m\vartheta(t)\|_{L(\Gamma)} &\leq C\,\|\vartheta(t)\|_{L(\Omega)}^{1-\frac{1}{2^k}} \|\vartheta(t)\|_{H^{2^k m}(\Omega)}^{\frac{1}{2^{k+1}}} \|\vartheta(t)\|_{H^{2^k(m+1)}(\Omega)}^{\frac{1}{2^{k+1}}} \\
&\leq C\left\|\mathcal{A}^{2^{k-1}m}\vec{x}(t)\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}} \left\|\mathcal{A}^{2^{k-1}(m+1)}\vec{x}(t)\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}} \left\|\vartheta(t)\right\|_{L(\Omega)}^{1-\frac{1}{2^k}} \\
&= C\left\|\mathcal{A}^{2^{k-1}m}e^{\mathcal{A}\frac{t}{2}}e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}} \left\|\mathcal{A}^{2^{k-1}(m+1)}e^{\mathcal{A}\frac{t}{2}}e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}} \left\|\vartheta(t)\right\|_{L(\Omega)}^{1-\frac{1}{2^k}}.
\end{aligned} \tag{1.22}$$

□

At this point, we can invoke the well known pointwise estimate that is valid for any generator of an analytic semigroup: for all time $t > 0$ and integer $m = 1, 2, \dots$,

$$\|\mathcal{A}^m e^{\mathcal{A}t}\|_{\mathcal{L}(\mathbf{H})} \leq \frac{C^m}{t^m}, \tag{1.23}$$

where constant C is independent of m (see, e.g., Reference 33, p. 70). Applying this estimate to the chain Eq. (1.22), we have

$$\|D_m\vartheta(t)\|_{L(\Gamma)} \leq \frac{C}{t^{\frac{m}{2}+\frac{1}{4}}} \left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^k}} \|\vartheta(t)\|_{L(\Omega)}^{1-\frac{1}{2^k}}.$$

This gives (Lemma 1.1, Step 1) (Step 2) is obtained in the very same way, by setting $g = \phi_t$ in Eq. (1.22) and then invoking the containment Eq. (1.21). For (Step 3), we have along the same lines, by means of the trace interpolation inequality in Reference 38 and the containment Eq. (1.21),

$$\begin{aligned} \|D_1\phi_{tt}(t)\|_{L^2(\Gamma)} &\leq C\,\|\phi_{tt}(t)\|^{\frac{1}{2}}_{H^1(\Omega)}\,\|\phi_{tt}(t)\|^{\frac{1}{2}}_{H^2(\Omega)} \leq C\left\|\mathcal{A}^{\frac{1}{2}}\vec{x}_t(t)\right\|^{\frac{1}{2}}_{\mathbf{H}}\|\mathcal{A}\vec{x}_t(t)\|^{\frac{1}{2}}_{\mathbf{H}} \\ &\leq C\|\mathcal{A}^2\vec{x}(t)\|^{\frac{1}{2}}_{\mathbf{H}} \leq \frac{C}{t^{\frac{7}{4}}}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac{1}{2}}_{\mathbf{H}}, \end{aligned}$$

which completes the proof.

1.5 Proof of Theorem 1.1(1)

1.5.1 Estimating the Mechanical Velocity

In what follows, we will have need of the polynomial weight $h(t)$, defined by

$$h(t) \equiv t^s(T-t)^s. \tag{1.24}$$

For the proof of Theorem 1.1(1), we will take $s \equiv 6$.

In terms of the the solution $[\phi, \phi_t, \vartheta]$ of Eq. (1.9) and its corresponding energy $\mathcal{E}(t)$, the necessary inequality for the case of thermal control is

$$\sqrt{\mathcal{E}(T)} \leq C_T\,\|\vartheta\|_{L^2(Q)}\,. \tag{1.25}$$

It is the derivation of this inequality that will drive the proof of Theorem 1.1.

We will start by applying the Laplacian to both sides of the heat equation in Eq. (1.9). This gives

$$\Delta\vartheta_t - \Delta^2\vartheta - \alpha\Delta^2\phi_t = 0 \quad \text{in } \Omega.$$

From this expression and the free boundary conditions in Eq. (1.9), we have that the velocity term ϕ_t satisfies the following elliptic problem for all $t > 0$:

$$\begin{cases} \Delta^2\phi_t(t) = \dfrac{1}{\alpha}\Delta\vartheta_t(t) - \dfrac{1}{\alpha}\Delta^2\vartheta(t) \quad \text{in } \Omega \\ \begin{cases} \Delta\phi_t(t) + (1-\mu)B_1\phi_t(t) = -\alpha\vartheta_t \\ \dfrac{\partial\Delta\phi_t(t)}{\partial\nu} + (1-\mu)\dfrac{\partial B_2\phi_t(t)}{\partial\tau} - \phi_t(t) = \alpha\lambda\vartheta_t \end{cases} \quad \text{on } \Gamma. \end{cases} \tag{1.26}$$

Using this Green's map defined in Eq. (1.15), we have from Eq. (1.26) that the velocity ϕ_t may be written explicitly as

$$\phi_t(t) = \frac{1}{\alpha}\mathring{\mathbf{A}}^{-1}\left[\Delta\vartheta_t(t) - \Delta^2\vartheta(t)\right] - \alpha G_1\gamma_0(\vartheta_t(t)) + \alpha\lambda G_2\gamma_0(\vartheta_t(t)). \tag{1.27}$$

From this, we have

$$\begin{aligned} &\int_0^T h\,\|\phi_t\|^2_{L^2(\Omega)}\,dt \\ &= \int_0^T h\left(\frac{1}{\alpha}\mathring{\mathbf{A}}^{-1}[\Delta\vartheta_t(t) - \Delta^2\vartheta(t)] - \alpha G_1\gamma_0(\vartheta_t(t)) + \alpha\lambda G_2\gamma_0(\vartheta_t(t)), \phi_t\right)_{L^2(\Omega)} dt, \end{aligned} \tag{1.28}$$

where $h(t)$ is the polynomial weight described in Eq. (1.24).

Analysis of the right-hand side of Eq. (1.28).

1.

$$\int_0^T h\left([G_1 - \lambda G_2]\,\gamma_0\vartheta_t, \phi_t\right)_{L^2(\Omega)} dt = -\int_0^T h'([G_1 - \lambda G_2]\,\gamma_0\vartheta, \phi_t)_{L^2(\Omega)}\,dt - \int_0^T h([G_1 - \lambda G_2]\,\gamma_0\vartheta, \phi_{tt})_{L^2(\Omega)}\,dt. \tag{1.29}$$

a. By the regularity posted in Eq. (1.16) and an application of Lemma 1.1 (with $m = 0$ and $k = 2$, say) we have

$$\left|\int_0^T h'([G_1 - \lambda G_2]\,\gamma_0\vartheta, \phi_t)_{L^2(\Omega)}\,dt\right| \le C\int_0^T |h'|\,\|\vartheta\|_{L^2(\Gamma)}\,\|\phi_t\|_{L^2(\Omega)}\,dt$$
$$\le C\int_0^T \frac{|h'|}{t^{\frac{1}{4}}}\left(\frac{h(t)}{h(t)}\right)^{\frac{5}{8}}\|\vartheta(t)\|^{\frac{3}{4}}_{L^2(\Omega)}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac{5}{4}}_{\mathbf{H}}dt.$$

Invoking Hölder's inequality to this right hand side, with Hölder conjugates $(\frac{8}{3}, \frac{8}{5})$, we obtain now the estimate

$$\left|\int_0^T h'([G_1 - \lambda G_2]\,\gamma_0\vartheta, \phi_t)_{L^2(\Omega)}\,dt\right|$$
$$\le C_\epsilon T^{\frac{26}{3}}\int_0^T h(t)\,\|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \tag{1.30}$$

b. Proceeding as above, with $m = 0$ and $k = 2$ in Lemma 1.1, we have

$$\left|\int_0^T h([G_1 - \lambda G_2]\,\gamma_0\vartheta, \phi_{tt})_{L^2(\Omega)}\,dt\right|$$
$$\le C\int_0^T \frac{h(t)}{t^{\frac{1}{4}}}\|\vartheta(t)\|^{\frac{3}{4}}_{L^2(\Omega)}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac{1}{4}}_{\mathbf{H}}\|\mathcal{A}\vec{x}_t(t)\|_{L^2(\Omega)}\,dt$$
$$\le C\int_0^T \frac{h(t)}{t^{\frac{5}{4}}}\|\vartheta(t)\|^{\frac{3}{4}}_{L^2(\Omega)}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac{5}{4}}_{\mathbf{H}}dt$$
$$\le C_\epsilon T^{\frac{26}{3}}\int_0^T h(t)\,\|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \tag{1.31}$$

Combining Eq. (1.30) and Eq. (1.31) now gives

$$\left|\int_0^T h([G_1 - \lambda G_2]\,\gamma_0\vartheta_t, \phi_t)_{L^2(\Omega)}\,dt\right| \le C_\epsilon T^{\frac{20}{3}}\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \tag{1.32}$$

2. Next,

$$\int_0^T h\left(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta_t), \phi_t\right)_{L^2(\Omega)}dt = -\int_0^T h'\left(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta), \phi_t\right)_{L^2(\Omega)}dt - \int_0^T h\left(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta), \phi_{tt}\right)_{L^2(\Omega)}dt. \tag{1.33}$$

a. By Green's theorem and the Lemma 1.1, with $m = 0$ and $k = 2$, we have

$$\begin{aligned}
&\left|\int_0^T h'\big(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta), \phi_t\big)_{L^2(\Omega)} dt\right| \\
&= \left|\int_0^T h'\left(\vartheta, \left(\frac{\partial}{\partial\nu} + I\right)\mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Gamma)} dt - \int_0^T h'\big(\vartheta, \Delta\,\mathring{\mathbf{A}}^{-1}\phi_t\big)_{L^2(\Omega)} dt\right| \\
&\le C_\epsilon T^{\frac{26}{3}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt.
\end{aligned} \tag{1.34}$$

b. Likewise, by Green's Theorem, the analyticity of the semigroup and Lemma 1.1, with $m = 0$ and $k = 2$, we have

$$\left|\int_0^T h(t)\big(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi_{tt}\big)_{L^2(\Omega)} dt\right| \le C_\epsilon T^{\frac{26}{3}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.35}$$

Applying the estimates Eq. (1.34) and Eq. (1.35) to Eq. (1.33) now yields

$$\left|\int_0^T h(\Delta\vartheta_t, \phi_t)_{L^2(\Omega)}\, dt\right| \le C_\epsilon T^{\frac{26}{3}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.36}$$

3. By the Green's identity posted in Eq. (1.10), we have

$$\begin{aligned}
\int_0^T h\big(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\big)_{L^2(\Omega)}\, dt &= \int_0^T h(t)(\vartheta, \phi_t)_{L^2(\Omega)}\, dt \\
&= \int_0^T h(t)\Bigg[\left(\left[\frac{\partial\Delta}{\partial\nu} + (1-\mu)\frac{\partial B_2}{\partial\tau}\right]\vartheta,\ \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Gamma)} \\
&\quad - \left([\Delta + (1-\mu)B_1]\,\vartheta, \frac{\partial}{\partial\nu}\,\mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Gamma)}\Bigg] dt.
\end{aligned}$$

Applying once more the Lemma 1.1 (e.g., with $m = 3$, $k = 3$) we have

$$\left|\int_0^T h\big(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\big)_{L^2(\Omega)}\, dt\right| \le C_\epsilon T^8 \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.37}$$

Combining the expression Eq. (1.28) with the estimates of Eqs. (1.32), (1.36), and (1.37) gives us the following estimate for the mechanical velocity:

LEMMA 1.2
With $s = 6$ in Eq. (1.24), the solution $[\phi, \phi_t, \vartheta]$ of (1.9) satisfies the following estimate for all $\epsilon > 0$:

$$\int_0^T h(t)\,\|\phi_t\|^2_{L^2(\Omega)}\, dt \le C_\epsilon T^8 \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T \mathcal{E}\left(\frac{t}{2}\right) dt.$$

1.5.2 Estimating the Mechanical Displacement

Here, we shall show the following:

LEMMA 1.3
The solution $[\phi, \phi_t, \vartheta]$ of Eq. (1.9) satisfies the following estimate for all $\epsilon, \delta > 0$:

$$\int_0^T h(t)\big\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big\|^2_{L^2(\Omega)}dt \leq CT^{\frac{13}{2}-\delta}\,\|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt + \epsilon\int_0^T h(t)\mathcal{E}(t)dt.$$

PROOF OF LEMMA 1.3 We start by applying the multiplier $h(t)\phi(t)$ to the mechanical component in Eq. (1.9). We arrive at the relation

$$\begin{aligned}
&\int_0^T h(t)\big\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big\|^2_{L^2(\Omega)}dt\\
&= \int_0^T h'(t)(\phi_t, \phi)_{L^2(\Omega)}\,dt + \int_0^T h(t)\,\|\phi_t\|^2_{L^2(\Omega)}\,dt\\
&\quad + \alpha\lambda\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta, \mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt - \alpha\lambda\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta, \mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt\\
&\quad - \alpha\int_0^T h(t)(\vartheta, \Delta\phi)_{L^2(\Omega)}dt + \alpha\int_0^T h(t)\left((\vartheta, \lambda\phi + \frac{\partial\phi}{\partial\nu}\right)_{L^2(\Gamma)}\,dt.
\end{aligned} \tag{1.38}$$

∎

Now, using the elliptic regularity posted in Eq. (1.16) and the usage of Lemma 1.1, with $m = 0$ and $k = 3$, we obtain

$$\begin{aligned}
&\Bigg|\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta, \mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt - \alpha\lambda\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta, \mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt\\
&\qquad - \alpha\int_0^T h(t)(\vartheta, \Delta\phi)_{L^2(\Omega)}\,dt + \alpha\int_0^T h(t)\left((\vartheta, \lambda\phi + \frac{\partial\phi}{\partial\nu}\right)_{L^2(\Gamma)}\,dt\Bigg|\\
&\quad \leq C_\epsilon T^{\frac{26}{3}}\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.
\end{aligned} \tag{1.39}$$

Combining this estimate with that in Lemma 1.2 then gives the preliminary estimate

$$\begin{aligned}
\int_0^T h(t)\big\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big\|^2_{L^2(\Omega)}dt \leq &\left|\int_0^T h'(t)(\phi_t, \phi)_{L^2(\Omega)}\,dt\right|\\
&+ CT^8\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.
\end{aligned} \tag{1.40}$$

Apparently, we must estimate the first term on the right-hand side of Eq. (1.40). To this end, we use the pointwise expression for ϕ_t in Eq. (1.27):

$$\int_0^T h'(\phi_t, \phi)_{L^2(\Omega)}\,dt = \int_0^T h'\left(\frac{1}{\alpha}\mathring{\mathbf{A}}^{-1}(\Delta\vartheta_t - \Delta^2\vartheta) - \alpha G_1\gamma_0\vartheta_t + \alpha\lambda G_2\gamma_0\vartheta_t, \phi\right)_{L^2(\Omega)}\,dt \tag{1.41}$$

1. The abstract Green's Theorem gives

$$\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt = \int_0^T h'\Bigg[(\vartheta,\phi)_{L^2(\Omega)} + \left(\left[\frac{\partial\Delta}{\partial\nu} + (1-\mu)\frac{\partial B_2}{\partial\tau}\right]\vartheta, \mathring{\mathbf{A}}^{-1}\phi\right)_{L^2(\Gamma)} - \left([\Delta + (1-\mu)B_1]\,\vartheta, \frac{\partial}{\partial\nu}\mathring{\mathbf{A}}^{-1}\phi\right)_{L^2(\Gamma)}\Bigg]dt.$$

Applying now the Lemma 1.1 with $m = 3, 2$ yields

$$\left|\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt\right| \le C\int_0^T \frac{|h'|}{t^{\frac{7}{4}}}\|\vartheta\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}dt.$$

Let $k \ge 4$. Then, because $h'(t) = 6t^5(T-2t)(T-t)^5$, we can apply now Hölder's inequality with Hölder conjugates $(2\frac{2^k}{2^k-1}, \frac{1}{\frac{1}{2}+2^{-k-1}})$ so as to have

$$\left|\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt\right| \le C_\epsilon\int_0^T t^{\frac{2^{k-1}-6}{2^k-1}}\,|T-2t|^{\frac{2^{k+1}}{2^k-1}}\,(T-t)^{\frac{2^{k+2}-6}{2^k-1}}\,\|\vartheta\|_{L^2(\Omega)}^2\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt \le CT^{\frac{13\times2^{k-1}-12}{2^k-1}}\int_0^T \|\vartheta\|_{L^2(\Omega)}^2\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt$$

(again this inequality being valid for $k \ge 4$). Now for any $\delta > 0$, we can rechoose integer k large enough so as to have $\frac{13\times2^{k-1}-12}{2^k-1} \ge \frac{13}{2} - \delta$. This gives, then, for $T < 1$,

$$\left|\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt\right| \le CT^{\frac{13}{2}-\delta}\int_0^T \|\vartheta\|_{L^2(\Omega)}^2\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \quad (1.42)$$

(This is the term which ultimately dictates the singularity.)

2. Next,

$$\int_0^T h'[(G_1-\lambda G_2)\gamma_0\vartheta_t, \phi]_{L^2(\Omega)}dt = -\int_0^T h''[(G_1-\lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}\,dt - \int_0^T h'[(G_1-\lambda G_2)\gamma_0\vartheta, \phi_t]_{L^2(\Omega)}\,dt. \quad (1.43)$$

   a. By the regularity posted in Eq. (1.16) and Lemma 1.1,

$$\left|\int_0^T h''[(G_1-\lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}\,dt\right| \le C\int_0^T \frac{|h''|}{t^{\frac{1}{4}}}\,\|\vartheta\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}\,\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}dt.$$

   Applying Hölder's inequality to the right-hand side, with Hölder conjugates $(2\frac{2^k}{2^k-1}, \frac{1}{\frac{1}{2}+2^{-k-1}})$ now yields

$$\left|\int_0^T h''[(G_1-\lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}dt\right| \le C\int_0^T \frac{|h''|}{t^{\frac{1}{4}}}\left(\frac{h(t)}{h(t)}\right)^{\frac{1}{2}+2^{-k-1}} \times \|\vartheta\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}\,\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}dt \le C_\epsilon T^{3\frac{5\times2^{k-1}-4}{2^k-1}}\int_0^T \|\vartheta\|_{L^2(\Omega)}^2 + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.$$

Because for any $\delta > 0$, we can choose integer k large enough so that $3\frac{5\times 2^{k-1}-4}{2^k-1} \geq \frac{15}{2} - \delta$, we then get

$$\left| \int_0^T h''[(G_1 - \lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}\, dt \right| \leq C_\epsilon T^{\frac{15}{2}-\epsilon} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.44}$$

b. In the same way as above, we have for integer k large enough in Lemma 1.1,

$$\left| \int_0^T h'[(G_1 - \lambda G_2)\gamma_0\vartheta, \phi_t]_{L^2(\Omega)}\, dt \right| \leq C_\epsilon T^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.45}$$

The estimates Eqs. (1.44) and (1.45), applied to the relation Eq. (1.43) now give

$$\left| \int_0^T h'[(G_1 - \lambda G_2)\gamma_0\vartheta_t, \phi]_{L^2(\Omega)}\, dt \right| \leq C_\epsilon T^{\frac{15}{2}-\epsilon} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.46}$$

for integer k large enough.

3.

$$\int_0^T h'\left(\mathring{\mathbf{A}}^{-1}\Delta\vartheta_t, \phi\right)_{L^2(\Omega)} dt = -\int_0^T h''(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}\, dt - \int_0^T h'\left(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Omega)} dt \tag{1.47}$$

a. By Green's Theorem and Lemma 1.5, we have in a fashon similar to that in (*1.a.*),

$$\begin{aligned} \left| \int_0^T h''(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}\, dt \right| &= \left| -\int_0^T h''(\theta, \Delta\mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)} \right. \\ &\left. + \int_0^T h''\left[\theta, \left(\frac{\partial}{\partial\nu} + \lambda\right)\mathring{\mathbf{A}}^{-1}\phi\right]_{L^2(\Gamma)} \right| \leq C_\epsilon T^{3\frac{5\times 2^{k-1}-4}{2^k-1}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} \\ &+ \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt \leq CT^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt, \end{aligned} \tag{1.48}$$

for integer k large enough.

b. In the same way,

$$\left| \int_0^T h'\left(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Omega)} dt \right| \leq CT^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.49}$$

Eqs. (1.47), (1.48), and (1.49) together give the estimate

$$\left| \int_0^T h'\left(\Delta\vartheta_t, \mathring{\mathbf{A}}^{-1}\phi\right)_{L^2(\Omega)} dt \right| \leq CT^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.50}$$

Combining Eqs. (1.40), (1.41), (1.46), (1.50), and (1.42) will complete the proof of Lemma 1.3.

1.5.3 Conclusion of the Proof of Theorem 1.1(1)

Combining Lemmas 1.2 and 1.3 gives the following estimate for the energy:

$$\int_0^T \mathcal{E}(t)\,dt \le C_\epsilon T^{\frac{13}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt;$$

or after changing limits of integration,

$$\int_0^{\frac{T}{2}} [(1-\epsilon)h(t) - 2\epsilon h(2t)]\,\mathcal{E}(t)\,dt + (1-\epsilon)\int_{\frac{T}{2}}^T h(t)\mathcal{E}(t)dt \le C_\epsilon T^{\frac{13}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt.$$

For $\epsilon > 0$ small enough, this yields then

$$\int_{\frac{T}{2}}^T h(t)\mathcal{E}(t)\,dt \le C_\epsilon T^{\frac{13}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt.$$

Using the dissipation inherent in the thermoelastic system (i.e., $\mathcal{E}(t) \le \mathcal{E}(s)$ for $s \le t$), we finally obtain

$$\mathcal{E}(T) \le C T^{\frac{13}{2}-\delta-13} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt.$$

This establishes the inequality Eq. (1.25), with $C_T = CT^{-q}$, where $q = \frac{13}{4} - \frac{\delta}{2}$, for any $\delta > 0$. This concludes the proof of Theorem 1.1(1).

1.6 Proof of Theorem 1.1(2)

1.6.1 A First Supporting Estimate

In what follows, we will again make use of the polynomial weight $h(t)$ in Eq. (1.24), with $s = 4$ therein.

In the present case of mechanical control, the necessary inequality (Eq. (1.12)) becomes

$$\sqrt{\mathcal{E}(T)} \le C_T \|\phi_t\|_{L^2(Q)}\,. \tag{1.51}$$

to be valid for all finite energy solutions to Eq. (1.9).

We start by establishing the following estimate:

PROPOSITION 1.2
The solution $[\phi, \phi_t, \vartheta]$ *of Eq.* (1.9) *satisfies the relation*

$$\left|\int_0^T h(t)\left(A_R^{-1}\vartheta_t, \vartheta\right)_{L^2(\Omega)} dt\right| \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)}\,dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.$$

PROOF OF PROPOSITION 1.2 From the mechanical component of Eq. (1.9) we have, after an extra differentiation in time, the expression $-\alpha\Delta\vartheta_t = \frac{\partial^3}{\partial t^3}\phi + \Delta^2\phi_t$; whence we obtain

$$A_R^{-1}\vartheta_t = \frac{1}{\alpha}A_R^{-2}\frac{\partial^3}{\partial t^3}\phi + \frac{1}{\alpha}A_R^{-2}\Delta^2\phi_t,$$

where the positive definite, self-adjoint operator $A_R : D(A_R) : L^2(\Omega) \to L^2(\Omega)$ is as defined in Eq. (1.19). Subsequently, we will have the following relation:

$$\int_0^T h(t)\big(A_R^{-1}\vartheta_t, \vartheta\big)_{L^2(\Omega)}\, dt = \frac{1}{\alpha}\int_0^T h(t)\left(\frac{\partial^3}{\partial t^3}\phi, A_R^{-2}\vartheta\right)_{L^2(\Omega)} dt + \frac{1}{\alpha}\int_0^T h(t)\big(\Delta^2\phi_t, A_R^{-2}\vartheta\big)_{L^2(\Omega)}\, dt. \tag{1.52}$$

∎

We need to estimate the right-hand side of this expression.

1. For the first term on the right-hand side of Eq. (1.52), integration by parts gives

$$\int_0^T h(t)\left(\frac{\partial^3}{\partial t^3}\phi, A_R^{-2}\vartheta\right)_{L^2(\Omega)} dt = \int_0^T h(t)\big(\phi_t, A_R^{-2}\vartheta_{tt}\big)_{L^2(\Omega)}\, dt + 2\int_0^T h'(t)\big(\phi_t, A_R^{-2}\vartheta_t\big)_{L^2(\Omega)}\, dt + \int_0^T h''(t)\big(\phi_t, A_R^{-2}\vartheta\big)_{L^2(\Omega)}\, dt. \tag{1.53}$$

We proceed to scrutinize each term on the right-hand side. To this end, we introduce the (Robin) map $R \in \mathcal{L}[L^2(\Gamma), L^2(\Omega)]$, defined by

$$Rf = g \Leftrightarrow \Delta g = 0 \text{ on } \Omega \quad \text{and} \quad \frac{\partial g}{\partial \nu} + \lambda g = f \text{ on } \Gamma \tag{1.54}$$

(by elliptic regularity, we have in fact that $R \in \mathcal{L}[H^s(\Gamma), H^{s+\frac{3}{2}}(\Omega)]$ for all real s). Using this quantity with the heat equation in Eq. (1.9), we will then have the relations

$$\begin{aligned} A_R^{-2}\vartheta_t &= -A_R^{-1}\vartheta - \alpha A_R^{-1}\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_t; \\ A_R^{-2}\vartheta_{tt} &= \vartheta + \alpha\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_t - \alpha A_R^{-1}\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_{tt}. \end{aligned} \tag{1.55}$$

a. From Eq. (1.56), we have

$$\left|\int_0^T h(t)\big(\phi_t, A_R^{-2}\vartheta_{tt}\big)_{L^2(\Omega)}\, dt\right| \le \int_0^T h(t) \times \left|\left\{\phi_t, \vartheta + \alpha\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_t\right\}_{L^2(\Omega)}\right| dt + \int_0^T h(t)\left|\left\{\phi_t, \alpha A_R^{-1}\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_{tt}\right\}_{L^2(\Omega)}\right| dt. \tag{1.56}$$

To handle the most problematic term on the right-hand side of this expression (with again $\vec{x}(t) = [\phi(t), \phi_t(t), \vartheta(t)]$), we use the singular trace estimate in Lemma 1.1(3):

$$\begin{aligned}
\int_0^T h(t) \left| \left(\phi_t, A_R^{-1} R \frac{\partial}{\partial \nu} \phi_{tt} \right)_{L^2(\Omega)} \right| dt &\le C \int_0^T h(t) \|\phi_t\|_{L^2(\Omega)} \left\| \frac{\partial}{\partial \nu} \phi_{tt} \right\|_{L^2(\Gamma)} dt \\
&\le C \int_0^T \frac{h(t)}{t^{\frac{3}{4}} t} \|\phi_t\|_{L^2(\Omega)} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}(0) \right\|_{\mathbf{H}} dt \\
&\le C_\epsilon \int_0^T \frac{h(t)}{t^{\frac{7}{2}}} \|\phi_t\|^2_{L^2(\Omega)} dt \\
&\quad + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt \\
&\le C_\epsilon T^{\frac{9}{2}} \int_0^T \|\phi_t\|^2_{L^2(\Omega)} dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt.
\end{aligned}$$

Applying this estimate to Eq. (1.56) and treating in like fashion the other terms on the right-hand side thereof, we have

$$\left| \int_0^T h(t) \left(\phi_t, A_R^{-2} \vartheta_{tt} \right)_{L^2(\Omega)} dt \right| \le C_\epsilon T^{\frac{9}{2}} \int_0^T \|\phi_t\|^2_{L^2(\Omega)} dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.57}$$

b. Using the first relation in Eq. (1.56), we have, analogously to what was obtained in (*1.a*),

$$\begin{aligned}
&\left| \int_0^T h''(t) \left(\phi_t, A_R^{-2} \vartheta \right)_{L^2(\Omega)} dt + 2 \int_0^T h'(t) \left(\phi_t, A_R^{-2} \vartheta_t \right)_{L^2(\Omega)} dt \right| \\
&\quad \le C \int_0^T \left[|h''(t)| + \frac{|h'(t)|}{t^{\frac{3}{4}}} \right] \|\phi_t\|_{L^2(\Omega)} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}(0) \right\|_{\mathbf{H}} dt \\
&\quad \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)} dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt.
\end{aligned} \tag{1.58}$$

Combining Eqs. (1.56) and (1.58) now gives

$$\left| \int_0^T h(t) \left(\frac{\partial^3}{\partial t^3} \phi, A_R^{-2} \vartheta \right)_{L^2(\Omega)} dt \right| \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)} dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.59}$$

2. By the "Green's" formula in Eq. (1.10), we have

$$\begin{aligned}
\int_0^T h(t) \left(\Delta^2 \phi_t, A_R^{-2} \vartheta \right)_{L^2(\Omega)} dt &= \int_0^T h(t) a\left(\phi_t, A_R^{-2} \vartheta \right)_{L^2(\Omega)} dt \\
&\quad + \int_0^T h(t) \left(\alpha \lambda \vartheta_t + \phi_t, A_R^{-2} \vartheta \right)_{L^2(\Gamma)} dt - \int_0^T h(t) \left[\Delta \phi_t + (1-\mu) B_1 \phi_t, \frac{\partial}{\partial \nu} A_R^{-2} \vartheta \right]_{L^2(\Gamma)} dt \\
&= - \int_0^T h(t) \left[\Delta \phi_t + (1-\mu) B_1 \phi_t, \left(\lambda I + \frac{\partial}{\partial \nu} \right) A_R^{-2} \vartheta \right]_{L^2(\Gamma)} dt \\
&\quad + \int_0^T h(t) \left(\phi_t, \Delta^2 A_R^{-2} \vartheta \right)_{L^2(\Omega)} dt + \int_0^T h(t) \left\{ \frac{\partial}{\partial \nu} \phi_t, [\Delta + (1-\mu) B_1] A_R^{-2} \vartheta \right\}_{L^2(\Gamma)} dt \\
&\quad - \int_0^T h(t) \left\{ \phi_t, \left[\frac{\partial \Delta}{\partial \nu} + (1-\mu) \frac{\partial B_2}{\partial \tau} - I \right] A_R^{-2} \vartheta \right\}_{L^2(\Gamma)} dt.
\end{aligned} \tag{1.60}$$

For the first term on the right-hand side of Eq. (1.60), we apply the Lemma 1.1(1) (with $m = 2$ and $D_2 \equiv \Delta + (1-\mu)B_1$ therein) so as to have

$$\left| \int_0^T h(t) \left[\Delta\phi_t + (1-\mu)B_1\phi_t, \left(\lambda I + \frac{\partial}{\partial\nu} \right) A_R^{-2}\vartheta \right]_{L^2(\Gamma)} dt \right|$$
$$\leq C \int_0^T \frac{h(t)}{t^{\frac{5}{4}}} \|\phi_t\|_{H^2(\Omega)}^{1-\frac{1}{2^k}} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}_0 \right\|_{\mathbf{H}}^{\frac{1}{2^k}} \|\vartheta\|_{L^2(\Omega)}\, dt \leq C \int_0^T \frac{h(t)}{t^{\frac{5}{4}}} \|\phi_t\|_{H^2(\Omega)}^{1-\frac{1}{2^k}} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}_0 \right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}.$$

Now letting $k = 2$, say, we can invoke Hölder's inequality, with Hölder conjugates $\left(\frac{8}{3}, \frac{5}{3}\right)$, to obtain the estimate

$$\left| \int_0^T h(t) \left(\Delta\phi_t + (1-\mu)B_1\phi_t, \left[\lambda I + \frac{\partial}{\partial\nu} \right] A_R^{-2}\vartheta \right)_{L^2(\Gamma)} dt \right|$$
$$\leq C_\epsilon T^{\frac{14}{3}} \int_0^T h(t) \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}(t/2)\, dt. \tag{1.61}$$

Applying this estimate to the right-hand side of Eq. (1.60) and subsequently handling the other terms thereof in a similar way—via the use of Lemma 1.1—we will have

$$\left| \int_0^T h(t) \left(\Delta^2\phi_t, A_R^{-1}\vartheta \right)_{L^2(\Omega)} dt \right|$$
$$\leq C_\epsilon T^{\frac{14}{3}} \int_0^T h(t) \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}(t)\, dt. \tag{1.62}$$

Combining Eqs. (1.52), (1.59), and (1.62) concludes the proof of Proposition 1.2.

1.6.2 Conclusion of the Proof of Theorem 1.1(2)

1. *Estimating the Thermal Component.* Applying the multiplier $h(t)A_R^{-1}\vartheta(t)$ to the heat component of the system Eq. (1.9) and subsequently invoking Proposition 1.2, we have

$$\begin{aligned}
\int_0^T h(t) \|\vartheta\|_{L^2(\Omega)}^2 &= -\int_0^T h(t)\left(A_R^{-1}\vartheta_t, \vartheta\right) dt \\
&\quad - \alpha \int_0^T h(t) \left\{ \left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu} \right) \right] \phi_t, \vartheta \right\} dt \\
&\leq C_\epsilon T^4 \int_0^T \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt \\
&\quad + \int_0^T h(t) \left\| \lambda\phi_t + \frac{\partial}{\partial\nu}\phi_t \right\|_{L^2(\Gamma)} \|\vartheta\|_{L^2(\Omega)}\, dt.
\end{aligned} \tag{1.63}$$

Via the Lemma 1.1 (with $m = 1$, $D_1 = \lambda I + \frac{\partial}{\partial\nu}$, and $k = 1$, say), we can estimate the third term on the right-hand side of Eq. (1.63) as

$$\int_0^T h(t) \left\| \lambda\phi_t + \frac{\partial}{\partial\nu}\phi_t \right\|_{L^2(\Gamma)} \|\vartheta\|_{L^2(\Omega)}\, dt \leq CT^5 \int_0^T \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt.$$

Combining this estimate with Eq. (1.63), we now obtain

$$\int_0^T h(t) \|\vartheta\|_{L^2(\Omega)}^2 \leq C_\epsilon T^4 \int_0^T \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.64}$$

2. *Estimating the Mechanical Component.* Here, we apply the multiplier intrinsic to uncoupled plates and beams. To wit, from the mechanical component of Eq. (1.9), we have via $h(t)\phi(t)$ and an invocation of the Green's Theorem Eq. (1.10) the expression

$$\int_0^T \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} dt = -\alpha \int_0^T h(t)(\vartheta, \Delta\phi)\, dt + \int_0^T h'(t)(\phi_t, \phi)\, dt + \int_0^T h(t)\, \|\phi_t\|^2_{L^2(\Omega)}\, dt. \tag{1.65}$$

Applying the estimate of Eq. (1.64) (available for the thermal component) now gives

$$\int_0^T \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} dt \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.66}$$

Combining the estimates of Eqs. (1.64) and (1.66) now give the estimate for the energy

$$\int_0^T h(t)\mathcal{E}(t)\, dt \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt.$$

With this in hand, we can proceed as in the previous case so as to have the observability inequality Eq. (1.51), with $C_T = T^{-\frac{5}{2}}$. Subsequently, we will determine that in the present case of mechanical control, one has $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$. This concludes the proof of Theorem 1.1(2) with free boundary conditions and one control.

1.7 Proof of Theorem 1.1(3)

Here we set the index $s = 2$ in Eq. (1.24). In this present case of dual—mechanical and thermal—control, the necessary inequality is

$$\sqrt{\mathcal{E}(T)} \le C_T(\|\phi_t\|_{L^2(Q)} + \|\vartheta\|_{L^2(Q)}), \tag{1.67}$$

where again $[\phi, \phi_t, \vartheta]$ solve the homogeneous system Eq. (1.9). Using the relation Eq. (1.65), we have

$$\begin{aligned}(1-\epsilon)\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} dt &\le C_\epsilon \int_0^T \left(h(t) + \frac{[h'(t)]^2}{h(t)}\right)\left[\|\phi_t\|^2_{L^2(\Omega)} + \|\vartheta\|^2_{L^2(\Omega)}\right] dt \\ &\le CT^2 \int_0^T \left[\|\phi_t\|^2_{L^2(\Omega)} + \|\vartheta\|^2_{L^2(\Omega)}\right] dt.\end{aligned}$$

This then gives

$$\int_0^T h(t)\mathcal{E}(t)\, dt \le CT^2 \int_0^T \left[\|\phi_t\|^2_{L^2(\Omega)} + \|\vartheta\|^2_{L^2(\Omega)}\right] dt,$$

whence we obtain the inequality Eq. (1.67). From here, we can use the usual algorithmic argument so as to have $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{3}{2}})$. This concludes the proof of Theorem 1.1(3).

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Chapter 2

Interior and Boundary Stabilization of Navier-Stokes Equations

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Abstract We report on very recent work on the stabilization of the steady-state solutions to Navier-Stokes equations on an open bounded domain $\Omega \subset R^d$, $d = 2, 3$, by either interior or else boundary control.

More precisely, as to the interior case, we obtain that the steady-state solutions to Navier-Stokes equations on $\Omega \subset R^d$, $d = 2, 3$, with no-slip boundary conditions, are locally exponentially stabilizable by a finite-dimensional feedback controller with support in an arbitrary open subset $\omega \subset \Omega$ of positive measure. The (finite) dimension of the feedback controller is minimal and is related to the largest algebraic multiplicity of the unstable eigenvalues of the linearized equation.

Second, as to the boundary case, we obtain that the steady-state solutions to Navier-Stokes equations on a bounded domain $\Omega \subset R^d$, $d = 2, 3$, are locally exponentially stabilizable by a boundary closed-loop feedback controller, acting tangentially on the boundary $\partial\Omega$, in the Dirichlet boundary conditions. If $d = 3$, the nonlinearity imposes and dictates the requirement that stabilization must occur in the space $[H^{\frac{3}{2}+\epsilon}(\Omega)]^3$, $\epsilon > 0$, a high topological level. A first implication thereof is that, for $d = 3$, the boundary feedback stabilizing controller *must* be infinite dimensional. Moreover, it generally acts on the entire boundary $\partial\Omega$. Instead, for $d = 2$, where the topological level for stabilization is $[H^{\frac{3}{2}-\epsilon}(\Omega)]^2$, the boundary feedback stabilizing controller can be chosen to act on *an arbitrarily small* portion of the boundary. Moreover, still for $d = 2$, it may even be *finite* dimensional, and this occurs if the linearized operator is diagonalizable over its finite-dimensional unstable subspace.

2.1 Introduction

We hereby report on recent joint work on the stabilization of steady-state solutions to Navier-Stokes equations on an open bounded domain $\Omega \subset R^d$, $d = 2, 3$, by either *interior* feedback control or else *boundary* feedback control. The case of interior control is taken from the joint work with Triggiani in Reference 4. The case of boundary control is taken from the joint work with Lasiecka and Triggiani in Reference 3. To enhance readability, we provide independent accounts of each case.

2.2 Part I: Interior Control [4]

2.2.1 Introduction

The controlled Navier-Stokes equations

Consider the controlled Navier-Stokes equations (see Reference 6, p. 45, and Reference 13, p. 253 for the uncontrolled case $u \equiv 0$) with the non-slip Dirichlet B.C.:

$$\begin{aligned} y_t(x,t) - \nu\Delta y(x,t) + (y\cdot\nabla)y(x,t) &= m(x)u(x,t) + f_e(x) \\ &\quad + \nabla p(x,t) \text{ in } Q = \Omega\times(0,\infty), \end{aligned} \tag{2.1}$$

$$\begin{aligned} \nabla\cdot y &= 0 && \text{in } Q; \\ y &= 0 && \text{on } \Sigma = \partial\Omega\times(0,\infty); \\ y(x,0) &= y_0(x) && \text{in } \Omega. \end{aligned}$$

Here, Ω is an open, smooth, bounded domain of R^d, $d = 2, 3$; m is the characteristic function of an open smooth subset $\omega \subset \Omega$ of positive measure; u is the control input; and $y = (y_1, y_2, \ldots, y_d)$ is the state (velocity) of the system. The function $v = mu$ can be viewed itself as an internal controller with support in $Q_\omega = \omega\times(0,\infty)$. The functions y_0, $f_e \in [L^2(\Omega)]^d$ are given, the latter being a body force, whereas p is the unknown pressure.

Let $(y_e, p_e) \in [(H^2(\Omega)]^d \cap V) \times H^1(\Omega)$ be a steady-state solution to Eq. (2.1), that is,

$$-\nu\Delta y_e + (y_e\cdot\nabla)y_e = f_e + \nabla p_e \quad \text{in } \Omega; \tag{2.2}$$

$$\begin{aligned} \nabla\cdot y_e &= 0 && \text{in } \Omega; \\ y_e &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The steady-state solution is known to exist for $d = 2, 3$, (see Reference 6, Theorem 7.3, p. 59). Here [6, p. 9], [13, p. 18]

$$\begin{aligned} V &= \left\{y \in \left[H_0^1(\Omega)\right]^d;\ \nabla\cdot y = 0\right\}, \quad \text{with norm } \|y\|_V \equiv \|y\| \\ &= \left\{\int_\Omega |\nabla y(x)|^2 d\Omega\right\}^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Literature

According to some recent results of Imanuvilov [9] (see also Reference 1) any such solution y_e is locally exactly controllable on every interval $[0, T]$ with controller u with support in Q_ω. More precisely, if the distance $\|y_e - y_0\|_{H^2(\Omega)}$ is sufficiently small, then there is a solution (y, p, u) to Eq. (2.1) of appropriate regularity such that $y(T) \equiv y_e$. The steering control is *open-loop* and depends on the initial condition. Subsequently, Reference 2 proved that any steady-state solution y_e is locally exponentially stabilizable by means of an *infinite-dimensional* feedback controller by using the controllability of the linear Stokes equation. In contrast, here we shall prove, via the state decomposition technique of References 14 and 15 and the first-order stabilization Riccati equation method developed in our previous work (Reference 2; see also Reference 5 still in the parabolic case, as well as Reference 11 in the hyperbolic case), that any steady-state solution y_e is locally exponentially stabilizable by a *finite-dimensional closed-loop* feedback controller of the form

$$u = -\sum_{i=1}^{2K}(R_N(y-y_e), \psi_i)_\omega\psi_i, \tag{2.4}$$

where $R_N \in \mathcal{L}(\mathcal{D}(A^{\frac{1}{4}})] \cap \mathcal{L}[\mathcal{D}(A^{\frac{1}{2}}); H)$ is the solution of the algebraic Riccati Eq. (2.19) below associated with the linearized system of Eq. (2.14) below and $\{\psi_i\}_{i=1}^{2K}$ is an explicitly constructed (in (3.3.5) of Reference 4) system of functions related to the space of eigenfunctions corresponding to the unstable eigenvalues of such linearized system. Here A is the Stokes operator defined by Eq. (2.6); H the space in Eq. (2.5); and $(\cdot\,,\,\cdot)_\omega$ is the scalar product in $[L^2(\omega)]^d$. The present closed-loop feedback stabilization result has two main features, besides being finite dimensional:

1. It is more precise and less restrictive concerning the vectors y_0 and y_e than the open-loop version provided by the local exact controllability result established in Reference 9 or the closed-loop stabilization in Reference 2 (in that smallness of the distance between y_0 and y_e is measured in the $\mathcal{D}(A^{\frac{1}{4}})$-norm, i.e., the $[H^{\frac{1}{2}}(\Omega)]^d$-norm, see the set $\mathcal{V}_\rho$ in Eq. (2.22) below rather than in the $[H^2(\Omega)]^d$-norm, as recalled above, where A is defined in Eq. (2.6)).

2. It is independent of the Carleman inequality for the Stokes equation, which is necessary for the proof of local controllability.

There is a large body of literature on the stabilization problem of steady-state solutions to Navier-Stokes equations. Here we confine ourselves to mention only a few of the papers (References 2 and 7), which are more related to this present work. We also refer to the recent paper of Fursikov [8] for a study of a boundary—rather than interior—problem for the Navier-Stokes equations, which, however, does not pertain to the topic of feedback stabilization in the established sense, as in the present paper.

Notation

Here we shall use the standard notation for the spaces of summable functions and Sobolev spaces on Ω. In particular, $H^s(\Omega)$ is the Sobolev space of order s with the norm denoted by $\|\cdot\|_s$. The following notation will also be used:

$$\nabla \cdot y = \text{div}\, y,\ (y\cdot\nabla)y = y_i D_i y_j = y\cdot\nabla y_j,\quad j = 1,\dots,d,\quad D_i = \frac{\partial}{\partial x_i};$$

$$\begin{aligned} H &= \{y \in [L^2(\Omega)]^d;\ \nabla\cdot y = 0,\ y\cdot n = 0 \text{ on } \partial\Omega\} \quad \text{(see Reference 6, p. 7)};\\ H^\perp &= \{y \in [L^2(\Omega)]^d : y = \nabla p,\ p \in H^1(\Omega)\},\ [L^2(\Omega)]^d = H + H^\perp, \end{aligned} \tag{2.5}$$

$H^\perp$ being the orthogonal complement of H in $[L^2(\Omega)]^d$ (see Reference 13, p. 15) with summation convention to be used throughout the paper, presently in $i = 1,\dots,d$, where n is the outward normal to the boundary $\partial\Omega$ of Ω. We shall denote by $P : [L^2(\Omega)]^d \to H$ the orthogonal Leray projector (Reference 6, p. 9), and moreover (Reference 6, p. 31),

$$Ay = -P\Delta y,\quad \forall\, y \in \mathcal{D}(A) = [H^2(\Omega)]^d \cap V,\quad V = \mathcal{D}(A^{\frac{1}{2}}), \tag{2.6}$$

which is a self-adjoint positive definite operator in H with compact (resolvent) A^{-1} on H (Reference 6, p. 32). Accordingly, the fractional powers $A^s, 0 < s < 1$, are well defined (Reference 6, p. 33). We have $V = \mathcal{D}(A^{\frac{1}{2}})$ (Reference 6, p. 33). Furthermore, we define $B : V \to V'$ by (Reference 6, p. 47, p. 54), (Reference 13, p. 162),

$$By = P[(y\cdot\nabla)y],\quad (By, w) = b(y, y, w),\quad \forall\, y, w \in V, \tag{2.7}$$

where the trilinear form is defined by (Reference 6, p. 49), (Reference 13, p. 161)

$$b(y, z, w) = \int_\Omega y_i(D_i z_j)w_j dx = \int_\Omega \langle y\cdot\nabla z, w\rangle_{R^d} d\Omega,\quad y, w \in H,\ z \in V. \tag{2.8}$$

We shall denote by $(\cdot, \cdot)$ the scalar product in both H and $[L^2(\Omega)]^d$. Similarly, we shall denote by the same symbol $|\cdot|$ the norm of both $[L^2(\Omega)]^d$ and H and by $\|\cdot\|$ the norm of the space V as defined in Eq. (2.3).

Preliminaries

In the notation introduced above, Eq. (2.1) can be equivalently rewritten in abstract form as

$$\frac{dy}{dt} + \nu Ay + By = P(mu + f_e); \qquad y(0) = y_0 \in H, \tag{2.9}$$

because the procedure of applying P to Eq. (2.1) eliminates the pressure from the equations in Reference 6, p. 47, the orthogonal space $H^\perp$ to H being made up of $[L^2(\Omega)]^d$-functions which are the gradients of $H^1(\Omega)$-functions by Eq. (2.5). Moreover, $y \in H$ for Eq. (2.1) implies $Py_t = y_t$.

2.2.2 Main Results

Assumptions

1. The boundary $\partial\Omega$ of Ω is a finite union of $d-1$ dimensional C^2-connected manifolds. Moreover, the boundary $\partial\omega$ of ω is of class C^2.

2. The steady-state solution (y_e, p_e) defined in Eq. (2.2) belongs to $([H^2(\Omega)]^d \cap V) \times H^1(\Omega)$, where we recall from Eq. (2.6) that then $y_e \in \mathcal{D}(A)$. (For $d = 2, 3$, this property is guaranteed by Reference 6, Theorem 7.3, p. 59) on y_e, for $f_e \in H$, followed by Reference 6, Theorem 3.11, p. 30 on p_e, for sufficiently smooth $\partial\Omega$.)

Preliminaries. The translated problem

By the substitutions $y \to y_e + y$, $p \to p_e + p$, we are readily led via Eq. (2.1), Eq. (2.2) to the study of *null stabilization* of the equation

$$y_t - \nu\Delta y + (y\cdot\nabla)y + (y_e\cdot\nabla)y + (y\cdot\nabla)y_e = mu + \nabla p \text{ in } Q; \tag{2.10}$$

$$\begin{aligned} \nabla\cdot y &= 0 \quad \text{in } Q;\\ y &= 0 \quad \text{on } \Sigma;\\ y(x,0) = y^0(x) &= y_0(x) - y_e(x). \end{aligned}$$

By use of Eq. (2.5) on H, Eq. (2.6) and Eq. (2.7) on A and B, we see that Eq. (2.10), after application of P, can be rewritten abstractly as

$$\frac{dy}{dt} + \nu Ay + A_0 y + By = P(mu),\ t > 0; \qquad y(0) = y^0 \tag{2.11}$$

[compare with Eq. (2.9), again $Py_t = y_t$, because $y \in H$ by Eq. (2.10)], where we have now introduced the operator $A_0 \in \mathcal{L}(V; H)$,

$$A_0 y = P[(y_e\cdot\nabla)y + (y\cdot\nabla)y_e], \quad \mathcal{D}(A_0) = V = \mathcal{D}(A^{\frac{1}{2}}), \tag{2.12a}$$

or equivalently, recalling Eq. (2.7),

$$(A_0 y, z) = b(y_e, y, z) + b(y, y_e, z), \quad \forall\, y \in V,\ z \in H. \tag{2.12b}$$

The operator A_0 in Eq. (2.12) is well-defined $H \supset V = \mathcal{D}(A_0) \to H$. This follows from the estimate

$$|A_0 y| \le C_1 \|y_e\|_2 \|y\|, \quad \forall\, y \in V = \mathcal{D}(A_0) = \mathcal{D}(A^{\frac{1}{2}}), \tag{2.13}$$

which is obtained directly by use of the definition of Eq. (2.12b).

The linearized problem

Next, we consider the following linearized system of the translated model in Eq. (2.10) or Eq. (2.11):

$$\frac{dy}{dt} + \nu Ay + A_0 y = P(mu),\ t > 0; \qquad y(0) = y^0 \in H. \tag{2.14}$$

We have already noted in Eq. (2.6) that the operator $-\nu A$ ($\nu > 0$, the viscosity coefficient) is negative self-adjoint and has compact resolvent on H. Thus, $-\nu A$ generates an analytic (self-adjoint) C_0-semigroup on H. It then follows from here and from $\mathcal{D}(A_0) = V = \mathcal{D}(A^{\frac{1}{2}})$, as noted in Eq. (2.6) and in Eq. (2.13), that: *the perturbed operator*

$$\mathcal{A} = -(\nu A + A_0), \quad \textit{with domain } \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) = [H^2(\Omega)]^d \cap V \tag{2.15}$$

likewise has compact resolvent and generates an analytic C_0-semigroup on H. This is well known. It follows from the above claim that the operator $\mathcal{A}$ has a finite number N of eigenvalues λ_j with Re $\lambda_j \geq 0$ (the unstable eigenvalues). The eigenvalues are repeated according to their algebraic multiplicity ℓ_j. Let $\{\varphi_j\}_{j=1}^N$ be a corresponding system of generalized eigenfunctions, $\varphi_j = \varphi_j^1 + i\varphi_j^2$, $j = 1, \ldots, N$ of $\mathcal{A}$. (See Reference 10, p. 41, p. 181.) More precisely, we shall denote by M the number of *distinct* unstable eigenvalues, so that $\ell_1 + \ell_2 + \cdots + \ell_M = N$. In order to state our first result, we finally need to introduce the following finite-dimensional real spaces X_N^α, $\alpha = 1, 2$ as well as the following natural number K:

$$X_N^\alpha = \text{span}\{\varphi_j^\alpha\}_{j=1}^N; \qquad K = \max\{\ell_j; \qquad 1 \leq j \leq M\}. \tag{2.16}$$

Main results. Linearized problem Eq. (2.14)

We first state the following feedback stabilization result for the linearized system in Eq. (2.14).

THEOREM 2.1

Let $\epsilon > 0$ be arbitrary but fixed, and let $\gamma_0 = |Re\lambda_{N+1}| - \epsilon$. Then, for each λ, $0 \leq \lambda \leq \gamma_0$, there are functions $\{\psi_i\}_{i=1}^K \subset X_N^1$, $\{\psi_i\}_{i=K+1}^{2K} \subset X_N^2$ and a linear self-adjoint operator $R_N : \mathcal{D}(R_N) \subset H \to H$ such that for some constants $0 < a_1 < a_2 < \infty$ and $C_1 > 0$, we have:

1.

$$a_1|A^{\frac{1}{4}}y|^2 \leq (R_N y, y) \leq a_2|A^{\frac{1}{4}}y|^2, \quad \forall\, y \in \mathcal{D}(A^{\frac{1}{4}}), \tag{2.17}$$

so that $\mathcal{D}(A^{\frac{1}{4}}) \subset \mathcal{D}(R_N^{\frac{1}{2}})$;

2.

$$|R_N y| \leq C_1\|y\|, \quad \forall\, y \in V = \mathcal{D}(A^{\frac{1}{2}}); \tag{2.18}$$

3. *R_N satisfies the following algebraic Riccati equation:*

$$-((\mathcal{A} + \lambda)y, R_N y) + \frac{1}{2}\sum_{i=1}^{2K}(\psi_i, R_N y)_\omega^2 = \frac{1}{2}|A^{\frac{3}{4}}y|^2, \quad \forall\, y \in \mathcal{D}(A). \tag{2.19}$$

The vectors $\{\psi_i\}_{i=1}^{2K}$ are explicitly constructed in (3.3.5) of Lemma 4 of Reference 4. Moreover, with $2K \leq N$, the feedback controller,

$$u = -\sum_{i=1}^{2K}(R_N y, \psi_i)_\omega \psi_i \tag{2.20a}$$

once inserted in Eq. (2.14)*, exponentially stabilizes the corresponding closed-loop system of Eq.* (2.14)*. The margin of stability for such a closed-loop system is* λ*. [See Remark 3.3.1 of Reference 4 for the effective number of controls* $2K \leq N$*.] More specifically, this means that the solution of*

$$\frac{dy}{dt} + \nu Ay + A_0 y + P\left[m \sum_{i=1}^{2K} (R_N y, \psi_i)_\omega \psi_i\right] = 0, \; y(0) = y^0 \in \mathcal{D}(A^{\frac{1}{4}}) \tag{2.20b}$$

satisfies

$$|A^{\frac{1}{4}} y(t)| \leq C_\lambda e^{-\lambda t} |A^{\frac{1}{4}} y^0|, \; t \geq 0. \tag{2.20c}$$

Nonlinear system Eq. (2.9)

We next use the stabilizer in Theorem 2.1 to the linearized system of Eq. (2.14) of the translated problem (2.1), or (2.2), to obtain the sought-after closed loop, local, feedback stabilization of the steady-state solution y_e to the Navier-Stokes Eq. (2.9).

THEOREM 2.2

With reference to Theorem 2.1, the feedback controller

$$u = -\sum_{i=1}^{2K} (R_N(y - y_e), \psi_i)_\omega \psi_i \tag{2.21}$$

(where the vectors ψ_i *are defined in (3.3.5) of Lemma 4 in Reference 4), once inserted in the Navier-Stokes system Eq.* (2.9)*, exponentially stabilizes the steady-state solution* y_e *to Eq.* (2.1) *in a neighborhood*

$$\mathcal{V}_\rho = \{y_0 \in D(A^{\frac{1}{4}}); \qquad |A^{\frac{1}{4}}(y_0 - y_e)| < \rho\} \tag{2.22}$$

of y_e*, for suitable* $\rho > 0$*. More precisely, if* $\rho > 0$ *is sufficiently small, then for each* $y_0 \in \mathcal{V}_\rho$ *there exists a weak solution* $y \in L^\infty(0, T; H) \cap L^2(0, T; V)$*,* $\frac{dy}{dt} \in L^{\frac{4}{3}}(0, T; V')$ *for* $d = 3$*, and* $\frac{dy}{dt} \in L^2(0, T; V')$ *for* $d = 2$*,* $\forall\, T > 0$*, to the closed-loop system*

$$\frac{dy}{dt} + \nu Ay + By + P\left\{m \sum_{i=1}^{2K} (R_N(y - y_e), \psi_i)_\omega \psi_i\right\} = Pf_e, \; t \geq 0, \; y(0) = y^0, \tag{2.23}$$

obtained from inserting the control of Eq. (2.21) *in Eq.* (2.9)*, such that the following two properties hold:*

1.

$$\int_0^\infty e^{2\lambda t} \left|A^{\frac{3}{4}}(y(t) - y_e)\right|^2 dt \leq C_2 \left|A^{\frac{1}{4}}(y_0 - y_e)\right|^2; \tag{2.24}$$

2.

$$\left|A^{\frac{1}{4}}(y(t) - y_e)\right| \leq C_3 e^{-\lambda t} \left|A^{\frac{1}{4}}(y_0 - y_e)\right|, \quad \forall\, t \geq 0. \tag{2.25}$$

We refer to Reference 6, p. 71 for definition of weak solutions to equations of the form in Eq. (2.23) and the asserted regularity. If $d = 2$, the solution to Eq. (2.23) is strong and unique (see Reference 6, p. 83).

The pressure p

Theorem 2.2 implies the following result giving corresponding asymptotic properties of the pressure p.

THEOREM 2.3

The solution y provided by Theorem 2.2 satisfies also the equation

$$y_t - \nu \Delta y + (y \cdot \nabla) y + m \left\{ \sum_{i=1}^{2K} (R_N(y - y_e), \psi_i)_\omega \psi_i \right\} = \nabla p + f_e \text{ in } Q \equiv \Omega \times (0, \infty); \tag{2.26}$$

$$\begin{aligned} \nabla \cdot y &\equiv 0 && \text{in } Q; \\ y &\equiv 0 && \text{on } \Sigma = \partial\Omega \times (0, \infty); \\ y(x, 0) &= y_0(x) && \text{in } \Omega. \end{aligned}$$

Moreover, the following relations hold true for the pressure p:

1. *for $d = 2$, we have*

$$\int_0^\infty t |p(t) - p_e|^2_{[H^1(\Omega)]^d} dt \le C \left| A^{\frac{1}{4}}(y_0 - y_e) \right|^2 \left[1 + \left| A^{\frac{1}{4}}(y_0 - y_e) \right|^2 \right]; \tag{2.27}$$

2. *for $d = 3$, we have*

$$\int_0^\infty |p(t) - p_e|^2_{[L^2(\Omega)]^d/R} dt \le C \left| A^{\frac{1}{4}}(y - y_e) \right|^2 \Big] \left[1 + \left| A^{\frac{1}{4}}(y_0 - y_e) \right|^2 \right]. \tag{2.28}$$

2.3 Part II: Boundary Control [3]

2.3.1 Introduction

Boundary controlled Navier-Stokes equations

We consider the controlled Navier-Stokes equations (see Reference 6, p. 45, and Reference 13, p. 253 for the uncontrolled case $u \equiv 0$) with boundary control u in the Dirichlet B.C.:

$$\begin{cases} y_t(x,t) - \nu_0 \Delta y(x,t) + (y \cdot \nabla) y(x,t) = f_e(x) + \nabla p(x,t) & \text{in } G; \quad (2.29a) \\ \nabla \cdot y = 0 & \text{in } G; \quad (2.29b) \\ y = u & \text{on } \Sigma; \quad (2.29c) \\ y(x,0) = y_0(x) & \text{in } \Omega. \quad (2.29d) \end{cases}$$

Here, $G = \Omega \times (0, \infty)$; $\Sigma = \partial\Omega \times (0, \infty)$ and Ω is an open smooth bounded domain of R^d, $d = 2, 3$; $u \in L^2(0, T; [L^2(\partial\Omega)]^d)$ is the boundary control input; and $y = (y_1, y_2, \dots, y_d)$ is the state (velocity) of the system. The constant $\nu_0 > 0$ is the viscosity coefficient. The functions y_0, $f_e \in [L^2(\Omega)]^d$ are given, the latter being a body force, whereas p is the unknown pressure. Because of the divergence theorem: $\int_\Omega \nabla \cdot y \, d\Omega = \int_\Gamma y \cdot \nu \, d\Gamma$, [$\Gamma = \partial\Omega$, ν = unit outward normal to $\partial\Omega$], we must require (at least) the integral boundary compatibility condition: $\int_\Gamma u \cdot \nu \, d\Gamma = 0$ on the control function u. Actually, a more stringent condition has to be imposed, in our final results: $u \cdot \nu \equiv 0$ on Σ, to sustain the pointwise boundary compatibility condition contained in the definition of the critical state space H in Eq. (2.34) below. To summarize we shall then assume

$$\text{either } u \cdot \nu \equiv 0 \text{ on } \Sigma; \qquad \text{or at least } \int_\Gamma u \cdot \nu \, d\Gamma \equiv 0, \text{ a.e. } t > 0, \tag{2.29e}$$

as it will be specified on a case-by-case basis.

Steady-state solutions and space V: same as in Eq. (2.2), Eq. (2.3)

Goal

Our goal is to construct a boundary control u, subject to the boundary compatibility condition (c.c.) given by Eq. (2.29e) in the strong pointwise form $u \cdot \nu \equiv 0$ on $\partial\Omega$, and, moreover, in feedback form $u = u(y - y_e)$ via some linear operator $y \to u$, such that, once $u(y)$ is substituted in the translated problem in Eq. (2.29c), the resulting well-posed, closed-loop system in Eqs. (2.29a–d) possesses the following desirable property: the steady-state solutions y_e defined in Eq. (2.2) are locally exponentially stable. In particular, motivated by our prior effort in Reference 4 to be described below, we seek to investigate if and when the feedback controller $u = u(y - y_e)$ can be chosen to be finite-dimensional, and, moreover, to act on an arbitrarily small portion (of positive measure) of the boundary $\Gamma = \partial\Omega$.

Orientation. Use of the Optimal Control Problem and Algebraic Riccati Theory

($d = 3$) We emphasize here only the more demanding case of $d = 3$. A preliminary difficulty (for $d = 2, 3$) is the requirement in Eq. (2.29e) that the boundary control u must always be tangential at each point of the boundary. It is standard that this requirement is intrinsically built in the definition of the state space H (above in Eq. (2.5) Part I) of the velocity vector y, which is critical to eliminate the second unknown of the Navier-Stokes model, the pressure term ∇p (see the orthogonal complement $H^\perp$ in Eq. (2.5) above, Part I), by virtue of the Leray projection P. Evolution of the velocity must occur in H. Accordingly, we must then have that the boundary controls be pointwise tangential: $u \cdot \nu \equiv 0$ on Σ in Eq. (2.29e). Next, a second difficulty, this time for $d = 3$, is that the nonlinearity of the Navier-Stokes equation dictates and forces the requirement that stabilization must occur in the space $[H^{\frac{3}{2}+\epsilon}(\Omega)]^3$, $\epsilon > 0$, see Eq. (5.18a–b) of Reference 3. This is a high topological level, of which we shall have to say more below. A third source of difficulty consists of deciding how to inject "dissipation" into the Navier-Stokes model, in fact, as required, through *a boundary* tangential controller expressed in feedback form. Here, motivated by Reference 4 and, in turn, by optimal control theory in Reference 12, in order to inject dissipation into the Navier-Stokes system as to force local exponential boundary stabilization of its steady-state solutions, we choose the strategy of introducing an optimal control problem (OCP) with a quadratic cost functional, over an infinite time-horizon, for the linearized Navier-Stokes model subject to tangential Dirichlet-boundary control u (i.e., satisfying $u \cdot \nu \equiv 0$ on Σ). One then seeks to express the boundary feedback, closed-loop controller of the optimal solution of the OCP, in terms of the Riccati operator arising in the corresponding algebraic Riccati theory. As a result, the same Riccati-based boundary feedback optimal controller that is obtained in the linearized OCP is then selected and implemented also on the full Navier-Stokes system. This controller in feedback form is both dissipative as well as "robust" (with respect to a certain class of perturbations). For $d = 3$, however, the OCP must be resolved at the *high* $[H^{\frac{3}{2}+\epsilon}(\Omega)]^3$-topological level, within the class of Dirichlet *boundary* controls in $L^2(0, \infty; (\partial\Omega)^3)$, which are further constrained to be *tangential to the boundary*.

Thus, the OCP faces two additional difficulties that set it apart and definitely outside the boundaries of established optimal control theory for parabolic systems with boundary controls:

1. The high degree of unboundedness of the *boundary* control operator of order $(\frac{3}{4} + \epsilon)$, as expressed in terms of fractional powers of the basic free-dynamics generator; and

2. The high degree of unboundedness of the "penalization" or "observation" operator of order also $(\frac{3}{4} + \epsilon)$, as expressed in terms of fractional powers of the basic free-dynamics generator.

This yields a "combined index" of unboundedness *strictly greater than* $\frac{3}{2}$. By contrast, the established (and rich) optimal control theory of boundary control parabolic problems and corresponding algebraic Riccati theory requires a "combined index" of unboundedness *strictly less than* 1 (see Reference 12, Vol. 1, in particular, pp. 501–503), which is the maximum limit handled by perturbation theory of analytic semigroups. To implement this program, however, one must first overcome, at the very outset, the preliminary stumbling obstacle of showing that the present highly nonstandard OCP—with the aforementioned high level of combined unboundedness in control and observation operators and further restricted within the class of *tangential* boundary controllers—is, in fact, *nonempty*. This result is achieved in Theorem 3.5.1 of Reference 3 in full generality (and in Proposition 3.7.1 of Reference 3 under the assumption that the linearized operator is diagonalizable over the finite-dimensional unstable subspace).

Thus, after this result, the study of the OCP may then begin. Because of the aforementioned intrinsic difficulties of the OCP with a combined index of unboundedness $>\frac{3}{2}$, one cannot (and cannot hope to) recover in full all desirable features of the corresponding algebraic Riccati theory that are available when the combined index of unboundedness in control and observation operators is *strictly less than* 1 (see Reference 12 and references therein). For instance, existence of a solution (Riccati operator) of the algebraic Riccati equation is here asserted on only the domain of the square of the generator of the optimal feedback dynamics (Proposition 4.5.1 of Reference 3) and not on the domain of the free-dynamics operator, as it would be required by or at least desirable from the viewpoint of the OCP. However, in our present treatment, the OCP is a means to extract dissipation and stability not an end in itself. And indeed, the present study of the algebraic Riccati theory, with a combined index of unboundedness in control and observation operator *strictly above* $\frac{3}{2}$ (rather than *strictly less than* 1), does manage at the end to draw out the key sought-after features of interest—dissipativity and decay—for the resulting optimal solution in feedback form of the OCP for the linearized Navier-Stokes equation. All this is accomplished in Section 4 of Reference 3.

The subsequent step of the strategy is then to select and use the same Riccati-based, boundary feedback operator, which was found to describe the optimal solution of OCP of the *linearized* Navier-Stokes equation, directly into the full Navier-Stokes model. For $d = 3$, the heavy groundwork for the feedback stabilization of the linearized problem via optimal control theory then makes the resulting analysis of well-posedness (in Section 5 of Reference 3) and stabilization (in Section 6) of the Navier-Stokes model more amenable than would otherwise be the case.

To this end, key use is made of the algebraic Riccati equation satisfied by the Riccati operator that describes the stabilizing control in closed-loop feedback form.

Literature

Reference 3 is a successor to Reference 4, which instead considered the *interior* stabilization problem of the Navier-Stokes equations, that is, the problem of Eq. (2.1), Part I, with (i) nonslip boundary condition $y \equiv 0$ on $\Sigma \equiv \partial\Omega \times (0, \infty)$ in place of the boundary controlled condition of Eq. (2.29c); and (ii) interior control $m(x)u(x, t)$ on the right-hand side of Eq. (2.29a), where $m(x)$ is the characteristic function of an *arbitrary* open subset $\omega \subset \Omega$ of positive measure. In this case, Reference 4 (Part I) proves that (the linearized problem is exponentially stabilizable, hence that) the steady-state solutions y_e to the Navier-Stokes equations are locally exponentially stabilizable by a *finite-dimensional* feedback controller, in fact, *of minimal size*, see Part I. In addition, one may select the *finite-dimensional* feedback controller to be expressed in terms of a Riccati operator (solution of an algebraic Riccati equation, which arises in an optimization problem associated with the linearized equation). We shall need to invoke this interior stabilization problem (though not in its full strength) in Section 3.5 of Reference 3.

The work in the literature that is most relevant to our present paper is that of Fursikov, see Reference 8 (of which we become aware after completing Reference 4), which culminates a series of papers quoted therein. A statement of the main contribution of Reference 8, as it pertains to the linearized problem of Eq. (2.32) below, is contained in Reference 8, Theorems 3.3 and 3.5, pp. 104–105].

One should note, however, that the aforementioned controller for the problem of Eq. (2.3) below given in Reference 8 is not a feedback controller in the standard sense. Instead, our main results (in Reference 4 as well as) in the present paper construct genuine, authentic, and real feedback controls (Riccati-based, in fact, hence with some feature of "robustness") that use at time t only the state information on Ω at time t. The present paper, therefore, encounters a host of technical problems not present in Reference 8: from the need for the genuine feedback control u to satisfy the pointwise compatibility condition $u \cdot \nu \equiv 0$ on Σ, to the high topological level $[H^{\frac{3}{2}+\epsilon}(\Omega)]^3$ at which stabilization must occur in our case [as dictated by the nonlinearity for $d = 3$, see Eq. (5.18a–b) of Reference 3, vs. the H^1-topology decay obtained in Reference 8], to the treatment of the Riccati theory for a corresponding OCP with a combined "index of unboundedness" in control and observation operators exceeding $\frac{3}{2}$ —thus $\frac{1}{2}+2\epsilon$ beyond the (rich) theory of the literature (see Reference 12) as explained in the Orientation on p. 36.

Main contributions of the present paper. Qualitative summary of main results

A qualitative description of the main results of the present paper follows next. First of all, the preset goal is achieved: *with no assumptions whatsoever (except mild assumptions on the domain), we prove here that the steady-state solutions to Navier-Stokes equations on* $\Omega \subset \mathbb{R}^d$*,* $d = 2, 3$*, are locally exponentially stabilizable by a closed-loop boundary feedback controller acting in the Dirichlet boundary conditions in the required topologies (Theorem 2.3 for* $d = 3$ *and Theorem 2.6 for* $d = 2$ *of Reference 3).* The feedback controller is expressed in terms of a Riccati operator (solution of a suitable algebraic Riccati equation): as such, via standard arguments (e.g., Reference 4) this feedback controller is "robust" with respect to a certain class of exogenous perturbations.

More precisely, the following main results are established in the present paper:

1. For the general cases $d = 2, 3$, an infinite-dimensional, closed-loop boundary feedback, stabilizing controller is constructed, as acting (for general initial data) on the entire boundary $\partial\Omega$ for $d = 3$ or on an arbitrarily small portion of the boundary for $d = 2$.

2. By contrast, for $d = 2$ and under a finite-dimensional spectral assumption FDSA = Eq. (3.6.2) of Reference 3 (diagonalizability of the restriction of the linearized operator over the finite-dimensional unstable subspace), the feedback controller can be chosen to be *finite-dimensional*, with dimension related to properties of the unstable eigenvalues, and, moreover, still to act on an *arbitrarily small portion* of the boundary.

3. For $d = 3$, local exponential feedback stabilization of the steady-state solutions to Navier-Stokes equations is *not* possible with a *finite-dimensional* boundary feedback controller (except for a meager set of special initial conditions).

4. The pathology noted in (iii) for $d = 3$ is due to the nonlinearity (see Eq. (5.18a–b) of Reference 3) which (by Sobolev embedding and multiplier theory for $d = 3$) forces the the requirement that solutions of the linearized problem be considered at the high regularity space $H^{\frac{3}{2}+\epsilon}(\Omega) \cap H$, $\epsilon > 0$ under initial conditions $y_0 \in H^{\frac{1}{2}+\epsilon}(\Omega) \cap H$ and $L^2(0, \infty; [L^2(\Gamma)]^d)$-boundary controls u. In turn, this high regularity space $H^{\frac{3}{2}+\epsilon}(\Omega)$ causes the occurrence of the compatibility condition $y_0|_\Gamma = u(0)$ at $t = 0$ on the boundary to be satisfied. Thus, for $d = 3$, the constructed feedback controller must be infinite-dimensional in general.

5. By contrast, the *linearized* problem for $d = 2, 3$ is exponentially stabilizable with a closed-loop boundary, *finite-dimensional* feedback controller acting on an *arbitrarily small portion* of the boundary up to the topological level $[H^{\frac{3}{2}-\epsilon}(\Omega)]^d$ and with initial conditions $y_0 \in [H^{\frac{1}{2}-\epsilon}(\Omega)]^d \cap H$ under the same FDSA = Eq. (3.6.2) of Reference 3.

Notation and preliminaries

Same as in Part I.

2.3.2 Main Results (Case $d = 3$)

The following assumptions will be in effect throughout the chapter.

Assumptions

1. The boundary $\partial\Omega$ of Ω is a finite union of $d-1$ dimensional C^2-connected manifolds.
2. The steady-state solution (y_e, p_e) defined in Eq. (2.2) Part I, belongs to $([H^2(\Omega)]^d \cap V) \times H^1(\Omega)$. (For $d = 2, 3$, this property is guaranteed by Reference 6 [Theorem 7.3, p. 59] on y_e, for $f_e \in H$, followed by Reference 6 [Theorem 3.11, p. 30] on p_e, for sufficiently smooth $\partial\Omega$.)

Preliminaries. The translated nonlinear Navier-Stokes problem

By the substitutions $y \to y_e + y$, $p \to p_e + p$, and $u \to y_e|_\Gamma + u$ with ($y_e|_\Gamma = 0$ being the Dirichlet trace of y_e on $\Gamma \equiv \partial\Omega$), we are readily led via Eq. (2.1) and Eq. (2.2) to study the boundary *null stabilization* of the equation

$$\begin{cases} y_t - \nu_0\Delta y + (y\cdot\nabla)y + (y_e\cdot\nabla)y + (y\cdot\nabla)y_e = \nabla p & \text{in } Q; \quad (2.30a)\\ \nabla\cdot y = 0 & \text{in } Q; \quad (2.30b)\\ y = u & \text{on } \Sigma; \quad (2.30c)\\ y(x,0) = y^0(x) = y_0(x) - y_e(x) & \text{in } \Omega. \quad (2.30d)\end{cases}$$

Abstract model of the Navier-Stokes problem (2.30) projected on *H*

We shall see in Section 3.1 of Reference 3 that, under the pointwise compatibility condition (c.c.) $u\cdot\nu = 0$ on Σ of Eq. (2.29e) (whereby then $Py_t = y_t$), application of the Leray projection P on Eq. (2.30a–d) leads to a corresponding equation in H, without the pressure terms, whose abstract version can be written as

$$y_t - \mathcal{A}y + By = -\mathcal{A}Du, \in [D(\mathcal{A})]' \quad y(0) = y^0 \in H, \quad u\cdot\nu \equiv 0 \text{ on } \Sigma, \tag{2.31}$$

where the infinitesimal generator $\mathcal{A}$ and the nonlinear operator B are defined in Eq. (2.15) and Eq. (2.7), respectively, of Part I. Moreover, the operator D: $[L^2(\Gamma)]^d \to [H^{\frac{1}{2}}(\Omega)]^d \cap H \in \mathcal{D}(A^{\frac{1}{4}-\epsilon})$ is defined in Eq. (3.1.3) of Reference 3. The $\mathcal{A}$ in Eq. (2.31) is actually the extension $H \to [D(\mathcal{A}^*)]'$.

The translated linearized problem. PDE version

The translated linearized problem corresponding to Eq. (2.30) is then

$$\begin{cases} y_t - \nu_0\Delta y + (y_e\cdot\nabla)y + (y\cdot\nabla)y_e = \nabla p & \text{in } Q; \quad (2.32a)\\ \nabla\cdot y = 0 & \text{in } Q; \quad (2.32b)\\ y = u & \text{on } \Sigma; \quad (2.32c)\\ y(x,0) = y^0(x) & \text{in } \Omega. \quad (2.32d)\end{cases}$$

Abstract model of Problem (2.32) projected on *H*

Its *abstract* version on H is then

$$y_t = \mathcal{A}y - \mathcal{A}Du \in [\mathcal{D}(\mathcal{A}^*)]', \quad y(0) = y^0 \in H; \qquad u\cdot\nu \equiv \text{ on } \Sigma. \tag{2.33}$$

Main results

Case $d = 3$. The linearized model We begin with the translated linearized Problem (2.32) or its projected version (2.33). For the first result—the main result on Problem (2.33)—essentially no assumptions are required.

THEOREM 2.4
With reference to the linearized Problem (2.33), Part II, the following results hold true:

1. *Let $d = 3$ and assume further that Ω is simply connected. Then, given any $y^0 \in W \equiv [H^{\frac{1}{2}+\epsilon}(\Omega)]^3 \cap H$, $\epsilon > 0$ arbitrary, there exists an open-loop, infinite-dimensional boundary control $u \in L^2(0,\infty;(L^2(\Gamma))^3)$, $u \cdot \nu \equiv 0$ on Σ, such that the corresponding solution y of Eq.* (2.33) *satisfies $y \in L^2(0,\infty;(H^{\frac{3}{2}+\epsilon}(\Omega))^3 \cap H) \cap H^{\frac{3}{4}+\frac{\epsilon}{2}}(0,\infty;H)$. Moreover, if y^0 vanishes on the portion Γ_0 of the boundary $\Gamma = \partial\Omega$, then u may be required to act on only the complementary part $\Gamma_1 = \Gamma / \Gamma_0$ of the boundary. In particular, if y^0 vanishes on all of Γ, then u may be required to have an arbitrarily small support Γ_1, meas$(\Gamma_1) > 0$.* (*This is Theorem 3.5.1 along with Remark 3.5.1, illustrated by Figures 3.5.1 and 3.5.2 in Reference 3.*)

2. *Let $d = 3$. Then, the control u claimed in (i) cannot generally be finite dimensional except for a meager set of special initial conditions.* (*This is Proposition 3.1.3 of Reference 3.*)

Case $d = 3$. Original Navier-Stokes model of Eq. (2.29) We now report the main result of the present paper, which provides the sought-after closed-loop boundary feedback control for the original Navier-Stokes Eq. (2.29) [or its projected version in Eq. (2.31)], which *exponentially stabilizes the stationary solution y_e of Eq.* (2.29) *in a neighborhood of y_e*. The stabilizing feedback control that we shall find is "robust," as it is expressed in terms of a Riccati operator R, which arises in an associated corresponding OCP. To state our (local) stabilizing result, we need to introduce the set

$$\mathcal{V}_\rho \equiv \left\{ y_0 \in W \equiv [H^{\frac{1}{2}+\epsilon}(\Omega)]^3 \cap H : |y_0 - y_e|_W < \rho \right\} \tag{2.34}$$

of initial conditions y_0 of Eq. (2.1), whose distance in the norm of W from a stationary solution y_e is less than $\rho > 0$. Here, $\epsilon > 0$ arbitrary is fixed once and for all.

THEOREM 2.5 (Main Theorem)
Let $d = 3$ and assume further that Ω is simply connected. If $\rho > 0$ in Eq. (2.34) *is sufficiently small, then: for each $y_0 \in \mathcal{V}_\rho$, there exists a unique fixed-point, mild, semigroup solution y of the following closed-loop problem:*

$$\begin{cases} y_t(x,t) - \nu_0 \Delta y(x,t) + (y \cdot \nabla) y(x,t) = f_e(x) + \nabla p(x,t) & \text{in } G; \quad (2.35a) \\ \nabla \cdot y = 0 & \text{in } G; \quad (2.35b) \\ y = \nu_0 \dfrac{\partial}{\partial \nu} R(y - y_e) & \text{on } \Sigma; \quad (2.35c) \\ y(x,0) = y_0(x) & \text{in } \Omega. \quad (2.35d) \end{cases}$$

obtained from Eq. (2.29) *by replacing u with the boundary feedback control $u = y_e + \nu_0 \frac{\partial}{\partial \nu} R(y - y_e)$ having the following regularity and asymptotic properties:*

1.

$$(y - y_e) \in C([0,\infty); W) \cap L^2(0,\infty; [H^{\frac{3}{2}+\epsilon}(\Omega)]^3 \cap H) \tag{2.36}$$

continuously in $y_0 \in W \equiv [H^{\frac{1}{2}+\epsilon}(\Omega)]^3 \cap H$:

$$|y(t) - y_e|_W^2 + \int_0^\infty |y(t) - y_e|^2_{[H^{\frac{3}{2}+\epsilon}(\Omega)]^3 \cap H} dt \le C |y_0 - y_e|_W^2, \ t \ge 0. \tag{2.37}$$

(This follows from Theorem 5.1 and Corollary 5.5 of Reference 3, via the translation $y \to y_e$, etc., performed above Problem (2.1).)

2. *There exist constants $M \geq 1$, $\omega > 0$ (independent of $\rho > 0$) such that such solution $y(t)$ satisfies*

$$|y(t) - y_e|_W \leq Me^{-\omega t}|y_0 - y_e|_W, \quad t \geq 0. \tag{2.38}$$

(This follows from Theorem 6.1(i) of Reference 3, via the translation $y \to y - y_e$, etc., performed above Problem (2.1).)

Here R is a Riccati operator, in the sense that it (arises in the OCP of Section 4.1 and) satisfies the Algebraic Riccati Equation (4.5.1) of Reference 3. The operator R is positive self-adjoint on H and, moreover, $R \in \mathcal{L}(W; W')$ where W' is the dual of W with respect to H as a pivot space. In addition (Proposition 4.1.4 of Reference 3),

$$c|x|_W^2 \leq (Rx, x)_H \leq C|x|_W^2, \; 0 < c < C < \infty, \quad \forall\, x \in W,$$

so that the $|R^{\frac{1}{2}}x|$-norm is equivalent to the W-norm. By a solution to Eq. (2.17)*, we mean, of course, a weak solution (see, e.g., References 6 and 15). (This part is Theorem 5.1 of Reference 3.)*

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Chapter 3

On Approximating Properties of Solutions of the Heat Equation

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Abstract By the maximal principle for the heat equation, a solution corresponding to zero initial data and produced by a positive Dirichlet boundary control is positive (i.e., belongs to the cone of positive functions). The notice is devoted to the question: Is the set of such solutions dense in the cone? The answer turns out to be negative: in 1-D case we construct an explicit example of a positive function separated from this set by a positive L_2-distance.

3.1 Introduction

By the maximal principle for the heat equation, solutions (states) corresponding to zero initial data and produced by positive Dirichlet boundary controls are positive (i.e., belong to the cone of positive functions). This notice is devoted to the question: Is the set of such states dense in the cone? The answer turns out to be negative: in 1-D case we construct an explicit example of a positive function separated from this set by a positive L_2-distance.

The question was raised in the framework of an approach to inverse problems (the BC method [2]). However, as we hope, it is of some independent interest for the boundary control theory.

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3.2 Dynamical System

Consider the system

$$u_t - u_{xx} = 0, \quad (x,t) \in (0,1) \times (0,T); \tag{3.1}$$

$$u|_{t=0} = 0, \quad x \in [0,1]; \tag{3.2}$$

$$u|_{x=0} = f_0, \quad u|_{x=1} = f_1, \quad t \in [0,T]; \tag{3.3}$$

with the Dirichlet boundary control $f = \text{col}\{f_0(t), f_1(t)\}$; let $u = u^f(x,t)$ be the solution.

The (real Hilbert) space of controls

$$\mathcal{F}^T := L_2([0,T]; \mathbf{R}^2) = \left\{\text{col}(f_0, f_1) | f_0, f_1 \in L_2[0,T]\right\}$$

plays the role of the outer space of the system in Eqs. (3.1) to (3.3). Introduce also the set

$$\mathcal{S}^T := \{f \in C^\infty([0,T]; \mathbf{R}^2) \,|\, \text{supp}\, f \subset (0,T]\}$$

of smooth controls vanishing near $t = 0$. For $f \in \mathcal{S}^T$ the system has a unique classical solution $u^f \in C^\infty([0,1] \times [0,T])$. The well-known fact is that the map $f \mapsto u^f$ acts continuously from $\mathcal{F}^T$ into $C([0,T]; H^{-1}[0,1])$ (see, e.g., Reference 1).

The inner space (space of states) of the system is

$$\mathcal{H} := L_2(0,1).$$

The "input-state" map $f \mapsto u^f(\cdot, T)$ defined on $\mathcal{S}^T$ acts from $\mathcal{F}^T$ into $\mathcal{H}$; its range

$$\mathcal{U}^T := \{u^f(\cdot, T) \,|\, f \in \mathcal{S}^T\}$$

is the reachable set of system in Eqs. (3.1) to (3.3). As is well known, this set is dense in the inner space:

$$\text{clos}\,\mathcal{U}^T = \mathcal{H}, \quad T > 0, \tag{3.4}$$

that is, the system is approximatelly controllable at all times (see, e.g., Reference 1).

3.3 Main Result

The outer space contains the cone of smooth positive controls

$$\mathcal{S}^T_+ := \{f \in \mathcal{S}^T \mid f_0, f_1 \geq 0\}$$

producing a cone of states

$$\mathcal{U}^T_+ := \{u^f(\cdot, T) \,|\, f \in \mathcal{S}^T_+\}.$$

The set

$$\mathcal{U}_+ := \bigcup_{T>0} \mathcal{U}^T_+$$

is also a cone in $\mathcal{H}$.

The inner space contains the cone of positive functions

$$\mathcal{C}_+ := \{y \in \mathcal{H} \mid y \geq 0\}.$$

By the maximal principle for the heat equation (see, e.g., Reference 3), the embedding

$$\mathcal{U}_+ \subset \mathcal{C}_+ \tag{3.5}$$

holds. In this notice we clarify this relation as follows.

THEOREM 3.1
The embedding of Eq. (3.5) *is not dense:*

$$\mathcal{C}_+ \setminus \operatorname{clos} \mathcal{U}_+ \neq \{\emptyset\}.$$

Moreover, we construct an example of $c \in \mathcal{C}_+$ satisfying

$$\operatorname{dist}(c, \mathcal{U}_+) > 0. \tag{3.6}$$

For the reader's convenience, the proof is divided on a few parts contained in Sections 3.4 to 3.8.

3.4 Dual System, Green Function

The system

$$v_t - v_{xx} = 0, \quad (x, t) \in (0, 1) \times (0, T); \tag{3.7}$$

$$v|_{t=0} = y, \quad x \in [0, 1]; \tag{3.8}$$

$$v|_{x=0} = v|_{x=1} = 0, \quad t \in [0, T]; \tag{3.9}$$

is said to be dual to the system in Eqs. (3.1) to (3.3); its solution $v = v^y(x, t)$ is well defined for any distribution $y \in \mathcal{D}'(0, 1)$ and is a function of the class $C^\infty([0, 1] \times (0, T])$. Integration by parts easily leads to the duality relation

$$\langle y, u^f(\cdot, T)\rangle = \int_0^T \left[f_0(t)\, v_x^y(0, T - t) - f_1(t)\, v_x^y(1, T - t)\right] dt \tag{3.10}$$

for any $f \in \mathcal{S}^T$ and $y \in \mathcal{D}'(0, 1)$. The values $v_x^y(0, \cdot)$ and $-v_x^y(1, \cdot)$ have the physical meaning of the heat flows through the endpoints $x = 0$ and $x = 1$.

Let δ_ξ be the Dirac measure supported at $\xi \in (0, 1)$; the solution

$$G(x, \xi; t) := v^{\delta_\xi}(x, t)$$

is the Green function the system in Eqs. (3.7) to (3.9). Recall its well-known representation through the heat kernel:

$$G(x, \xi; t) = \frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{m=\infty} \left[e^{-\frac{(x-\xi+2m)^2}{4t}} - e^{-\frac{(x+\xi+2m)^2}{4t}}\right] \tag{3.11}$$

and the representation by Fourier:

$$G(x, \xi; t) = \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \sin \pi n\xi \ \sin \pi n x \tag{3.12}$$

(see, e.g., Reference 3).

3.5 Basic Lemma

Let $\Delta' = [\alpha', \beta']$, $\Delta'' = [\alpha'', \beta'']$, $\Delta''' = [\alpha''', \beta''']$ be three closed disjoint segments in $(0, 1)$ (so that $0 < \alpha' < \beta' < \alpha'' < \beta'' < \alpha''' < \beta''' < 1$).

LEMMA 3.1

There exists $\theta > 0$ (small enough, determined by the segments) such that the inequalities

$$G_x(0, \xi'; t) - \theta G_x(0, \xi''; t) + G_x(0, \xi'''; t) > 0, \quad t > 0 \tag{3.13}$$

and

$$-[G_x(1, \xi'; t) - \theta G_x(1, \xi''; t) + G_x(1, \xi'''; t)] > 0, \quad t > 0 \tag{3.14}$$

hold for any $\xi' \in \Delta'$, $\xi'' \in \Delta''$, $\xi''' \in \Delta'''$.

PROOF

1. The well-known fact following from the maximal priciple is that the flow through the endpoints produced by δ_ξ is *strictly* positive:

$$G_x(0, \xi; t) > 0, \quad G_x(1, \xi; t) > 0, \quad t > 0 \tag{3.15}$$

(see, e.g., Reference 3).

2. Fix a $\xi \in (0, 1)$; representation of Eq. (3.11) easily gives:

$$\begin{aligned}\sqrt{\pi} t G_x(0, \xi; t) &= \sum_{m=-\infty}^{m=\infty} \frac{\xi + 2m}{2\sqrt{t}} e^{-\left(\frac{\xi+2m}{2\sqrt{t}}\right)^2} \\ &= \frac{\xi}{2\sqrt{t}} e^{-\left(\frac{\xi}{2\sqrt{t}}\right)^2} - \frac{2-\xi}{2\sqrt{t}} e^{-\left(\frac{2-\xi}{2\sqrt{t}}\right)^2} + \frac{\xi+2}{2\sqrt{t}} e^{-\left(\frac{\xi+2}{2\sqrt{t}}\right)^2} - \cdots .\end{aligned} \tag{3.16}$$

Because $\tau e^{-\tau^2}$ decreases monotonically as $\tau > \frac{1}{\sqrt{2}}$, taking

$$0 < t < \frac{\xi^2}{2} \tag{3.17}$$

we have

$$\frac{1}{\sqrt{2}} \le \frac{\xi}{2\sqrt{t}} < \frac{2-\xi}{2\sqrt{t}} < \frac{\xi+2}{2\sqrt{t}} < \dots,$$

so that Eq. (3.16) turns out to be an alternating series whose terms monotonically tend to zero. Therefore, Eq. (3.17) provides the estimates

$$\frac{\xi}{2\sqrt{t}} e^{-\left(\frac{\xi}{2\sqrt{t}}\right)^2} - \frac{2-\xi}{2\sqrt{t}} e^{-\left(\frac{2-\xi}{2\sqrt{t}}\right)^2} < \sqrt{\pi} t G_x(0, \xi; t) < \frac{\xi}{2\sqrt{t}} e^{-\left(\frac{\xi}{2\sqrt{t}}\right)^2}. \tag{3.18}$$

3. Choose $\xi' \in \Delta'$, $\xi'' \in \Delta''$, $\xi''' \in \Delta'''$. Restricting the times by

$$0 < t < \frac{\alpha'^2}{2}$$

and taking into account Eqs. (3.15), (3.17), and Eq. (3.18), one has

$$\sqrt{\pi}t[\,G_x(0,\xi';t) - G_x(0,\xi'';t) + G_x(0,\xi''';t)]$$
$$> \frac{\xi'}{2\sqrt{t}}e^{-\left(\frac{\xi'}{2\sqrt{t}}\right)^2} - \frac{2-\xi'}{2\sqrt{t}}e^{-\left(\frac{2-\xi'}{2\sqrt{t}}\right)^2} - \frac{\xi''}{2\sqrt{t}}e^{-\left(\frac{\xi''}{2\sqrt{t}}\right)^2}$$
$$= \frac{1}{2\sqrt{t}}\left\{\left[\frac{\xi'}{2}e^{-\left(\frac{\xi'}{2\sqrt{t}}\right)^2} - (2-\xi')e^{-\left(\frac{2-\xi'}{2\sqrt{t}}\right)^2}\right] + \left[\frac{\xi'}{2}e^{-\left(\frac{\xi'}{2\sqrt{t}}\right)^2} - \xi''e^{-\left(\frac{\xi''}{2\sqrt{t}}\right)^2}\right]\right\}.$$

For small times the terms $\frac{\xi'}{2}\,e^{-(\frac{\xi'}{2\sqrt{t}})^2}$ are dominating, and the differences in the square brackets are positive: simple calculations show that

$$G_x(0,\xi';t) - G_x(0,\xi'';t) + G_x(0,\xi''';t) > 0, \quad 0 < t < T_0, \tag{3.19}$$

where

$$T_0 := \min\left\{\frac{\alpha'^2}{2}, \frac{(\alpha''-\beta')\alpha'}{\ln\frac{2\beta''}{\alpha'}}, \frac{1-\beta'}{\ln\frac{2\,(2-\alpha')}{\alpha'}}\right\}.$$

4. Changing the roles of the endpoints, quite analogous analysis for $x = 1$ gives

$$-[G_x(1,\xi';t) - G_x(1,\xi'';t) + G_x(1,\xi''';t)] > 0, \quad 0 < t < T_1, \tag{3.20}$$

where

$$T_1 := \min\left\{\frac{(1-\beta''')^2}{2}, \frac{(\alpha'''-\beta'')(1-\beta''')}{\ln\frac{2\,(1-\alpha'')}{1-\beta'''}}, \frac{\alpha'''}{\ln\frac{2\,(1+\beta''')}{1-\beta'''}}\right\}.$$

Combining Eq. (3.19) and Eq. (3.20), we get

$$(-1)^j\,[\,G_x(j,\xi';t) - G_x(j,\xi'';t) + G_x(j,\xi''';t)\,] > 0, \quad 0 < t < T_*, \tag{3.21}$$

where $j = 0, 1$ and $T_* = \min\{T_0, T_1\}$.

5. The asymptotic

$$(-1)^j\,G_x(j,\xi;t) = 2e^{-\pi^2 t}\sin\pi\xi[1 + \mathrm{o}(1)], \quad t\to\infty$$

following from Eq. (3.12) allows to find a large enough $T^* > 0$ providing

$$(-1)^j[G_x(j,\xi';t) - \theta^* G_x(j,\xi'';t) + G_x(j,\xi''';t)] > 0, \quad t > T^* \tag{3.22}$$

with a constant θ^*,

$$0 < \theta^* < \frac{\sin\pi\xi' + \sin\pi\xi'''}{\sin\pi\xi''}.$$

By virtue of Eq. (3.15), the flows $(-1)^j\,G_x(j,\xi';t)$, and $(-1)^j\,G_x(j,\xi''';t)$ are strictly positive as $T_* < t < T^*$ (uniformely with respect to $\xi' \in \Delta', \xi''' \in \Delta'''$), whereas $G_x(j,\xi'';t)$ is bounded (uniformely with respect to $\xi'' \in \Delta''$). Therefore, one can find θ_* ensuring the inequality

$$(-1)^j\,[\,G_x(j,\xi';t) - \theta_*\,G_x(j,\xi'';t) + G_x(j,\xi''';t)\,] > 0, \quad T_* < t < T^*. \tag{3.23}$$

Joining Eqs. (3.21), (3.22), and (3.23), we arrive as Eqs. (3.13) and (3.14) with $\theta = \min\{1, \theta^*, \theta_*\}$. The lemma is proved. ∎

The physical meaning of this result is quite transparent. If a point source $\theta\delta_{\xi''}$ of a small enough amplitude is surrounded sources $\delta_{\xi'}$ and $\delta_{\xi'''}$, the resulting heat flow through the endpoints produced by the source $\delta_{\xi'} - \theta\delta_{\xi''} + \delta_{\xi'''}$ turns out to be positive at all times.

3.6 Inequalities for u^f

The duality relation of Eq. (3.10) together with Lemma 3.1 imply some a pointwise positivity condition for states $u^f(\cdot, T)$ produced by positive f's. Here ξ', ξ'', ξ''', and θ are the same as in Lemma 3.1.

LEMMA 3.2
For any $f \in \mathcal{S}_+^T$ *the inequality*

$$u^f(\xi', T) - \theta\, u^f(\xi'', T) + u^f(\xi''', T) > 0, \quad T > 0 \tag{3.24}$$

is valid.

PROOF Taking $f \in \mathcal{S}_+^T$ and using Eq. (3.10), one has

$$\begin{aligned} &u^f(\xi', T) - \theta u^f(\xi'', T) + u^f(\xi''', T) \\ &\quad = \langle \delta_{\xi'} - \theta\delta_{\xi''} + \delta_{\xi'''}, u^f(\cdot, T)\rangle \\ &\quad = \sum_{j=0}^{1} \int_0^T f_j(t)(-1)^j [G_x(j, \xi'; T-t) - \theta G_x(j, \xi''; T-t) + G_x(j, \xi'''; T-t)]\, dt > 0 \end{aligned}$$

by virtue of Eqs. (3.13) and (3.14). The lemma is proved. ∎

It is worth to recall that θ is determined by the system of segments Δ', Δ'', Δ''' and to note that it is available for *all positive* T's. As we guess, Eq. (3.24) is a supplement to the well-known Harnack ineqalities (see Reference 3, the end of Chapter 3).

Let us derive one more relation for $u^f(\cdot, T)$. Take

$$\Delta' = \left[\frac{1}{7}, \frac{2}{7}\right], \qquad \Delta'' = \left[\frac{3}{7}, \frac{4}{7}\right], \qquad \Delta''' = \left[\frac{5}{7}, \frac{6}{7}\right]$$

and put $\xi' = \xi \in \Delta'$, $\xi'' = \xi + \frac{2}{7} \in \Delta''$, $\xi''' = \xi + \frac{4}{7} \in \Delta'''$; let θ correspond to these segments in the sense of Lemma 3.1. By virtue of (3.24), for $f \in \mathcal{S}_+^T$ we have

$$u^f(\xi, T) - \theta u^f\left(\xi + \frac{2}{7}, T\right) + u^f\left(\xi + \frac{4}{7}, T\right) > 0, \quad T > 0,$$

which leads to the inequality:

$$(u^f(\cdot, T), \chi) > 0, \quad T > 0, \tag{3.25}$$

where

$$\chi := \begin{cases} 1 & \textit{in } \Delta'; \\ -\theta & \textit{in } \Delta''; \\ 1 & \textit{in } \Delta'''; \\ 0 & \textit{out of } \Delta' \cup \Delta'' \cup \Delta'''. \end{cases}$$

3.7 One More Lemma

Here we discuss a fact of a general character.

LEMMA 3.3
Let $\mathcal{C}$ and $\mathcal{U} \subset \mathcal{C}$ be two cones in a Hilbert space $\mathcal{H}$. If there exist $c_+,\ c_- \in \mathcal{C}$ such that $c_+ \perp c_-$ and $(u, c_+ - c_-) \geq 0$ for all $u \in \mathcal{U}$ then

$$\operatorname{dist}(c_-, \mathcal{U}) \geq \frac{\|c_-\|^2}{\|c_+ - c_-\|} \geq 0,$$

and, hence,

$$\mathcal{C} \setminus \operatorname{clos}\mathcal{U} \neq \{\emptyset\}.$$

PROOF Taking $u \in \mathcal{U}$, one can represent

$$u - c_- = \left(u - c_-, \frac{c_+ - c_-}{\|c_+ - c_-\|}\right) \frac{c_+ - c_-}{\|c_+ - c_-\|} + h$$

with $h \perp (c_+ - c_-)$, which yields

$$\begin{aligned}\|u - c_-\| &\geq \left|\left(u - c_-, \frac{c_+ - c_-}{\|c_+ - c_-\|}\right)\right| \\ &= [(u, c_+ - c_-) + \|c_-\|^2] \frac{1}{\|c_+ - c_-\|} \\ &\geq \frac{\|c_-\|^2}{\|c_+ - c_-\|} > 0\end{aligned}$$

and shows that $\mathcal{U}$ is not dense in $\mathcal{C}$. The lemma is proved. ∎

3.8 Completing the Proof of Theorem 3.1

Let $\chi_-,\ \chi_+ \in \mathcal{C}_+$ be positive and negative parts of the function χ from Section 3.6, so that

$$\chi_+ = \begin{cases} 1 & \textit{in } \Delta' \cup \Delta'''; \\ 0 & \textit{out of } \Delta' \cup \Delta'''; \end{cases}$$

and

$$\chi_- = \begin{cases} \theta & \textit{in } \Delta''; \\ 0 & \textit{out of } \Delta''; \end{cases}$$

whereas $\chi = \chi_+ - \chi_-$. Taking into account Eq. (3.25) and applying Lemma 3.3, one easily gets

$$\operatorname{dist}(\chi_-, \mathcal{U}_+) \geq \frac{\|\xi_-\|^2}{\|\xi\|} = \frac{\theta^2}{\sqrt{14 + 7\theta^2}} > 0.$$

This proves Theorem 3.1 and provides an example of $c = \chi_-$ mentioned at the end of Section 3.3.

So, the states belonging to $\mathcal{U}_+$ are not suitable for approximating the functions whose supports are separated from the endpoints (like χ_-). At the same time, the set $\mathcal{U}_+$ is rich enough in the following sense. Recall that a cone $\mathcal{K}$ in a Hilbert space $\mathcal{G}$ is said to be *generating* if the algebraic difference

$$\mathcal{K} - \mathcal{K} := \{v - w \,|\, v, w \in \mathcal{K}\}$$

is dense in $\mathcal{G}$. The cone $\mathcal{U}_+$ possesses such the property. Indeed, marking positive and negative parts of functions by "±" and denoting $f_\pm := \text{col}\{f_{0\pm}, f_{1\pm}\}$, one has

$$\begin{aligned}\mathcal{U}^T &= \{u^f(\cdot, T) \,|\, f \in \mathcal{S}^T\} = \{u^{f_+ - f_-}(\cdot, T) \,|\, f_\pm \in \mathcal{S}_+^T\} \\ &= \{u^{f_+}(\cdot, T) - u^{f_-}(\cdot, T) \,|\, f_\pm \in \mathcal{S}_+^T\} \\ &= \{u^f(\cdot, T) \,|\, f \in \mathcal{S}_+^T\} - \{u^g(\cdot, T) \,|\, g \in \mathcal{S}_+^T\} \subset \mathcal{U}_+ - \mathcal{U}_+.\end{aligned}$$

As result, the approximate controllability Eq. (3.4) implies

$$\text{clos}\,(\mathcal{U}_+ - \mathcal{U}_+) = \mathcal{H}\,.$$

3.9 Comments

- Most likely, Theorem 3.1 is valid in any dimension. Nevertheless, an attempt to construct a multidimensional version of function χ on the same way, that is, surrounding a small subdomain of low (negative) temperature by a massive layer of high (positive) temperature providing a positive resulting flow through the boundary of the domain, encounters the following difficulty. The first inequality in Eq. (3.17) is based on alternating property of the series in Eq. (3.15), which has no direct analog in high dimensions. Therefore, estimation of the flow from below is a problem.
- The author discussed the issue of this notice with participants of the Conference on Control Theory for PDE held in Georgetown University, Washington, May 30–June 3, 2003. I would like to thank R. Triggiani for the kind invitation and my colleagues for valuable consultations and advice.
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Chapter 4

Kolmogorov's ε-Entropy for a Class of Invariant Sets and Dimension of Global Attractors for Second-Order Evolution Equations with Nonlinear Damping

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Abstract This chapter contains several abstract results related to the finite fractal dimension of invariant compact sets arising in the study of global attractors for second-order (in time) evolution equations with nonlinear damping.

4.1 Introduction

The main aim of this chapter is to present an approach for studying the dimensionality of attractors arising in second-order evolutions with nonlinear damping. Although this issue has been well studied in the case of first-order evolutions of parabolic type, it is much less investigated in the case of second-order evolutions, which are predominantly of hyperbolic type. In fact, specific examples that we have in mind include semilinear wave and plate equations with nonlinear boundary or internal damping. It is known that the estimates for estimating the dimension of attractors in nonlinear wave dynamics with *nonlinear dissipation* comprise a very delicate issue that has been largely open. This can be best testified by the fact that, until very recently, results available in the literature, even in the case

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of *the full interior* dissipation (i.e., damping $g(u_t)$ supported in the entire domain), dealt either with one-dimensional (1-D) wave equations [10] or with equations subject to severe restrictions (linear bound) imposed on the semilinear terms [1], [21]. Most recently, there has been renewed interest in this problem, and positive results for semilinear wave equation in dimension higher than one with *interior dissipation* have been established [16, 19].

Problems with *nonlinear boundary* dissipation in the context of long-time behavior is an altogether different matter, as recognized in Reference 20, page 353. The issue is not only that the dissipation needs to be propagated from the boundary to the interior, in order to guarantee existence of global attractors [5], but also that *boundary* damping is represented by a (distributional) operator that is no longer defined on the phase state space. In addition, the resulting semigroup is no longer time reversible (unlike the case of interior damping, including localized interior damping).

One of the most flexible approaches to finite dimensionality of the attractors is based on establishing some kind of "squeezing" property of the flow on the attractor (see, e.g., References 4, 9, 11, 14, 23 among others). Although this approach has been successful in studying the dimensionality of attractors in parabolic flows and some rather special hyperbolic flows, the conditions required by standard "squeezing" inequalities are too stringent to accommodate hyperbolic-like dynamics with nonlinear dissipation (particularly boundary or localized dissipation). It turns out that the impetus provided by recent developments in boundary control theory provides a new insight into the problem and allows one to formulate certain generalizations of squeezing properties that can then be proved by PDE methods spurred by recent developments in control theory. It is the aim of this paper to explore the above connection. In particular, we shall show that establishing certain types of observability-stabilizability estimates suffices for the verification of appropriate variants of squeezing properties.

To explain our ideas, let us begin with a typical example of a "squeezing property" such as the one given in the following Ladyzhenskaya's theorem.

THEOREM 4.1 (Ladyzhenskaya [14])
Let M be a compact set in a Hilbert space H. Assume that V is a Lipschitz continuous mapping in H such that $V(M) \supseteq M$ and there exists a finite-dimensional projector P in H such that

$$\|(I - P)(Vv_1 - Vv_2)\| \le \eta \|v_1 - v_2\|, \quad v_1, v_2 \in M, \tag{4.1}$$

where $\eta < 1$. Then, the fractal dimension $\dim_f M$ of the set M is finite.

Assumption of Eq. (4.1), roughly speaking, means that the mapping V squeezes the set M along the space $(I - P)H$. The negative invariance of M gives us that $M \subseteq V^k M$ for all $k \in \mathbb{N}$. Thus, the set M must be initially squeezed. This property is expressed by the assertion on the finite dimensionality of M.

The squeezing inequality of Eq. (4.1) implies the relation

$$\|Vv_1 - Vv_2\| \le \eta \|v_1 - v_2\| + \|P(Vv_1 - Vv_2)\|, \quad v_1, v_2 \in M. \tag{4.2}$$

As we shall see below, the statement of Theorem 4.1 remains valid if we replace the inequality Eq. (4.1) by its weaker version of Eq. (4.2). An important consequence of this observation is that the inequality Eq. (4.1) has some common treads with observability–stabilizability inequalities studied in the context of control theory. To explore this connection, let us think of V as the flow $T(t_0)$ for some value of time $t_0 > 0$. A typical form of stabilizability inequality is the following inequality [17]

$$\|T(t_0)v_1 - T(t_0)v_2\| \le (I + p)^{-1}(\|v_1 - v_2\|) + (I + p)^{-1}(LOT(v_1, v_2)), \tag{4.3}$$

where the scalar function p is monotone increasing and vanishes at the origin. The function p is determined from the behavior of the nonlinear damping at the origin. If the damping is linear at the origin, the function p is linear as well. The lower order terms denote terms "below" the energy level and are usually given in the form

$$LOT(v_1, v_2) = C \sup_{t \in [0, t_0]} ||T(t)v_1 - T(t)v_2||_{H_1},$$

where the injection $H \subset H_1$ is compact. This means that the LOT terms are compact with respect to the topology generating the flow. If one aims at proving uniform stability of the dynamical system, then the goal is to show that the LOT terms can be dispensed with. This is typically done by a suitable compactness-uniqueness argument. In such case, the attractor is trivial and reduces to the equilibrium point. If, instead, the uniqueness property does not hold, then the inequality of Eq. (4.3) allows one to show compactness of the attractor. To make relation with the inequality of Eq. (4.2), let us consider the special case when p is linear, which is the case when the damping is linear at the origin. So we have $(I + p)^{-1}(\|v\|) = \eta\|v\|$, where $\eta < 1$. The inequality of Eq. (4.2) requires that the lower order term describing *trajectories* be replaced by finite dimensional projections of the point evaluation (at t_0) of these trajectories. Thus, we observe some link between finite dimensionality of an invariant set and observability-stabilizability type estimates on this set. The "gap" between the "squeezing" property of Eq. (4.2) and the stabilizability inequality of Eq. (4.3) is twofold : (i) the appearance of the nonlinear relation $(I + p)^{-1}$ in Eq. (4.3) and (ii) the structure of the "lower" order terms in the inequality, where Eq. (4.2) requires that these terms be finite dimensional and evaluated on the phase space (rather than trajectories as in Eq. (4.3)). This raises the following rather natural question: *What kind of abstract observability-stabilizability estimates will guarantee finite dimensionality of an invariant set?*

In other words, our aim is to decrease the gap between the stabilizability inequality of Eq. (4.3) and the squeezing inequality of Eq. (4.2). As we shall see later by means of the counterexample, it is not possible to entirely close the gap. The general form of nonlinearity of the first term in Eq. (4.3) essentially precludes finite dimensionality. However, even in this case, we will be able to provide a computational procedure (in Theorem 4.2) for a more accurate estimate of the dimension. This is why we use Kolmogorov's ε-entropy together with the notion of fractal dimension. In the context of long-time dynamics governed by nonlinear PDEs, ε-entropy was used earlier for characterization of "thickness" of global attractors for reaction-diffusion equations [2, 3] and for (linearly) damped hyperbolic equations [8, 25] in unbounded domains. As to the structure of lower order terms in the second part of inequality Eq. (4.3), we shall see that the original squeezing property can be greatly generalized to accommodate the stabilizability inequality. This will be the content of Theorem 4.5, in which we show that the stabilizability inequality with a linear damping at the origin does imply finite fractal dimension of a bounded invariant set. The abstract results mentioned above will lead (Section 4.7) to new results on finite dimensionality of attractors for nonlinear wave and plate equations with *nonlinear boundary dissipation*.

4.2 Main Abstract Results

We start with the definitions of ε-entropy and fractal dimension. We refer to Reference 13 for details concerning the concept of entropy.

DEFINITION 4.1 Let K be a compact set in a Banach space X. Then *Kolmogorov's ε-entropy* $H_\varepsilon(K)$ of the set K in X is the value

$$H_\varepsilon(K) = \ln N(K, \varepsilon), \quad \varepsilon > 0,$$

where $N(K, \varepsilon)$ is the minimal number of closed sets of the diameter not greater than 2ε, which cover the compact K. The *fractal dimension* $\dim_f K$ of K is defined by the formula

$$\dim_f M = \limsup_{\varepsilon \to 0} \frac{H_\varepsilon(K)}{\ln(1/\varepsilon)} . \tag{4.4}$$

It follows directly from the definition that a compact set K has finite fractal dimension if and only if his ε-entropy satisfies the estimate $H_\varepsilon(K) \leq C_1 \ln \frac{1}{\varepsilon} + C_2$ with some constants C_1 and C_2.

Roughly speaking, ε-entropy characterizes "thickness" of compact sets. Examples of calculations of ε-entropy for various sets can be found in Reference 13 and 24 (see also the references therein). For example (see Reference 24, Sect. 4.10.3), the ε-entropy $H_\varepsilon(K)$ of the set $K = \{u \in H^s(\mathcal{O}) : \|u\|_s \leq 1\}$ as a compact set in $L_2(\mathcal{O})$ has the form $H_\varepsilon(K) \sim \varepsilon^{-\frac{\dim \mathcal{O}}{s}}$, $s > 0$.

Our first main abstract result is the following assertion.

THEOREM 4.2

Let X be a separable Hilbert space and M be a bounded closed set in X. Assume that there exists a mapping $V : M \mapsto X$ such that

1. $M \subseteq VM$;
2. *V is Lipschitz on M, that is, there exists $L > 0$ such that*

$$\|Va_1 - Va_2\| \leq L\|a_1 - a_2\|, \quad a_1, a_2 \in M; \tag{4.5}$$

3. *There exist pseudometrics ϱ_1 and ϱ_2 on X such that*

$$\|Va_1 - Va_2\| \leq g(\|a_1 - a_2\|) + h([\varrho_1(a_1, a_2)^2 + \varrho_2(Va_1, Va_2)^2]^{1/2}) \tag{4.6}$$

for all $a_1, a_2 \in M$, where $g, h : \mathbf{R}_+ \mapsto \mathbf{R}_+$ are continuous nondecreasing functions such that

$$g(0) = 0; \quad g(s) < s, \ s > 0; \quad s - g(s) \text{ is nondecreasing}; \tag{4.7}$$

and the function $h(s)$ is strictly increasing in the interval $[0, s_0]$ for some $s_0 > 0$ and $h(0) = 0$;

4. *For any $q > 0$ and for any closed bounded set $B \subset M$ the maximal number $m(B, q)$ of elements $x_j^B \in B$ such*

$$\varrho_1(x_j^B, x_i^B)^2 + \varrho_2(Vx_j^B, Vx_i^B)^2 > q^2, \quad i \neq j, \ i, j = 1, \dots, m(B, q), \tag{4.8}$$

is finite.

Then M is a compact set and there exists $0 < \varepsilon_0 < 1$ such that for all $\varepsilon \leq \varepsilon_0 < 1$ Kolmogorov's ε-entropy $H_\varepsilon(M)$ admits the following estimate

$$H_\varepsilon(M) \leq \int_\varepsilon^{\varepsilon_0} \frac{\ln m(g_\delta^{-1}(s), q(s))}{s - g_\delta(s)} ds + H_{g_\delta(\varepsilon_0)}(M), \tag{4.9}$$

where $g_\delta(s) = \frac{1-\delta}{2} g(2s) + \delta s$ with arbitrary $\delta \in (0, 1)$, the function $q(s)$ is defined by the formula

$$q(s) = \frac{1}{2} h^{-1} \{\delta[2s - g(2s)]\}, \quad 0 < s < \varepsilon_0, \tag{4.10}$$

and

$$m(r,q) = \sup\{m(B,q) \,:\, B \subseteq M,\ \mathrm{diam} B \le 2r\}. \tag{4.11}$$

The proof of Theorem 4.2 is given in Section 4.3.

REMARK 4.1 It follows from the arguments given below that we can assume the validity of relation of Eq. (4.6) for all $a_1, a_2 \in M$ such that $\|a_1 - a_2\| \le \alpha(M) + \varepsilon_0$, with some positive ε_0, where $\alpha(M)$ is the Kuratowski α-measure of noncompactness, which is defined by the formula

$$\alpha(B) = \inf\{d \,:\, B \text{ has a finite cover of diameter } < d\} \tag{4.12}$$

on bounded sets of X (for the properties of α-measure see, e.g., Reference 12, p. 13). Similarly, we need property (4) for sets with the diameter less than $\alpha(M) + \varepsilon_0$. ∎

As we shall see later, the inequality in Eq. (4.6) with appropriate interpretation of the phase space is implied by stabilizability inequality of Eq. (4.3). The function $g(s)$ coincides with $(I+p)^{-1}(s)$. Also, compactness of $H \subset H_1$ will lead to compact psedometrics ρ_1 and ρ_2. On the other hand, if the pseudometrics ϱ_1 and ϱ_2 are precompact, then the number $m(B,q)$ is finite for any bounded B and $q > 0$. We recall that a pseudometric ϱ on a Banach space X is said to be *precompact* (with respect to the norm of X) if any bounded sequence (in norm) has a subsequence that is Cauchy with respect to ϱ. To prove the finiteness of $m(B,q)$ we can use contradiction argument. Indeed, assume that there is a sequence $\{x_j^B\} \subset B$ such that

$$\varrho_1\big(x_j^B, x_i^B\big)^2 + \varrho_2\big(Vx_j^B, Vx_i^B\big)^2 > q^2 > 0, \quad i \ne j,\ j,i = 1,2,\dots,$$

for some $q > 0$. Because pseudometrics ϱ_1 and ϱ_2 are precompact, we can choose a subsequence $\{j_n\}$ such that

$$\varrho_1\big(x_{j_n}^B, x_{j_k}^B\big) + \varrho_2\big(Vx_{j_n}^B, Vx_{j_k}^B\big) \to 0, \quad \text{as } n,k \to \infty,$$

which is impossible because of the previous relation.

Thus, stabilizability inequality of Eq. (4.3) guarantees that the hypotheses of Theorem 4.2 holds. Consequently, the formula for computations of Kolmogorov's ε-entropy (and of fractal dimension) given in Eq. (4.9) holds true. However, this fact alone does not necessarily imply that the fractal dimension is finite. In fact, as shown by a counterexample, generally this is not the case. It all depends on the effective estimate of the formula of Eq. (4.9). It turns out that in the case when more information on the behavior of the damping at the origin is given, this formula does imply a finite fractal dimension. The relevant results are formulated below.

THEOREM 4.3

In addition to the hypotheses of Theorem 4.2 we assume that

1. $m(r,q) \le m_0(r/q)$, *where* $m_0(s)$ *is a continuous nondecreasing function;*
2. $\lim_{s\to 0} \frac{g(s)}{s} = \gamma < 1$.

Then for sufficiently small $\varepsilon_0 > 0$ *we have the estimate*

$$H_\varepsilon(M) \le c_0 \int_\varepsilon^{\varepsilon_0} \ln m_0\left(\frac{c_1 s}{h^{-1}[2\delta(1-\gamma)s]}\right)\frac{ds}{s} + H_{g_\delta(\varepsilon_0)}(M), \tag{4.13}$$

where $\varepsilon \in (0,\varepsilon_0)$, $\delta \in (0,1)$ *and* c_0, c_1 *are positive constants independent of* ε.

PROOF It follows from Theorem 4.2 and from the relations

$$s - g_\delta(s) \sim (1-\delta)(1-\gamma)s, \quad g_\delta^{-1}(s) \sim \frac{s}{\delta + (1-\delta)\gamma}, \quad q(s) \sim \frac{1}{2}h^{-1}(2\delta(1-\gamma)s),$$

which hold in the limit $s \to 0$. ∎

This theorem implies the following assertion.

THEOREM 4.4
Let $h(s)$ be a linear function, $h(s) = h_0 \cdot s$. Then under the conditions of Theorem 4.3 the fractal dimension $\dim_f M$ of the set M is finite.

PROOF If $h(s) = h_0 \cdot s$, then by Eq. (4.13) we have that

$$H_\varepsilon(M) \le c_0 \ln m_0 \left(\frac{c_1 h_0}{2\delta(1-\gamma)}\right) \cdot \int_\varepsilon^{\varepsilon_0} \frac{ds}{s} + H_{g_\delta(\varepsilon_0)}(M) \le C_1 \ln \frac{1}{\varepsilon} + C_2.$$

Thus $\dim_f M < \infty$. ∎

In the special case when the dissipation is strictly sublinear at the origin, the following special case of Theorem 4.2 can be formulated:

THEOREM 4.5
Let X be a separable Hilbert space and A be a bounded closed set in X. Assume that there exists a mapping $V : A \mapsto X$ such that

1. *$A \subseteq VA$;*
2. *V is Lipschitz on A, that is, there exists $L > 0$ such that*

   $$\|Va_1 - Va_2\| \le L\|a_1 - a_2\| \text{ for all } a_1, a_2 \in A.$$

3. *There exist precompact seminorms $n_1(x)$ and $n_2(x)$ on X such that*

   $$\|Va_1 - Va_2\| \le g(\|a_1 - a_2\|) + K \cdot [n_1(a_1 - a_2) + n_2(Va_1 - Va_2)], \tag{4.14}$$

   for all $a_1, a_2 \in A$ where $K > 0$ is a constant and $g : \mathbf{R}_+ \mapsto \mathbf{R}_+$ are a continuous non-decreasing function satisfying Eq. (4.7).

If $\lim_{s\to 0} \frac{g(s)}{s} \equiv \gamma < 1$, then A is a compact set in X of the finite fractal dimension.

The proof of Theorem 4.5 is given in Section 4.5.

4.3 Proof of Theorem 4.2

Step 1: compactness of *M*. We first prove the following assertion.

LEMMA 4.1
Assume that $V : M \mapsto X$ is a continuous mapping possessing a property of Eq. (4.6) *where $g,\ h : \mathbf{R}_+ \mapsto \mathbf{R}_+$ are continuous functions, $h(0) = 0$, and the pseudometrics ϱ_1 and ϱ_2 satisfy hypothesis*

(iv) of Theorem 4.2. Then

$$\alpha(VB) \leq g(\alpha(B)) \ \textit{for any} \ \ B \subseteq M, \tag{4.15}$$

where $\alpha(B)$ is the Kuratowski α-measure of the set B.

PROOF By the definition of α-measure (see Eq. (4.12)) for any $\varepsilon > 0$ there exist sets $F_1, \dots, F_n$ such that

$$B = F_1 \cup \dots \cup F_n, \quad \operatorname{diam} F_i < \alpha(B) + \varepsilon.$$

Let $\mathcal{N} = \{x_i \ : \ i = 1, 2 \dots m\} \subset B$ be a finite set such that for every $y \in B$ there is $i \in \{1, 2, \dots, m\}$ with the property $\rho_1(y, x_i)^2 + \rho_2(Vy, Vx_i)^2 \leq \varepsilon^2$. This set exists by hypothesis (4) of Theorem 4.2. It means that

$$B = \cup_{i=1}^m C_i, \quad C_i = \{y \in B \ : \ \rho_1(y, x_i)^2 + \rho_2(Vy, Vx_i)^2 \leq \varepsilon^2\}.$$

Now we can write the representations

$$B = \cup_{i,j}(C_i \cap F_j) \qquad \text{and} \qquad VB = \cup_{i,j}[V(C_i \cap F_j)].$$

Using Eq. (4.6), it is easy to see that $\operatorname{diam}[V(C_j \cap F_i)] \leq g[(\alpha)(B) + \varepsilon] + h(2\varepsilon)$. This implies of Eq. (4.15). ∎

If the assumptions of Theorem 4.2 hold, then Eq. (4.15) implies that

$$\alpha(M) \leq \alpha(VM) \leq g[\alpha(M)].$$

Because $g(s) < s$ for $s > 0$, the later relation is possible only if $\alpha(M) = 0$. Thus M is compact.

Step 2: estimate for ε-entropy. Assume that $\{F_i \ : \ i = 1, \dots, N(M, \varepsilon)\}$ is the minimal covering of M by its closed subsets with a diameter less than 2ε. Let $\{x_j^i \ : \ j = 1, \dots, n_i\} \subset F_i$ be a maximal finite set such that Eq. (4.8) holds with $B = F_i$. By assumption (4) such a set exists and $n_i \leq m(\varepsilon, q)$, where $m(\varepsilon, q)$ is given by Eq. (4.11). We also have that

$$F_i \subset \bigcup_{j=1}^{n_i} B_j^i, \quad B_j^i \equiv \left\{v \in F_i \ : \ \rho_1\big(v, x_j^i\big)^2 + \rho_2\big(Vv, Vx_j^i\big)^2 \leq q^2\right\}.$$

Therefore

$$VM \subset \bigcup_{i=1}^{N(M,\varepsilon)} \bigcup_{j=1}^{n_i} VB_j^i.$$

If $y_1, y_2 \in B_j^i$, then from Eq. (4.6) we have

$$\|Vy_1 - Vy_2\| \leq g(\|y_1 - y_2\|) + h\left(\sum_{k=1,2}\left[\rho_1\big(y_k, x_j^i\big)^2 + \rho_2\big(Vy_k, Vx_j^i\big)^2\right]^{1/2}\right).$$

Thus $\operatorname{diam} VB_j^i \leq g(2\varepsilon) + h(2q)$ for any $\varepsilon > 0$ and $q > 0$. Therefore

$$N\bigg(VM, \frac{1}{2}[g(2\varepsilon) + h(2q)]\bigg) \leq m(\varepsilon, q) \cdot N(M, \varepsilon).$$

If we choose $q = q(\varepsilon)$, where $q(s)$ is defined by Eq. (4.10), then we obtain that

$$N(M, g_\delta(\varepsilon)) \le m(\varepsilon, q(\varepsilon)) \cdot N(M, \varepsilon),$$

where $g_\delta(\varepsilon) = \frac{1-\delta}{2} g(2\varepsilon) + \delta\varepsilon$. Let $\{\varepsilon_n\}$ be the sequence defined by the recurrence relation $\varepsilon_{n+1} = g_\delta(\varepsilon_n)$. It is clear that $\varepsilon_{n+1} < \varepsilon_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Thus we obtain

$$\ln N(M, \varepsilon_n) \le \sum_{k=0}^{n-1} \ln m[\varepsilon_k, q(\varepsilon_k)] + \ln N(M, \varepsilon_0). \tag{4.16}$$

If $\varepsilon_{k+1} = g_\delta(\varepsilon_k) \le \varepsilon < \varepsilon_k$, then $\varepsilon_k \le g_\delta^{-1}(\varepsilon)$ and $q(\varepsilon_k) > q(\varepsilon)$. Therefore, because $m(r, q)$ is nonincreasing with respect to q and nondecreasing with respect to r, we have that

$$\ln m[\varepsilon_k, q(\varepsilon_k)] \le \ln m\left[g_\delta^{-1}(\varepsilon), q(\varepsilon)\right], \quad \varepsilon_{k+1} \le \varepsilon < \varepsilon_k.$$

Thus

$$\ln m[\varepsilon_k, q(\varepsilon_k)] \le \int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{\ln m\left[g_\delta^{-1}(\varepsilon), q(\varepsilon)\right]}{\varepsilon_k - \varepsilon_{k+1}} d\varepsilon.$$

Because $\varepsilon_k - \varepsilon_{k+1} = \varepsilon_k - g_\delta(\varepsilon_k) > \varepsilon - g_\delta(\varepsilon)$, using Eq. (4.16) we obtain

$$\ln N\left[M, g_\delta^n(\varepsilon_0)\right] \le \int_{g_\delta^n(\varepsilon_0)}^{\varepsilon_0} \frac{\ln m\left[g_\delta^{-1}(\varepsilon), q(\varepsilon)\right]}{\varepsilon - g_\delta(\varepsilon)} d\varepsilon + \ln N(M, \varepsilon_0). \tag{4.17}$$

Now we take arbitrary $\varepsilon \in (0, \varepsilon_0)$. Because $g_\delta(s)$ is invertible, we can find $\widetilde{\varepsilon}_0$ in the interval $(\varepsilon_1, \varepsilon_0]$ and a number n such that $\varepsilon = g_\delta^n(\widetilde{\varepsilon}_0)$. Therefore it follows from Eq. (4.17) that

$$\ln N(M, \varepsilon) \le \int_{\varepsilon}^{\widetilde{\varepsilon}_0} \frac{\ln m\left(g_\delta^{-1}(\varepsilon), q(\varepsilon)\right)}{\varepsilon - g_\delta(\varepsilon)} d\varepsilon + \ln N(M, \widetilde{\varepsilon}_0).$$

This implies Eq. (4.9). The proof of Theorem 4.2 is complete.

4.4 Calculation of $\ln m(r, q)$

Theorem 4.2 shows the importance of the characteristic $m(r, q)$ for calculations of bounds for ε-entropy. In this section we find bounds for $\ln m(r, q)$ in several cases that are important from a point of view of applications.

PROPOSITION 4.1
Assume that the pseudometrics ρ_1 and ρ_2 has the form

$$\rho_1(x_1, x_2) = \|R_1(x_1 - x_2)\| \qquad \text{and} \qquad \rho_2(x_1, x_2) = \|R_2(x_1 - x_2)\|, \tag{4.18}$$

where R_1 and R_2 are compact operators in X. Then

- *for every closed set $F \subset M$ we have*

$$\ln m(F, q) \le H_{q/2}^{X\times X}(\widetilde{F}), \tag{4.19}$$

where $\widetilde{F} = \{(R_1 x; R_2 V x) : x \in F\} \subset X \times X$; here and below $H_\varepsilon^Y(A)$ is the ε-entropy of the set A in the space Y;

- *the relation*

$$\ln m(r,q) \le m_0(r/q) \equiv H^X_{\frac{q}{4r\sqrt{2}}}(R_1 B) + H^X_{\frac{q}{4rL\sqrt{2}}}(R_2 B) \tag{4.20}$$

holds, where B is a unit ball in X.

PROOF It is clear that

$$m(F,q) = \aleph\{w_i \in \widetilde{F} \;:\; \|w_i - w_j\|_{X\times X} > q,\; i \ne j\}$$

for every $F \subset M$, where $\aleph\{...\}$ denotes the maximal number of elements with the given properties. Therefore, by Reference 13, Theorem 4, we obtain Eq. (4.19).

Because $\widetilde{F} \subset R_1 F \times R_2 V B$, we have that

$$\widetilde{F} \subset R_1 B^1_{\mathrm{diam}F} \times R_2 B^2_{L\mathrm{diam}F},$$

where B^i_ρ denotes a ball in X of the radius ρ with the center at some point, $i = 1, 2$. Therefore

$$\begin{aligned} m(F,q) &\le \aleph\left\{w_i \in R_1 B^1_{\mathrm{diam}F} \times R_2 B^2_{L\mathrm{diam}F} \;:\; \|w_i - w_j\|_{X\times X} > q,\; i \ne j\right\}, \\ &= \aleph\left\{w_i \in R_1 B^0_1 \times R_2 B^0_L \;:\; \|w_i - w_j\|_{X\times X} > \frac{q}{\mathrm{diam}F},\; i \ne j\right\}, \end{aligned}$$

where B^0_ρ denotes a ball in X of the radius ρ with the center at zero. Thus

$$m(r,q) \le \aleph\left\{w_i \in R_1 B^0_1 \times R_2 B^0_L \;:\; \|w_i - w_j\|_{X\times X} > \frac{q}{2r},\; i \ne j\right\}.$$

Therefore, from Reference 13, Theorem 4, we have that

$$m(r,q) \le N^{X\times X}\left(R_1 B^0_1 \times R_2 B^0_L, \frac{q}{4r}\right),$$

where $N^{X\times X}(K_1 \times K_2, \varepsilon)$ is the minimal number of closed sets of the diameter not greater than 2ε, which cover the compact $K_1 \times K_2$ in the space $X \times X$. A simple calculation shows that

$$N^{X\times X}(K_1 \times K_2, \varepsilon) \le N(K_1, \varepsilon/\sqrt{2}) \cdot N(K_2, \varepsilon/\sqrt{2}).$$

Thus we obtain that the relation

$$m(r,q) \le N\left(R_1 B^0_1, \frac{q}{4r\sqrt{2}}\right) \cdot N\left(R_2 B^0_1, \frac{q}{4rL\sqrt{2}}\right),$$

which implies Eq. (4.20). ∎

Using the calculations of ε-entropy given in Reference 13, it is easy to obtain from Proposition 4.1 the following assertions.

COROLLARY 4.1

Assume that R_1 and R_2 are finite dimensional operators. Then

$$\ln m(r,q) \le C + (\dim R_1 + \dim R_2) \ln \frac{r}{q},$$

where the constant C does not depend on r and q, $r \ge q$.

COROLLARY 4.2
Assume (without loss of generality) that R_1 and R_2 are nonnegative compact operators with the eigenvalues

$$\lambda_1(R_1) \geq \lambda_2(R_2) \geq \dots \qquad \textit{and} \qquad \lambda_1(R_2) \geq \lambda_2(R_2) \geq \dots$$

If $\lambda_k(R_i) \leq c_0 k^{-\alpha}$, $i = 1, 2$, for some $\alpha > 0$, then

$$\ln m(r, q) \leq C \left(\frac{r}{q}\right)^{1/\alpha},$$

where the constant C does not depend on r and q ($r \geq q$).

4.5 Bounds for ε-Entropy and Proof of Theorem 4.5

Corollaries 4.1 and 4.2 leads to the following assertion concerning the ε-entropy of the set M.

THEOREM 4.6
In addition to the hypotheses of Theorem 4.2 we assume that

1. *the pseudometrics ρ_1 and ρ_2 appearing in Eq.* (4.6) *are of the form given in Eq.* (4.18).
2. *$h(s) = h_0 \cdot s^\beta$ for some $0 < \beta \leq 1$ and $\lim_{s\to 0} \frac{g(s)}{s} = \gamma < 1$.*
   - *If the operators R_1 and R_2 satisfy the conditions of Corollary 4.2, then*

     $$H_\varepsilon(M) \leq C_1(\alpha, \beta) \left(\frac{1}{\varepsilon}\right)^{\left(\frac{1}{\beta}-1\right)\frac{1}{\alpha}} + C_2 \;\; \textit{for} \;\; 0 < \beta < 1$$

     and $H_\varepsilon(M) \leq C_1(\alpha) \ln \frac{1}{\varepsilon} + C_2$ when $\beta = 1$.
   - *If R_1 and R_2 are finite-dimensional operators, then*

     $$H_\varepsilon(M) \leq C_1 \ln\frac{1}{\varepsilon} + C_2(\beta)\left(\frac{1}{\beta} - 1\right)\left[\ln\frac{1}{\varepsilon}\right]^2 + C_3, \quad 0 < \beta \leq 1.$$

PROOF We apply Theorem 4.3 with $h^{-1}(s) = (s/h_0)^{1/b}$. In the first case $\ln m_0(s) = c \cdot s^{1/\alpha}$ by Corollary 4.2. Therefore, Eq. (4.13) implies the desired estimate. In the second case by Corollary 4.1 we have that $\ln m_0(s) = c_1 + c_2 \ln s$ and again Eq. (4.13) implies the result. ∎

Thus, under the conditions of Theorem 4.6 we can guarantee the finiteness of fractal dimension of the set M only in the case $\beta = 1$ (i.e., when $h(s)$ is a linear function). This is the content of Theorem 4.5.

PROOF OF THEOREM 4.5 We shall apply the second part of Theorem 4.6. For this we need to show that the psedometrics ρ_1 and ρ_2 appearing in Eq. (4.6) are of the form as in Eq. (4.18) with R_i finite rank operators.

To this end it suffices to prove that for any $\delta > 0$, there exist a constant K_δ and finite dimensional orthoprojectors P_1^δ and P_2^δ in X such that

$$n_1(a_1 - a_2) \le g_\delta(\|a_1 - a_2\|) + K_\delta \|P_1^\delta(a_1 - a_2)\|, \quad a_1, a_2 \in M, \tag{4.21}$$

and

$$n_2(Va_1 - Va_2) \le g_\delta(\|a_1 - a_2\|) + K_\delta \|P_2^\delta(Va_1 - Va_2)\|, \quad a_1, a_2 \in M, \tag{4.22}$$

where $g_\delta(s) = \delta \cdot (s - g(s))$. We carry the proof by contradiction. Indeed, assume that Eq. (4.21) was not true. Then, there exist $\delta_0 > 0$, a sequence of positive numbers $\{c_m\}$ such that $c_m \to \infty$ as $m \to \infty$ and a sequence of orthoprojectors $\{P_m\}$ such that $P_m \to I$ strongly in X and

$$n_1(v_m) > g_\delta(\|v_m\|) + c_m \|P_m v_m\|, \quad m = 1, 2, \dots, \tag{4.23}$$

for some precompact sequence $\{v_m\} \subset X$ (we recall that M is a compact set). Thus, we can assume that there is $v \in X$ such that $v_m \to v$ as $m \to \infty$. Because $c_m \to \infty$, from Eq. (4.23) we have that $\|P_m v_m\| \to 0$ as $m \to \infty$. This implies that $v = 0$ and thus $\|v_m\| \to 0$ as $m \to \infty$. It is also clear from Eq. (4.23) that $\|v_m\| > 0$; therefore, Eq. (4.23) implies that

$$n_1(w_m) > \frac{g_\delta(\|v_m\|)}{\|v_m\|} + c_m \|P_m w_m\|, \quad m = 1, 2, \dots, \tag{4.24}$$

where $w_m = \frac{v_m}{\|v_m\|}$. Because $\|w_m\| = 1$, we can also assume that $w_m \to w$ weakly in X for some $w \in X$. From Eq. (4.24) we have that $\|P_m w_m\| \to 0$ as $m \to \infty$. Because $P_m \to I$ strongly and

$$P_m w = P_m(w - w_m) + P_m w_m \to 0 \ \text{ weakly in } \ X,$$

we conclude that $w = 0$. This implies that $n_1(w_m) \to 0$ as $m \to \infty$. Therefore, Eq. (4.24) gives us that

$$\frac{g_\delta(\|v_m\|)}{\|v_m\|} \le n_1(w_m) \to 0, \quad m \to \infty,$$

which is impossible because $\|v_m\| \to 0$ and $\lim_{s\to 0} \frac{g_\delta(s)}{s} = \delta(1 - \gamma) > 0$. Thus Eq. (4.21) holds.

To prove Eq. (4.22) we note that the same argument gives us that

$$n_2(Va_1 - Va_2) \le g_\delta(L^{-1}\|Va_1 - Va_2\|) + K_\delta \|P_2^\delta(Va_1 - Va_2)\|, \quad a_1, a_2 \in H.$$

Therefore, using Lipschitz property (2) and the monotonicity of g_δ, we obtain Eq. (4.22).

Using Eqs. (4.21) and (4.22) we can rewrite Eq. (4.14) in the form

$$\|Va_1 - Va_2\| \le \widetilde{g}_\delta\,(\|a_1 - a_2\|) + \widetilde{K} \cdot (\|P_1(a_1 - a_2)\|^2 + \|P_2(Va_1 - Va_2)\|^2)^{1/2}$$

for every $a_1, a_2 \in A$ and any $0 < \delta < 1$, where $\widetilde{g}_\delta\,(s) = (1 - \delta)g(s) + \delta \cdot s$, the constant $\widetilde{K}$ and the orthoprojectors P_1 and P_2 may depend on δ. Now we can apply the second part of Theorem 4.6 with $\beta = 1$ and $g(s) = \widetilde{g}_\delta(s)$ and conclude the proof. ∎

4.6 Critical Case—Counterexample

Now we consider the critical case when $\lim_{s\to 0} \frac{g(s)}{s} = \gamma = 1$. By Theorem 4.6 we can expect the result on finite dimensionality for the case of a linear function $h(s)$ only.

THEOREM 4.7
In addition to the hypotheses of Theorem 4.2 we assume that

1. $\lim_{s\to 0} \frac{g(s)}{s} = 1$ *and* $h(s) = h_0 \cdot s$ *is a linear function;*

2. $m(r, q) \le m_0(r/q)$*, where* $m_0(s)$ *is a continuous nondecreasing function.*

Then we have the estimate

$$H_\varepsilon(M) \le \frac{2}{1-\delta} \int_\varepsilon^{\varepsilon_0} \ln m_0 \left(\frac{4h_0 s}{\delta[2s - g(2s)]} \right) \frac{ds}{2s - g(2s)} + H_{g_\delta(\varepsilon_0)}(M),$$

for all $0 < \varepsilon \le \varepsilon_0$*, where* ε_0 *is small enough. If* $g''(0)$ *exists and* $g''(0) < 0$*, then*

$$H_\varepsilon(M) \le c_0 \int_\varepsilon^{\varepsilon_0} \ln m_0 \left(\frac{c_1}{s} \right) \frac{ds}{s^2} + c_2,$$

where c_0*,* c_1*, and* c_2 *are positive constants independent of* ε*.*

In particular, in the case when the pseudometrics ρ_1 *and* ρ_2 *has the form of Eq.* (4.18) *we have the estimate*

$$H_\varepsilon(M) \le C_1 \left(\frac{1}{\varepsilon} \right)^{1+1/\alpha} + C_2$$

provided the operators R_1 *and* R_2 *satisfy the conditions of Corollary 4.2 and*

$$H_\varepsilon(M) \le C_1 \frac{1}{\varepsilon} \left(1 + \ln \frac{1}{\varepsilon} \right) + C_2,$$

when R_1 *and* R_2 *are finite-dimensional operators.*

PROOF This is a direct corollary of Theorem 4.2. ∎

As we see, Theorem 4.7 provides us with estimates, which do not guarantee finiteness of fractal dimension. The following assertion shows that in general under the conditions of Theorem 4.8 we cannot obtain finite dimensionality of the set M.

THEOREM 4.8
We claim that in a separable Hilbert space X *there exist a compact set* A *and a mapping* $V : A \mapsto X$ *such that*

1. $VA = A$*;*

2. *Kolmogorov's* ε*-entropy* $H_\varepsilon(A)$ *admits the estimate* $H_\varepsilon(A) \ge c \cdot \varepsilon^{-1/\beta}$ *for some* $c > 0$ *and* $\beta > 1/2$ *(this implies that* $\dim_f A = \infty$*);*

3. V *is Lipschitz on* A*, that is, there exists* $L > 0$ *such that*

$$\|Va_1 - Va_2\| \le L\|a_1 - a_2\|, \quad a_1, a_2 \in A;$$

4. *there exists a compact operator* $R : X \mapsto X$ *such that*

$$\|Va_1 - Va_2\| \le g(\|a_1 - a_2\|) + \|R(a_1 - a_2)\|$$

   for all $a_1, a_2 \in A$*, where* $g : \mathbf{R}_+ \mapsto \mathbf{R}_+$ *is a continuous nondecreasing function satisfying Eq.* (4.7).

PROOF Without loss of generality we can assume that $X = l^2$.

Let

$$A = \{x = (x_1, x_2, \dots) \in l^2 \; : \; 0 \le x_i \le \alpha_i\},$$

where $\alpha_i > 0$ and $\alpha_i \sim \left(\frac{1}{i}\right)^\beta$ with $\beta > 1/2$. It is clear that A is a compact set of infinite fractal dimension. The estimate for $H_\varepsilon(A)$ given in (2) follows from Reference 13, Theorem 16.

We define a mapping $V : A \mapsto A$ in the following way. Let

$$g_i(s) = \frac{s}{1 + \gamma_i s}, \quad s \ge 0, \quad \gamma_i \ge \gamma_{i+1} > 0, \quad \lim_{i\to\infty} \gamma_i = 0.$$

We suppose $V(x) = (V_1(x_1), V_2(x_2), \dots)$, where

$$V_i(x_i) = \alpha_i g_i\left(\frac{x_i}{\alpha_i}\right) + (1 - g_i(1))x_i, \quad x = (x_1, x_2, \dots) \in A.$$

Invariance property (1) follows from continuity of each $g_i(s)$ and the properties $g_i(0) = 0$ and $g_i(\alpha_i) = \alpha_i$. Lipschitz continuity property (2) follows from the obvious relation

$$|V_i(x_i) - V_i(y_i)| \le 2|x_i - y_i|.$$

Thus, we need establish the stabilizability estimate in (4). We represent $V = G + R$, where $G(x) = (G_1(x_1), G_2(x_2), \dots)$ with

$$G_i(x_i) = \alpha_i g_i\left(\frac{x_i}{\alpha_i}\right), \quad x = (x_1, x_2, \dots) \in A,$$

and $R(x) = \{(1 - g_1(1)x_1, [1 - g_2(1)]x_2, \dots\}$ is a linear operator in l^2. Because $1 - g_i(1) \to 0$ as $i \to \infty$, R is a compact operator. Thus, we only need an estimate for $\|G(x) - G(y)\|$.

Because $g_i(|s|) = \frac{|s|}{1+\gamma_i|s|}$ is a metric on **R** for every i and $0 \le x_i, y_i \le \alpha_i$, it is easy to see that

$$\|G(x) - G(y)\|^2 = \sum_1^\infty \alpha_i^2 \left[g_i\left(\frac{x_i}{\alpha_i}\right) - g_i\left(\frac{y_i}{\alpha_i}\right)\right]^2 \le \sum_1^\infty \alpha_i^2 \left[g_i\left(\frac{|z_i|}{\alpha_i}\right)\right]^2,$$

where $z = x - y$. Hence

$$\|G(x) - G(y)\|^2 \le \|z\|^2 + \sum_{i=1}^\infty \alpha_i^2 \left\{\left[g_i\left(\frac{|z_i|}{\alpha_i}\right)\right]^2 - \left(\frac{|z_i|}{\alpha_i}\right)^2\right\}.$$

However,

$$[g_i(w)]^2 - w^2 = -\gamma_i w^3 \frac{2 + \gamma_i w}{(1 + \gamma_i w)^2} \le -\gamma_i w^3 \frac{2 + \gamma_1}{(1 + \gamma_1)^2}$$

for any $w \in [0, 1]$. Thus,

$$\|G(x) - G(y)\|^2 \le \|z\|^2 - \frac{2 + \gamma_1}{(1 + \gamma_1)^2} \cdot \sum_{i=1}^\infty \frac{\gamma_i}{\alpha_i}|z_i|^3.$$

Now we note that

$$\|z\|^2 = \sum_{i=1}^\infty |z_i|^2 = \sum_{i=1}^\infty \left[\frac{\alpha_i}{\gamma_i}|z_i|\right]^{1/2} \cdot \left[\frac{\gamma_i}{\alpha_i}|z_i|^3\right]^{1/2}$$
$$\le \left[\sum_{i=1}^\infty \frac{\alpha_i^2}{\gamma_i}\right]^{1/2} \cdot \left[\sum_{i=1}^\infty \frac{\gamma_i}{\alpha_i}|z_i|^3\right]^{1/2}.$$

This implies that

$$\sum_{i=1}^{\infty} \frac{\gamma_i}{\alpha_i} |z_i|^3 \geq \left[\sum_{i=1}^{\infty} \frac{\alpha_i^2}{\gamma_i} \right]^{-1} \|z\|^4.$$

Therefore, we obtain that $\|G(x) - G(y)\| \leq g(\|z\|)$ for $x, y \in A$, where

$$g(s) = s\sqrt{1 - c_0 s^2}, \quad c_0 = \frac{2 + \gamma_1}{(1 + \gamma_1)^2} \left[\sum_{i=1}^{\infty} \frac{\alpha_i^2}{\gamma_i} \right]^{-1}.$$

Here we choose $\{\gamma_i\}$ such that $\sum_{i=1}^{\infty} \alpha_i^2 \gamma_i^{-1} < \infty$. Clearly, we can modify g such that Eq. (4.7) will be satisfied. Thus (4) holds. ∎

4.7 Applications to Hyperbolic-Like Evolutions with a Nonlinear Boundary Dissipation

4.7.1 Attractors for Semilinear Wave Equations with Nonlinear Boundary Dissipation

Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a bounded, connected set with a smooth boundary Γ. The exterior normal on Γ is denoted by ν. We consider the following equation

$$w'' - \Delta w + f(w) = 0 \text{ in } Q = [0, \infty) \times \Omega \tag{4.25}$$

subject to the boundary condition

$$\partial_\nu w + w = -j(w') \text{ in } \sum = [0, \infty) \times \Gamma \tag{4.26}$$

and the initial conditions

$$w(0) = w_0 \qquad \text{and} \qquad w'(0) = w_1. \tag{4.27}$$

Here f and j are nonlinear functions subject to the following assumption.

(f) $f \in C^2(\mathbf{R})$ such that for $n = 3$ $|f''(s)| \leq c(1 + |s|^{1-\delta})$ for all s and for some $c > 0$ and $\delta > 0$. For $n = 2$ we assume $|f''(s)| \leq c(1 + |s|^p)$ for some $0 \leq p < \infty$. We also assume that $\liminf_{|s|\to\infty} \frac{f(s)}{s} > 0$.

(j) $j \in C^1(\mathbf{R})$, $j(0) = 0$, and there exist two positive constants m_1 and m_2 such that $m_1 \leq j'(s) \leq m_2$ for all $s \in \mathbf{R}$.

A longtime behavior of solutions to Eqs. (4.25) to (4.27) was a subject of two recent papers [5, 6]. It was proved in Reference 5 that problem in Eqs. (4.25) to (4.27) possesses a compact global attractor $\mathcal{A}$ in the space $\mathcal{H} = H^1(\Omega) \times L_2(\Omega)$. Our goal here is to demonstrate connection between the abstract theory presented in Section 4.2 and certain stabilizability inequalities established for the wave equation.

Let $[u(t), u'(t)]$ and $[v(t), v'(t)]$ be two trajectories of problem Eqs. (4.25) to (4.27), which belong to the attractor $\mathcal{A}$. For these trajectories we have proved in Reference 6 the following estimate:

$$E[u(t) - v(t)] \leq C_1 e^{-\beta t} E[u(0) - v(0)] + C_2 \max_{s\in[0,t]} \|u(s) - v(s)\|^2_{L_2(\Omega)} \tag{4.28}$$

for all $t \geq 0$, where $E[w(t)] = \frac{1}{2}(\int_\Omega |\nabla w(t)|^2 + \int_\Omega |w'(t)|^2 + \int_\Gamma |w(t)|^2)$. This is a special case of k-stabilizability estimate in Eq. (4.3) where we take $p(s) = m_1 s$ and t_0 sufficiently large. The point we wish to make is that this estimate allow us to apply Theorem 4.5 with $g(s) = (I + p)^{-1}(s)$ and the appropriately selected phase space. To see this, consider the space $X = \mathcal{H} \times H^1(Q_T)$ equipped with the norm

$$\|U\|_X^2 = \|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L_2(\Omega)}^2 + 2\int_0^T E(v(t))dt, \quad \text{where } U = (u_0, u_1, v).$$

Here $T > 0$ is a constant to be determined later. On the space X we define a seminorm

$$n_T(U) := \max_{0 \leq t \leq T} \|v(t)\|_{L_2(\Omega)}.$$

By the compactness of the imbedding (Reference 22, Corollary 9)

$$H^1(Q_T) = L_2[0, T; H^1(\Omega)] \cap H^1[0, T; L_2(\Omega)] \subset C[(0, T); L_2(\Omega)]$$

we obtain that $n_T(U)$ is a compact seminorm on X. Next we define the set A and the map V appearing in Theorem 4.5. Consider in the space X the set

$$\mathcal{A}_T = \{U \equiv [u_0, u_1, u(t) \text{ for } t \in [0, T]] \ : \ (u_0, u_1) \in \mathcal{A}\},$$

where $u(t)$ is the solution to Eqs. (4.25) to (4.27) with initial data $u(0) = u_0$, $u'(0) = u_1$ and $\mathcal{A}$ is the attractor. The operator $V_T : \mathcal{A}_T \mapsto X$ is now defined by the formula

$$V_T : (u_0, u_1, u(t)) \mapsto (u(T), u'(T), u(T + t)).$$

With the above notation, one can verify that all conditions of Theorem 4.5 are satisfied for an appropriate T. Details are technical and given in Reference 6. The final result proved in Reference 6 reads:

THEOREM 4.9
[6]Under the assumptions (f) and (j), the dynamical system generated by problem Eqs. (4.25) *to* (4.27) *in the space* $\mathcal{H} = H^1(\Omega) \times L_2(\Omega)$ *admits a compact global attractor* $\mathcal{A}$ *whose fractal dimension is finite.*

4.7.2 Attractors for von Karman Evolutions with a Nonlinear Boundary Dissipation

We now consider a nonlinear system of dynamic elasticity described by von Karman evolution with a nonlinear boundary dissipation.

Let $\Omega \subset \mathbf{R}^2$ be bounded domain with a sufficiently smooth boundary Γ. We assume that Γ consists of two disjoint parts Γ_0 and Γ_1. Consider the following von Karman model with boundary dissipation active on Γ_1 via the "free" boundary conditions [15]

$$u_{tt} + Bu_t + \Delta^2 u = [v(u) + F_0, u] + p \quad \text{in } \Omega \times (0, \infty). \tag{4.29}$$

The Airy stress function $v(u)$ satisfies the following elliptic problem

$$\Delta^2 v(u) + [u, u] = 0, \text{ in } \Omega, \quad \frac{\partial}{\partial n} v(u) = v(u) = 0 \quad \text{on } \Gamma. \tag{4.30}$$

The von Karman bracket $[u, v]$ is given by

$$[u, v] \equiv u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}.$$

The boundary conditions associated with Eq. (4.29) are of "free" type on Γ_1 and clamped on Γ_0:

$$\begin{aligned} \Delta u + (1-\mu)B_1 u = 0, \quad &\text{on } \Gamma_1,\\ \frac{\partial}{\partial n}\Delta u + (1-\mu)B_2 u - \mu_1 u - \beta u^3 = dj(u_t), \quad &\text{on } \Gamma_1,\\ u = \frac{\partial}{\partial n} u = 0, \quad &\text{on } \Gamma_0. \end{aligned} \tag{4.31}$$

The boundary operators B_1 and B_2 are given [15] by:

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$

$$B_2 u = \frac{\partial}{\partial \tau}\left[\left(\nu_1^2 - \nu_2^2\right) u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx})\right],$$

where $\nu = (\nu_1, \nu_2)$ is the outer normal to Γ, $\tau = (-\nu_2, \nu_1)$ is the unit tangent vector along $\partial\Omega$. The operator B in Eq. (4.29) is assumed linear and bounded on $L_2(\Omega)$. The nonlinear function $j \in C^1(\mathbf{R})$ is assumed monotone increasing. The parameters μ_1 and β are nonnegative, the constant $0 < \mu < 1$ has a meaning of the Poisson modulus, and the damping parameter d is positive.

Equations (4.29) and (4.30) of von Karman are well known in nonlinear elasticity and constitute a basic model describing nonlinear oscillations of a plate accounting for large displacements (see, e.g., References 15 and 18, where in this last reference damping via free boundary conditions is considered).

In this subsection we assume that

- The monotone function $j \in C^1(\mathbf{R})$ is assumed to satisfy $j(0) = 0$ along with the following bounds: there exist positive constants m, M such that

  $$m \le j'(s) \le M(1 + |s|^{p-1}) \ \text{ for } \ s \in \mathbf{R}, \ 1 \le p < \infty.$$

  If $p \ge 3$, then the following coercivity condition holds

  $$j(s)s \ge m_1|s|^{(p-1)r} - C, \quad |s| \ge 1, \tag{4.32}$$

  for some constants $m_1 > 0$, $r > 1$ and $C \ge 0$.

- We assume that Γ is star shaped, that is, there exists $x_0 \in \mathbf{R}^2$ such that

  $$(x - x_0)\nu \le 0 \ \text{ on } \Gamma_0 \qquad \text{and} \qquad (x - x_0)\nu \ge 0 \ \text{ on } \Gamma_1.$$

- The operator $B \in \mathcal{L}[L_2(\Omega)]$ is nonnegative and injective.

- $p \in L_2(\Omega)$ and $F_0 \in H^2_{\Gamma_1}(\Omega)$ when $\beta > 0$, where

  $$H^2_{\Gamma_1}(\Omega) \equiv \left\{ u \in H^2(\Omega), u = \frac{\partial}{\partial n} u = 0 \ on \ \Gamma_1 \right\}.$$

  If $\beta = 0$ we assume that $F_0 \in H^2_{\Gamma_1}(\Omega) \cap H^1_0(\Omega)$.

The following assertion is proved in Reference 7.

THEOREM 4.10

Under the assumptions listed above, Eqs. (4.29) *and* (4.30) *with boundary conditions Eq.* (4.31), *generates a continuous semiflow $T(t)$ in the space $\mathcal{H} \equiv H^2_{\Gamma_0}(\Omega) \times L_2(\Omega)$, where $H^2_{\Gamma_0}(\Omega)$ denotes space of $H^2(\Omega)$ functions subject to clamped boundary conditions on Γ_0. This semiflow $T(t)$ possesses a global, compact attractor $\mathcal{A}$ provided the damping parameter d is sufficiently large. The fractal dimension of the attractor $\mathcal{A}$ is finite.*

In this theorem the proof of finite dimensionality of the attractor $\mathcal{A}$ relies on the following "stabilizability" estimate proved in Reference 7

$$E(z(t)) \leq C_1 e^{-\omega t} E(z(0)) + C_2 \sup_{0 \leq \tau \leq t} \left\{ \|z(\tau)\|^2_{H^{2-\epsilon}(\Omega)} + \|z_t(\tau)\|^2_{H^{-\epsilon}(\Omega)} \right\},$$

where $z(t) \equiv u(t) - w(t)$, $[u(t), u'(t)]$ and $[w(t), w'(t)]$ are two trajectories of the problem of Eqs. (4.29) to (4.31), which belong to the attractor $\mathcal{A}$,

$$E(z(t)) = \int_\Omega [|z_t|^2 + a(z, z)] dx dy$$

and

$$\tilde{a}(u, v) \equiv u_{xx} v_{xx} + u_{yy} v_{yy} + \mu(u_{xx} v_{yy} + u_{yy} v_{xx}) + 2(1 - \mu) u_{xy} v_{xy}$$

is the usual second-order differential bilinear form associated with the free boundary conditions. The stabilizability estimate referred to above is, again, a special case of stabilizability estimate in Eq. (4.3), where $p(s) = ms$ and t_0 needs to be taken sufficiently large.

In this case we apply Theorem 4.5 with $g(s) = (I + p)^{-1}(s)$ and the phase space $X = H^2_{\Gamma_0}(\Omega) \times L_2(\Omega) \times W_2(0, T)$, where

$$W_2(0, T) = \left\{ z(t) : \int_0^T \left(\|z\|^2_{H^2(\Omega)} + \|z_t\|^2_{L_2(\Omega)} + \|z_{tt}\|^2_{H^{-2}(\Omega)} \right) dt < \infty \right\}$$

with

$$A := \{ U \equiv [u(0); u_t(0); u(t), t \in (0, T)] \; : \; [u(0); u_t(0)] \in \mathcal{A} \},$$

where $u(t)$ is the solution to the initial problem with initial data $[u(0); u_t(0)]$ and with the operator $V \; : \; A \mapsto X$ defined by the formula

$$V \; : \; [u(0); u_t(0); u(t)] \mapsto [u(T); u_t(T); u(T + t)].$$

We refer to Reference 7 for the details.

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Chapter 5

Extension of the Uniform Cusp Property in Shape Optimization

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Abstract The object of this paper is to first extend the definition of the *uniform cusp property* introduced in Reference 1 to a larger class of dominating *cusp functions* continuous only at the origin along with the $W^{1,p}$-compactness theorem for the family of all subsets of a bounded holdall verifying that property. The local C^0-graphs of sets with a compact boundary verifying a segment property are further characterized, and such sets are shown to satisfy a uniform cusp property for a dominating non-negative cusp function that is continuous only at the origin. Those characterizations are used in the last section to present a new sufficient condition for the compactness of the family of subsets of a bounded holdall, which are locally C^0-epigraphs and whose local C^0-graphs are dominated by a single cusp function. Finally, a streamlined version of the sufficient condition of Reference 3 is also given as a special case of this condition.

5.1 Introduction

A large class of domains Ω in $\mathbf{R}^N$ can be characterized by a local geometric property, the *segment property*, which is equivalent to the property that Ω be locally a C^0-epigraph. This property is

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sufficient to get the density of the function space $C^k(\overline{\Omega})$ in the Sobolev space $W^{m,p}(\Omega)$ for any $1 \leq m \leq k$, and it plays a key role in establishing the continuity of the solution of the Neumann problem for the Laplace operator with respect to the underlying domain Ω. In the same spirit it is well known that the property in which a set be locally a Lipschitzian epigraph is equivalent to the so-called *uniform cone property*. This property plays a key role in extension theorems and in establishing the continuity of the solution of the Dirichlet problem for the Laplace operator with respect to the underlying domain Ω. In addition, the family of such subsets of a fixed bounded open holdall D forms a compact family in all the metric topologies introduced in Reference 1 via distance and characteristic functions. Unfortunately, this compactness is not true for the weaker *uniform segment property*.

The object of this paper is to first extend the definition of the *uniform cusp property* introduced in Reference 1 (cf. Section 11 of this chapter) to a larger class of dominating nonnegative *cusp functions* continuous only at the origin along with the $W^{1,p}$-compactness theorem for the family of subsets of a bounded holdall verifying that property. In Section 5.3.3 the local C^0-graphs of sets with a compact boundary verifying a segment property are further characterized, and such sets are shown to satisfy a uniform cusp property for a dominating nonnegative cusp function that is continuous only at the origin. Those characterizations are exploited in the last section to present a new compactness theorem for the family of subsets of a bounded holdall, which are locally C^0-epigraphs and whose local C^0-graphs are dominated by a single cusp function. A streamlined version of the sufficient condition of Reference 3 is also given as a special case of that theorem.

NOTATION 5.1
Given an integer $N \geq 1$, m_N and H_{N-1} will denote the N-dimensional Lebesgue and $(N-1)$-dimensional Hausdorff measures. The inner product and the norm in $\mathbf{R}^{\mathrm{N}}$ will be written $x \cdot y$ and $|x|$. The complement $\{x \in \mathbf{R}^{\mathrm{N}} : x \notin \Omega\}$ *and the boundary $\overline{\Omega} \cap \overline{\complement\Omega}$ of a subset Ω of $\mathbf{R}^{\mathrm{N}}$ will be respectively denoted by $\complement\Omega$ or $\mathbf{R}^{\mathrm{N}} \setminus \Omega$ and by $\partial\Omega$ or Γ. The* distance function *$d_A(x)$ from a point x to a subset $A \neq \varnothing$ of $\mathbf{R}^{\mathrm{N}}$ is defined as* $\inf\{|y - x| : y \in A\}$.

5.2 Preliminaries

5.2.1 Topologies on Families of Sets

We first introduce some notation and recall a few results on metric topologies defined on spaces of equivalence classes of sets constructed from the characteristic function or the distance and oriented distance functions to a set.

Given $\Omega \subset \mathbf{R}^{\mathrm{N}}$, $\Gamma \neq \varnothing$, the *oriented distance function* is defined as

$$b_\Omega(x) \overset{\text{def}}{=} d_\Omega(x) - d_{\complement\Omega}(x). \tag{5.1}$$

It is Lipschitz continuous of constant 1, and ∇b_Ω exists and $|\nabla b_\Omega| \leq 1$ almost everywhere in $\mathbf{R}^{\mathrm{N}}$. Thus $b_\Omega \in W^{1,p}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$ for all p, $1 \leq p \leq \infty$. Recall that $b^+_\Omega = d_\Omega$, $b^-_\Omega = d_{\complement\Omega}$, and $|b_\Omega| = d_\Gamma$, and that $\chi_{\text{int}\,\Omega} = |\nabla d_{\complement\Omega}|$, $\chi_{\text{int}\,\complement\Omega} = |\nabla d_\Omega|$, and $\chi_\Gamma = 1 - |\nabla d_\Gamma|$ a.e. in $\mathbf{R}^{\mathrm{N}}$, where χ_A denotes the characteristic function of a subset A of $\mathbf{R}^{\mathrm{N}}$. Given a nonempty subset D of $\mathbf{R}^{\mathrm{N}}$, the family $C_b(D) = \{b_\Omega : \Omega \subset \overline{D} \text{ and } \Gamma \neq \varnothing\}$ is closed in $W^{1,p}(D)$. The following theorem is central. It shows that convergence and compactness in the metric on $C_b(D)$ associated with $W^{1,p}(D)$ will imply the same properties in the other topologies introduced in Reference 1 (Chapter 5, Theorem 5.1).

THEOREM 5.1
Let D be a bounded open subset of $\mathbf{R}^N$. The map

$$b_\Omega \mapsto (b_\Omega^+, b_\Omega^-, |b_\Omega|) = (d_\Omega, d_{\complement\Omega}, d_{\partial\Omega}) \, : \, C_b(D) \subset W^{1,p}(D) \to W^{1,p}(D)^3 \tag{5.2}$$

is continuous and for all p, $1 \le p < \infty$, the following maps are also continuous

$$b_\Omega \mapsto (\chi_{\partial\Omega}, \chi_{\operatorname{int}\Omega}, \chi_{\operatorname{int}\complement\Omega}) \colon W^{1,p}(D) \to L^p(D)^3. \tag{5.3}$$

5.2.2 Segment Properties

An *open segment* between two points x and y of $\mathbf{R}^N$ will be denoted

$$(x, y) \stackrel{\text{def}}{=} \{x + t(y - x) \, : \, \forall t, \; 0 < t < 1\}.$$

DEFINITION 5.1 *Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \varnothing$.*

1. Ω is said to satisfy the *segment property* if

$$\forall x \in \partial\Omega, \;\; \exists r > 0, \;\; \exists \lambda > 0, \;\; \exists d \in \mathbf{R}^N, \;\; |d| = 1$$

   such that

$$\forall y \in B(x, r) \cap \overline{\Omega}, \quad (y, y + \lambda d) \subset \operatorname{int}\Omega.$$

2. Ω is said to satisfy the *uniform segment property* if

$$\exists r > 0, \, \exists \lambda > 0 \;\text{ such that } \forall x \in \partial\Omega, \, \exists d \in \mathbf{R}^N, \; |d| = 1$$

   such that

$$\forall y \in B(x, r) \cap \overline{\Omega}, \quad (y, y + \lambda d) \subset \operatorname{int}\Omega.$$

THEOREM 5.2 (Reference 1, Chapter 2, Theorem 7.2)
If $\partial\Omega$ is compact, then the segment property and the uniform segment property of Definition 5.1 coincide.

5.3 Extension of the Uniform Cusp Property

5.3.1 Extended Definitions

The *uniform cusp property* introduced in Reference 1 (Chapter 5, Section 11) was specified by a continuous function $h : [0, \rho[\to \mathbf{R}$ such that

$$h(0) = 0, \quad h(\rho) = \lambda, \quad \forall \theta, \quad 0 < \theta < \rho, \quad 0 < h(\theta) < \lambda. \tag{5.4}$$

Recall that with h of the form $h(\theta) = \lambda \, (\theta/\rho)^\alpha$, $0 < \alpha \le 1$, we recover the *uniform cusp property* for $0 < \alpha < 1$ and the *uniform cone property* for $\alpha = 1$, $\rho = \lambda \, \tan\omega$ and $h(\theta) = \theta / \tan\omega$, which corresponds to an open cone in 0 of aperture ω, height λ, and axis $e_N = (0, \ldots, 0, 1)$. Letting $H = \{e_N\}^\perp$, it corresponds to the following open region in $\mathbf{R}^N$

$$\mathcal{O} = \left\{\zeta' + \zeta_N e_N \,:\, \zeta' \in H,\ |\zeta'| < \rho \quad \text{and} \quad h(|\zeta'|) < \zeta_N < \lambda\right\}, \tag{5.5}$$

which contains the segment $(0, \lambda e_N)$ but not 0.

This suggests to extend the family of functions h to the larger space

$$\mathcal{H} \overset{\text{def}}{=} \{h : [0, \infty[\to \mathbf{R} \,:\, h(0) = 0 \text{ and } h \text{ is continuous in } 0\} \tag{5.6}$$

by associating with $h \in \mathcal{H}$, $\rho > 0$, and $\lambda > 0$ the axi-symmetrical region

$$C(\lambda, h, \rho) \overset{\text{def}}{=} \left\{\zeta' + \zeta_N e_N \,:\, \zeta' \in H,\ |\zeta'| < \rho,\ \limsup h(|\zeta'|) < \zeta_N < \lambda\right\} \tag{5.7}$$

around the axis e_N in $\mathbf{R}^N$ as can be seen from the next lemma.

LEMMA 5.1

Given $\rho > 0$, $\lambda > 0$, and $h \in \mathcal{H}$, the region $C(\lambda, h, \rho)$ contains the segment $(0, \lambda e_N)$, does not contain 0, and is open.

PROOF By definition

$$(0, \lambda e_N) \subset \left\{\zeta_N e_N : 0 = \lim_{\xi' \to 0} h(|\xi'|) < \zeta_N < \lambda\right\} \subset C(\lambda, h, \rho),$$

because $h(0) = 0$ and h is continuous in 0. Also $0 \notin C(\lambda, h, \rho)$ because it would yield the contradiction $0 = \lim_{\xi' \to 0} h(|\xi'|) < \zeta_N = 0$. To show that $C(\lambda, h, \rho)$ is open, we fix a point $\zeta = \zeta' + \zeta_N e_N \in C(\lambda, h, \rho)$ and construct a neighborhood of ζ contained in $C(\lambda, h, \rho)$. Let

$$\varepsilon = \zeta_N - \bar{l}_{\zeta'} > 0, \quad \bar{l}_{\zeta'} \overset{\text{def}}{=} \limsup_{\xi' \to \zeta'} h(|\xi'|).$$

By definition of the limsup there exists $\rho > 0$ such that

$$\sup_{\substack{\xi' \in B(\zeta', \rho) \\ \xi' \neq \zeta'}} h(|\xi'|) < \bar{l}_{\zeta'} + \varepsilon/2.$$

For all $\xi' \neq \zeta'$ in $B(\zeta', \rho)$ and $\rho_{\xi'} \overset{\text{def}}{=} \min\{|\xi' - \zeta'|, \rho - |\xi' - \zeta'|\} > 0$

$$B(\xi', \rho_{\xi'}) \subset B(\zeta', \rho) \Rightarrow \bar{l}_{\xi'} < \bar{l}_{\zeta'} + \varepsilon/2, \quad \bar{l}_{\xi'} \overset{\text{def}}{=} \limsup_{\eta' \to \xi'} h(|\eta'|).$$

For all $\xi_N \in \mathbf{R}$ such that $|\xi_N - \zeta_N| < \varepsilon/2$, we get

$$\begin{aligned} \bar{l}_{\xi'} - \xi_N &= \bar{l}_{\zeta'} - \zeta_N + \bar{l}_{\xi'} - \bar{l}_{\zeta'} + \zeta_N - \xi_N \\ &= -\varepsilon + \bar{l}_{\xi'} - \bar{l}_{\zeta'} + |\zeta_N - \xi_N| < -\varepsilon + \varepsilon/2 + \varepsilon/2 = 0 \\ &\Rightarrow \xi' + \xi_N e_N \in C(\lambda, h, \rho), \end{aligned}$$

$B(\zeta, \min\{\rho, \varepsilon/2\}) \subset C(\lambda, h, \rho)$ and $C(\lambda, h, \rho)$ is open. ∎

Given a direction $d \in \mathbf{R}^N$, $|d| = 1$, the rotated region is defined as

$$C(\lambda, h, \rho, d) \stackrel{\text{def}}{=} \left\{ y \in \mathbf{R}^N : \begin{array}{l} |P_{H_d}(y)| < \rho \text{ and} \\ \limsup h(|P_{H_d}(y)|) < y \cdot d < \lambda \end{array} \right\}, \tag{5.8}$$

where $H_d = \{d\}^\perp$ is the hyperplane through 0 orthogonal to the direction d.

Recall the definition of the *orthogonal subgroup* of $N \times N$ matrices

$$\mathrm{O(N)} \stackrel{\text{def}}{=} \{A : {}^*\!A\, A = A\, {}^*\!A = I\}, \tag{5.9}$$

where ${}^*\!A$ is the transposed matrix of A. If $e_N = (0, \ldots, 0, 1)$ is given in $\mathbf{R}^N$, a direction can be specified by either a matrix (of rotation) $A \in \mathrm{O(N)}$ or the corresponding unit vector $d = Ae_N \in \mathbf{R}^N$. In view of this, there exists a matrix $A \in \mathrm{O(N)}$ such that $d = Ae_N$ and hence $C(\lambda, h, \rho, d) = AC(\lambda, h, \rho)$. Finally, the translation of $C(\lambda, h, \rho, d)$ to the point x will be denoted

$$C_x(\lambda, h, \rho, d) \stackrel{\text{def}}{=} x + C(\lambda, h, \rho, d).$$

LEMMA 5.2
For all $\lambda > 0$, $\rho > 0$, $h \in \mathcal{H}$, and $x \in \mathbf{R}^N$, the regions $C(\lambda, h, \rho)$ and $C_x(\lambda, h, \rho, d)$ are nonempty and open. Moreover the segment $(x, x + \lambda d)$ is contained in $C_x(\lambda, h, \rho, d)$.

The function h will be referred to as a *cusp function* and the space $\mathcal{H}$ as the *space of cusp functions*. The definition of the uniform cusp property in Reference 1 (Chapter 5, Section 11) can now be extended to the larger family $\mathcal{H}$.

DEFINITION 5.2 *Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \varnothing$.*

1. Ω is said to satisfy the *local uniform cusp property* if

$$\forall x \in \partial\Omega, \quad \exists h \in \mathcal{H},\ \exists \lambda > 0,\ \exists \rho > 0,\ \exists r > 0,\ \exists d \in \mathbf{R}^N,\ |d| = 1,$$

such that

$$\forall y \in B(x, r) \cap \overline{\Omega}, \quad C_y(\lambda, h, \rho, d) \subset \operatorname{int} \Omega.$$

2. Given $h \in \mathcal{H}$, Ω is said to satisfy the *h-local uniform cusp property* if

$$\forall x \in \partial\Omega, \quad \exists \lambda > 0,\ \exists \rho > 0,\ \exists r > 0,\ \exists d \in \mathbf{R}^N,\ |d| = 1,$$

such that

$$\forall y \in B(x, r) \cap \overline{\Omega}, \quad C_y(\lambda, h, \rho, d) \subset \operatorname{int} \Omega.$$

3. Ω is said to satisfy the *uniform cusp property* if

$$\exists h \in \mathcal{H},\ \exists \lambda > 0,\ \exists \rho > 0,\ \exists r > 0, \quad \forall x \in \partial\Omega,\ \exists d \in \mathbf{R}^N,\ |d| = 1,$$

such that

$$\forall y \in B(x, r) \cap \overline{\Omega}, \quad C_y(\lambda, h, \rho, d) \subset \operatorname{int} \Omega.$$

The three cases of Definition 5.2 differ only when $\partial\Omega$ is not compact.

THEOREM 5.3

If $\partial\Omega$ is compact, then the three uniform cusp properties of Definition 5.2 coincide.

PROOF It is sufficient to show that the local uniform cusp property implies the uniform cusp property. Because $\partial\Omega$ is compact, there exists a finite open subcover $\{B_i\}_{i=1}^m$, $B_i = B(x_i, r_{x_i})$, of $\partial\Omega$ for some finite sequence $\{x_i\}_{i=1}^m$ of points of $\partial\Omega$. We now claim that

$$\exists r > 0, \forall x \in \partial\Omega, \quad \exists i, 1 \le i \le m, \quad \text{such that } B(x,r) \subset B_i.$$

Define

$$\lambda \overset{\text{def}}{=} \min_{1\le i\le m} \lambda_i \quad \rho \overset{\text{def}}{=} \min_{1\le i\le m} \rho_i \quad \text{and} \quad h(\theta) \overset{\text{def}}{=} \max_{1\le i\le m} h_i(\theta).$$

Therefore, for each $x \in \partial\Omega$, there exists i, $1 \le i \le m$, such that $B(x,r) \subset B_i$ and hence there exists $d_i = d_{x_i}$, $|d_i| = 1$, such that

$$\forall y \in \overline{\Omega} \cap B_i, \quad C_y(\lambda_i, h_i, \rho_i, d_i) \subset \operatorname{int}\Omega.$$

Because $B(x,r) \subset B_i$ and $C(\lambda, h, \rho, d_i) \subset C(\lambda_i, h_i, \rho_i, d_i)$, for all $z \in C(\lambda, h, \rho, d_i)$,

$$\limsup_{z'\to z} h_i(|P_H(z')|) \le \limsup_{z'\to z} h(|P_H(z')|) < z \cdot d_i < \lambda \le \lambda_i,$$

we finally get

$$\forall y \in \overline{\Omega} \cap B(x,r), \quad C_y(\lambda, h, \rho, d_i) \subset C_y(\lambda_i, h_i, \rho_i, d_i) \subset \operatorname{int}\Omega.$$

So we pick $d_x = d_i$, and we conclude that Ω has the uniform cusp property. ∎

5.3.2 Compactness as a Special Case of the Uniform Fat Segment Property

It turns out that the uniform segment property is too *weak* or too *meager* to yield the compactness of the corresponding family of subsets of a bounded holdall D of $\mathbf{R}^N$ (cf. Reference 1, Chapter 5, Example 11.1, p. 256). This suggests *fattening* the open segment $(0, \lambda e_N)$ by replacing it with the uniform cusp property.

Given a bounded open subset D of $\mathbf{R}^N$, $\rho > 0$, $\lambda > 0$, $r > 0$, and $h \in \mathcal{H}$, consider the family

$$L(D,\lambda,h,\rho,r) \overset{\text{def}}{=} \left\{ \Omega \subset \overline{D} \; : \; \begin{array}{c} \Omega \text{ satisfies the uniform cusp} \\ \text{property for } (\lambda,h,\rho,r) \end{array} \right\} \tag{5.10}$$

in the sense of Definition 5.2 (3). The compactness Theorem 11.1 (Reference 1, Chapter 5) readily extends to $\mathcal{H}$.

THEOREM 5.4

Let D be a nonempty bounded open subset of $\mathbf{R}^N$ and $1 \le p < \infty$. Given $\rho > 0$, $\lambda > 0$, and $h \in \mathcal{H}$, the family

$$B(D,\lambda,h,\rho,r) \overset{\text{def}}{=} \{b_\Omega \; : \; \forall\Omega \in L(D,\lambda,h,\rho,r)\}$$

is either empty or compact in $C(\overline{D})$ and $W^{1,p}(D)$. As a consequence the families

$$\begin{aligned} B_d(D,\lambda,h,\rho,r) &\overset{\text{def}}{=} \{d_\Omega \; : \; \forall\Omega \in L(D,\lambda,h,\rho,r)\}, \\ B_d^c(D,\lambda,h,\rho,r) &\overset{\text{def}}{=} \{d_{\complement\Omega} \; : \; \forall\Omega \in L(D,\lambda,h,\rho,r)\}, \\ B_d^\partial(D,\lambda,h,\rho,r) &\overset{\text{def}}{=} \{d_{\partial\Omega} \; : \; \forall\Omega \in L(D,\lambda,h,\rho,r)\} \end{aligned}$$

are compact in $C(\overline{D})$ *and* $W^{1,p}(D)$*, and the following families are compact in* $L^p(D)$

$$X(D,\lambda,h,\rho,r) \stackrel{\text{def}}{=} \{\chi_\Omega : \forall\Omega \in L(D,\lambda,h,\rho,r)\},$$
$$X^c(D,\lambda,h,\rho,r) \stackrel{\text{def}}{=} \{\chi_{\complement\Omega} : \forall\Omega \in L(D,\lambda,h,\rho,r)\}.$$

The proof of the theorem will require the following two lemmas.

LEMMA 5.3
Given a bounded open subset D of $\mathbf{R}^{\mathrm{N}}$*, let* $\{\Omega_n\}$ *be a sequence of subsets of* $\overline{D}$ *such that* $\partial\Omega_n \neq \emptyset$ *and* $\mathrm{m}(\partial\Omega_n) = 0$*. Further assume that there exists* $\Omega \subset \bar{D}$ *such that* $\partial\Omega \neq \emptyset$ *and* $\mathrm{m}(\partial\Omega) = 0$*. Then*

$$b_{\Omega_n} \rightharpoonup b_\Omega \text{ in } W^{1,2}(D)\text{-weak} \Rightarrow b_{\Omega_n} \to b_\Omega \text{ in } W^{1,2}(D)\text{-strong}.$$

PROOF Because, for all $n \geq 1$, $\mathrm{m}(\partial\Omega_n) = 0 = \mathrm{m}(\partial\Omega)$, $|\nabla b_\Omega| = 1 = |\nabla b_{\Omega_n}|$ almost everywhere in D (cf. Reference 1, Theorem 3.2, p. 215). As a result

$$\int_D |\nabla b_{\Omega_n} - \nabla b_\Omega|^2\,dx = \int_D |\nabla b_{\Omega_n}|^2 + |\nabla b_\Omega|^2 - 2\nabla b_{\Omega_n}\cdot\nabla b_\Omega\,dx$$
$$= 2\int_D (1 - \nabla b_{\Omega_n}\cdot\nabla b_\Omega)\,dx \to 2\int_D \left(1 - |\nabla b_\Omega|^2\right)dx = 0.$$

Therefore $\nabla b_{\Omega_n} \to \nabla b_\Omega$ in $L^2(D)^N$-strong and $b_{\Omega_n} \to b_\Omega$ in $W^{1,2}(D)$-strong, because the convergence $b_{\Omega_n} \to b_\Omega$ in $L^2(D)$-strong follows from the weak convergence in $W^{1,2}(D)$. ∎

LEMMA 5.4 (Reference 1, Chapter 5, Section 10, Lemma 10.1)
Given a sequence $\{b_{\Omega_n}\} \subset C_b(D)$ *such that* $b_{\Omega_n} \to b_\Omega$ *in* $C(\overline{D})$ *for some* $b_\Omega \in C_b(D)$*, we have the following properties:*

$$\forall x \in \overline{\Omega},\quad \forall R > 0,\quad \exists N(x,R) > 0,\quad \forall n \geq N(x,R),\quad B(x,R)\cap\Omega_n \neq \emptyset,$$

and for all $x \in \overline{\complement\Omega}$*,*

$$\forall R > 0,\quad \exists N(x,R) > 0,\quad \forall n \geq N(x,R),\quad B(x,R)\cap\complement\Omega_n \neq \emptyset. \tag{5.11}$$

Moreover,

$$\forall x \in \partial\Omega,\quad \forall R > 0,\quad \exists N(x,R) > 0,\quad \forall n \geq N(x,R),$$
$$B(x,R)\cap\Omega_n \neq \emptyset \quad \text{and} \quad B(x,R)\cap\complement\Omega_n) \neq \emptyset,$$

and $B(x,R)\cap\partial\Omega_n \neq \emptyset$*.*

PROOF OF THEOREM 5.4 If the family $L(D,\lambda,h,\rho,r)$ is empty, there is nothing to prove. Assume that it is not empty.

1. *Compactness in* $C(\overline{D})$. Because for all Ω in $L(D,\lambda,h,\rho,r)$, Ω is locally a C^0-epigraph, $\partial\Omega \neq \emptyset$, $b_\Omega \in C_b(D)$, and $\mathrm{m}(\partial\Omega) = 0$. Consider an arbitrary sequence $\{\Omega_n\}$ in $L(D,\lambda,h,\rho,r)$. For $\overline{D}$ compact $C_b(D)$ is compact in $C(\overline{D})$ and there exists $\Omega \subset \overline{D}$ and a subsequence $\{\Omega_{n_k}\}$ such that $b_{\Omega_{n_k}} \to b_\Omega$ in $C(\overline{D})$. It remains to prove that $\Omega \in L(D,\lambda,h,\rho,r)$:

$$\forall x \in \partial\Omega,\ \exists A \in \mathrm{O(N)}\ \text{such that}\ \forall y \in \overline{\Omega}\cap B(x,r),\quad y + AC(\lambda,h,\rho) \subset \operatorname{int}\Omega.$$

From Lemma 5.4, for each $x \in \partial\Omega$, $\forall k \geq 1$, $\exists n_k \geq k$ such that

$$B\left(x, \frac{r}{2^k}\right) \cap \partial\Omega_{n_k} \neq \varnothing.$$

Denote by x_k an element of that intersection:

$$\forall k \geq 1, \quad x_k \in B\left(x, \frac{r}{2^k}\right) \cap \partial\Omega_{n_k}.$$

By construction $x_k \to x$. Next consider $y \in B(x,r) \cap \overline{\Omega}$. From the first part of the lemma, there exists a subsequence of $\{\Omega_{n_k}\}$, still denoted $\{\Omega_{n_k}\}$, such that

$$\forall k \geq 1, \quad B\left(y, \frac{r}{2^k}\right) \cap \Omega_{n_k} \neq \varnothing.$$

For each $k \geq 1$ denote by y_k a point of that intersection. By construction

$$y_k \in \overline{\Omega}_{n_k} \to y \in \overline{\Omega} \cap B(x,r).$$

There exists $K > 0$ large enough such that for all $k \geq K$, $y_k \in B(x_k, r)$. To see this, note that $y \in B(x,r)$ and that

$$\exists \rho > 0, \quad B(y,\rho) \subset B(x,r) \quad \text{and} \quad |y - x| + \frac{\rho}{2} < r.$$

Now

$$\begin{aligned} |y_k - x_k| &\leq |y_k - y| + |y - x| + |x - x_k| \\ &\leq \frac{r}{2^k} + r - \frac{\rho}{2} + \frac{r}{2^k} \leq r + \left[\frac{r}{2^{k-1}} - \frac{\rho}{2}\right] < r. \end{aligned}$$

Because $r/\rho > 1$ the result is true for

$$\frac{r}{2^{k-1}} - \frac{\rho}{2} < 0 \quad \Rightarrow k > 2 + \log(r/\rho).$$

So we have constructed a subsequence $\{\Omega_{n_k}\}$ such that for $k \geq K$

$$x_k \in \partial\Omega_{n_k} \to x \in \partial\Omega \quad \text{and} \quad y_k \in \overline{\Omega}_{n_k} \cap B(x_k, r) \to y \in \overline{\Omega} \cap B(x,r).$$

For each k, $\exists A_k \in \mathrm{O(N)}$, $A_k\,{}^*A_k = {}^*A_k\,A_k = I$, such that $y_k + A_k C(\lambda, h, \rho) \subset \operatorname{int}\Omega_{n_k}$. Pick another subsequence of $\{\Omega_{n_k}\}$, still denoted $\{\Omega_{n_k}\}$, such that

$$\exists A \in \mathrm{O(N)},\ A\,{}^*A = {}^*A\,A = I, \quad A_k \to A.$$

Now consider $z \in y + AC(\lambda, h, \rho)$. Because $y + AC(\lambda, h, \rho)$ is open

$$\exists \rho > 0, \quad B(z,\rho) \subset y + AC(\lambda, h, \rho),$$

and there exists $K' \geq K$ such that

$$\begin{gathered} \forall k \geq K', \quad B(z, \rho/2) \subset y_k + A_k C(\lambda, h, \rho) \subset \operatorname{int}\Omega_{n_k} = \complement\overline{\complement\Omega_{n_k}}, \\ \Rightarrow \complement B(z,\rho/2) \supset \overline{\complement\Omega_{n_k}} \quad \Rightarrow 0 < \rho/2 = d_{\complement B(z,\rho/2)}(z) \leq d_{\complement\Omega_{n_k}}(z) \to d_{\complement\Omega}(z) \\ \Rightarrow 0 < \rho/2 \leq d_{\complement\Omega}(z) \quad \Rightarrow z \in \complement\overline{\complement\Omega} = \operatorname{int}\Omega \quad \Rightarrow y + AC(\lambda, h, \rho) \subset \operatorname{int}\Omega. \end{gathered}$$

This proves that $\Omega \subset \overline{D}$ satisfies the uniform cusp property $[\Omega \in L(D, \lambda, h, \rho, r)]$.

2. *Compactness in* $W^{1,p}(D)$. From the discussion prior to Theorem 5.1 it is sufficient to prove the result for $p = 2$. Consider the subsequence $\{\Omega_{n_k}\} \subset L(D, \lambda, h, \rho, r)$ and let $\Omega \in L(D, \lambda, h, \rho, r)$ be the set previously constructed such as $b_{\Omega_{n_k}} \to b_\Omega$ in $C(\overline{D})$. Hence $b_{\Omega_{n_k}} \to b_\Omega$ in $L^2(D)$. Because $\overline{D}$ is compact for all $\Omega \subset \overline{D}$

$$\int_D |b_{\Omega_{n_k}}|^2\, dx \le \int_D \operatorname{diam}(D)^2\, dx \le \operatorname{diam}(D)^2\, \mathrm{m}(D),$$
$$\int_D |\nabla b_{\Omega_{n_k}}|^2\, dx \le \int_D dx = \mathrm{m}(D),$$

and there exists a subsequence, still denoted $\{b_{\Omega_{n_k}}\}$, which converges weakly to b_Ω. Because all the sets are locally C^0-epigraphs, $\mathrm{m}(\partial\Omega_{n_k}) = 0 = \mathrm{m}(\partial\Omega)$ and, by Lemma 5.3, $b_{\Omega_{n_k}} \to b_\Omega$ in $W^{1,2}(D)$-strong.

3. The compactness of the other families follows from the continuity of the maps

$$b_\Omega \mapsto (b_\Omega^+, b_\Omega^-, |b_\Omega|) = (d_\Omega, d_{\complement\Omega}, d_{\partial\Omega}) \,:\, C_b(D) \subset C(\overline{D}) \to C(\overline{D})^3$$

in Reference 1, Chapter 5, Theorem 2.1 (iii), p. 207 and

$$b_\Omega \mapsto (b_\Omega^+, b_\Omega^-, |b_\Omega|) = (d_\Omega, d_{\complement\Omega}, d_{\partial\Omega})\colon C_b(D) \subset W^{1,p}(D) \to W^{1,p}(D)^3,$$
$$b_\Omega \mapsto (\chi_{\partial\Omega}, \chi_{\operatorname{int}\Omega}, \chi_{\operatorname{int}\complement\Omega})\colon W^{1,p}(D) \to L^p(D)^3$$

in Reference 1, Chapter 5, Theorem 2.2 (iv) and (v), p. 226, and the fact that $\mathrm{m}(\partial\Omega) = 0$ implies $\chi_{\operatorname{int}\Omega} = \chi_\Omega$ and $\chi_{\operatorname{int}\complement\Omega} = \chi_{\complement\Omega}$ almost everywhere for $\Omega \in L(D, \lambda, h, \rho, r)$. □

5.3.3 Uniform Cusp Property and Local C^0-epigraphs

A set having the segment property has an $(N-1)$-dimensional boundary and is locally a C^0-epigraph in the following sense.

DEFINITION 5.3 *Let Ω be a subset of $\mathbf{R}^N$ such that $\partial\Omega \neq \emptyset$. The set Ω is said to be* locally a C^0-epigraph, *if for each $x \in \partial\Omega$ there exist*

1. *An open neighborhood $\mathcal{U}(x)$ of x*

2. *A direction $e_N(x) \in \mathbf{R}^N$, $|e_N(x)| = 1$, and the hyperplane $H(x) = \{e_N(x)\}^\perp$*

3. *A bounded open neighborhood $V_{H(x)}$ of 0 in $H(x)$ such that*

$$\mathcal{U}(x) \subset \{y \in \mathbf{R}^N \,:\, P_{H(x)}(y - x) \in V_{H(x)}\}, \tag{5.12}$$

where $P_{H(x)}$ is the orthogonal projection onto $H(x)$ and

4. *a C^0-mapping $a_x\colon V_{H(x)} \to \mathbf{R}$ such that*

$$\mathcal{U}(x) \cap \partial\Omega = \left\{ x + \zeta' + \zeta_N e_N(x) : \begin{array}{c} \zeta' \in V_{H(x)} \\ \zeta_N = a_x(\zeta') \end{array} \right\} \tag{5.13}$$

$$\mathcal{U}(x) \cap \operatorname{int}\Omega = \mathcal{U}(x) \cap \left\{ x + \zeta' + \zeta_N e_N(x) : \begin{array}{c} \zeta' \in V_{H(x)} \\ \zeta_N > a_x(\zeta') \end{array} \right\}. \tag{5.14}$$

NOTATION 5.2
Given the unit vector $e_N \in \mathbf{R}^{\mathrm{N}}$ *and the hyperplane* $H = \{e_N\}^{\perp}$, *there exists* $A_x \in \mathrm{O(N)}$ *such that* $e_N(x) = A_x e_N$, $H(x) = A_x H$, *and* $V_{H(x)} = V_{A_x H} = A_x V_x$ *for some* $V_x \subset H$. *It will also be useful to introduce the mappings* $\bar{a}_x = a_x \circ A_x : V_x \to \mathbf{R}$ *defined on* V_x *in the fixed hyperplane* H.

We quote Theorem 7.3 in Chapter 2 of Reference 1.

THEOREM 5.5

1. *If* Ω *satisfies the local uniform cusp property, then* Ω *is locally a* C^0*-epigraph. Let* $h_x \in \mathcal{H}$ *be the function,* $r_x > 0$, $\rho_x > 0$, *and* $\lambda_x > 0$ *the parameters,* d_x *the direction, and* $H(x) = \{d_x\}^{\perp}$ *the hyperplane through* 0 *orthogonal to* d_x *associated with the point* $x \in \partial\Omega$. *Then there exists* $\overline{\rho}_x$,
$$0 < \overline{\rho}_x \leq r_{x\lambda} \stackrel{\text{def}}{=} \min\{r_x, \lambda_x/2\}, \tag{5.15}$$
which is the largest radius such that
$$B_{H(x)}(0, \overline{\rho}_x) \subset \left\{P_{H(x)}(y-x) \,:\, \forall y \in B(x, r_{x\lambda}) \cap \partial\Omega\right\}.$$
The neighborhoods of Definition 5.3 can be chosen as
$$\boxed{\begin{gathered} V_{H(x)} \stackrel{\text{def}}{=} B_{H(x)}(0, \overline{\rho}_x) \quad \textit{and} \\ \mathcal{U}(x) \stackrel{\text{def}}{=} B(x, r_{x\lambda}) \cap \left\{y : P_{H(x)}(y-x) \in V_{H(x)}\right\}, \end{gathered}} \tag{5.16}$$
where $B_{H(x)}(0, \overline{\rho}_x)$ *is the open ball of radius* $\overline{\rho}_x$ *in the hyperplane* $H(x)$. *For each* $\zeta' \in V_{H(x)}$, *there exists a unique* $y_{\zeta'} \in \partial\Omega \cap \mathcal{U}(x)$ *such that* $P_{H(x)}(y_{\zeta'} - x) = \zeta'$ *and the mapping*
$$\zeta' \mapsto a_x(\zeta') \stackrel{\text{def}}{=} (y_{\zeta'} - x) \cdot d_x \,:\, V_{H(x)} \to \mathbf{R}$$
is well defined, bounded,
$$\forall \zeta' \in V_{H(x)}, \quad |a_x(\zeta')| < r_{\lambda x}, \tag{5.17}$$
and uniformly continuous in $V_{H(x)}$. *Moreover,*
$$\mathrm{m}_N[\partial\Omega \cap \mathcal{U}(x)] = 0, \tag{5.18}$$
where m_N *is the* N*-dimensional Lebesgue measure.*
2. *Conversely, if* Ω *is locally a* C^0*-epigraph,* Ω *and* $\complement\Omega$ *satisfy the segment property.*

The next theorem sharpens and completes Theorem 7.4 (i) in Chapter 2 of Reference 1 by making the connection with the uniform cusp property.

THEOREM 5.6
Fix $H = \{e_N\}^{\perp}$. *Let* Ω *be a subset of* $\mathbf{R}^{\mathrm{N}}$ *such that* $\partial\Omega \neq \varnothing$.

1. *If* $\partial\Omega$ *is compact and* Ω *has the segment property,* Ω *is locally a* C^0*-epigraph with the following properties. In each point* $x \in \partial\Omega$, *the associated neighborhoods* $V_{H(x)}$ *of* 0 *in* $H(x)$ *and* $\mathcal{U}(x)$ *of* x *in* $\mathbf{R}^{\mathrm{N}}$ *and the function* a_x *can be chosen with the following properties: there exist bounded open neighborhoods* V *of* 0 *in* H *and* U *of* 0 *in* $\mathbf{R}^{\mathrm{N}}$ *such that for each* $x \in \partial\Omega$ *there exists*

$A_x \in \mathrm{O(N)}$ such that $e_N(x) = A_x e_N$, $H(x) = A_x H$, $V_{H(x)} = A_x V$, and $\mathcal{U}(x) = x + A_x U$. The family of bounded uniformly continuous functions $a_x : V_{H(x)} \to \mathbf{R}$ can be chosen in such a way that the family $\bar{a}_x = a_x \circ A_x : V \to \mathbf{R}$ be (uniformly) equicontinuous with respect to $x \in \partial\Omega$, and there exists a nonnegative function $h \in \mathcal{H}$ such that

$$\forall x \in \partial\Omega,\ \forall \zeta' \in V, \quad |\bar{a}_x(\zeta)| \le h(|\zeta'|) \tag{5.19}$$

$$\forall \zeta_1', \zeta_2' \in V,\ \zeta_1' - \zeta_2' \in V,\ |\bar{a}_x(\zeta_2') - \bar{a}_x(\zeta_1')| \le \limsup h(|\zeta_2' - \zeta_1'|) \tag{5.20}$$

(note that $\limsup h$ *also belongs to $\mathcal{H}$). Moreover Ω satisfies the uniform cusp property of Definition 5.2 (3).*

2. *If Ω is locally a C^0-epigraph and there exists $A_x \in \mathrm{O(N)}$ such that $A_x^{-1}(\mathcal{U}(x)-x)$ is independent of $x \in \partial\Omega$, then there exists r such that, for all $x \in \partial\Omega$, $B(0, 2r) \subset \mathcal{U}(x) - x$ and Ω satisfies the uniform segment property for the pair $(r', \lambda') = (r, r)$.*

PROOF

1. In view of the compactness of $\partial\Omega$, we know from Theorem 5.2 that there exists $\lambda > 0$ and $r > 0$ such that Ω verifies the uniform segment property. So the conclusions of Theorem 5.5 (1) hold with $r_x = r$, $\lambda_x = \lambda$, $r_{x\lambda} = r_\lambda = \min\{r, \lambda/2\}$ for all $x \in \partial\Omega$. To each $x \in \partial\Omega$ is associated a matrix $A_x \in \mathrm{O(N)}$, a radius ρ_x, $0 < \rho_x \le r_\lambda$, and a bounded uniformly continuous function a_x defined on $V_{H(x)} = B_{H(x)}(0, \rho_x)$ such that $\partial\Omega$ is locally the graph of a_x in the neighborhood

$$\mathcal{U}(x) = B(x, r_\lambda) \cap \left\{y : P_{H(x)}(y - x) \in B_{H(x)}(0, \rho_x)\right\}$$

of x. Note that, by construction, $B(x, \rho_x) \subset \mathcal{U}(x)$ and that

$$\mathcal{U}(x) - x = A_x \left\{B(0, r_\lambda) \cap [z : P_H(z) \in B_H(0, \rho_x)]\right\}$$

depends only on r_λ and ρ_x up to a rotation around the origin.

(a) *Construction of the neighborhoods $V'_{H(x)}$ and $\mathcal{U}'(x)$ and the maps a'_x.* Because the mappings $\bar{a}_x : B_H(0, \rho_x) \to \mathbf{R}$ are uniformly bounded by r_λ, continuous in 0, and $\bar{a}_x(0) = 0$, for each α, $0 < \alpha \le r_\lambda/2$, there exists ρ'_x, $0 < \rho'_x \le \rho_x/2$, such that

$$\forall \zeta' \in B_H(0, \rho'_x), \quad |\bar{a}_x(\zeta')| < \alpha/2. \tag{5.21}$$

The family $\{B(x, \rho'_x) : x \in \partial\Omega\}$ is an open cover of $\partial\Omega$. Because $\partial\Omega$ is compact, there exists a finite sequence $\{x_i\}_{i=1}^m$ of points of $\partial\Omega$ such that $\{B_i = B(x_i, \rho'_{x_i}) : 1 \le i \le m\}$ is a finite subcover of $\partial\Omega$. At each point x_i, $\partial\Omega$ is locally the graph of the function $\bar{a}_i = \bar{a}_{x_i} : B_H(0, \rho'_{x_i}) \to \mathbf{R}$ in $\mathcal{U}_i = \mathcal{U}(x_i)$. By the same technique as in the proof of part (i) of Theorem 5.1 in Chapter 2 of Reference 1,

$$\exists \rho > 0, \quad \forall x \in \partial\Omega, \quad \exists i, 1 \le i \le m, \text{ such that } B(x, \rho) \subset B_i.$$

Hence $|x - x_i| < \rho'_{x_i} - \rho$ and $\rho < \rho'_{x_i}$. Define the following neighborhoods of 0 in H and $\mathbf{R}^{\mathrm{N}}$

$$V \overset{\text{def}}{=} B_H(0, \rho) \quad \text{and} \quad U \overset{\text{def}}{=} \left\{\zeta' + \zeta_N e_N : \zeta' \in B_H(0, \rho),\ |\zeta_N| < \alpha\right\}.$$

For each $x \in \partial\Omega$, there exists i, $1 \le i \le m$, such that $B(x, \rho) \subset B_i$. Define

$$\boxed{\begin{gathered} V'_x \overset{\text{def}}{=} V, \quad A_x \overset{\text{def}}{=} A_{x_i}, \quad \mathcal{U}'(x) \overset{\text{def}}{=} x + A_{x_i} U, \quad V'_{H(x)} \overset{\text{def}}{=} A_x V \\ a'_x(\zeta') \overset{\text{def}}{=} a_i(P_{H(x_i)}(x - x_i) + \zeta') - a_i(P_{H(x_i)}(x - x_i)) \end{gathered}}$$

and recall that, by construction of a_i and the fact that $x \in \partial\Omega \cap \mathcal{U}(x_i)$, we have $a_i(P_{H(x_i)}(x - x_i)) = A_{x_i} e_N \cdot (x - x_i)$. It is readily seen that V'_x and $A_x^{-1}(\mathcal{U}'(x) - x)$ are independent of $x \in \partial\Omega$. Moreover

$$\forall \zeta' \in B_H(0, \rho), \quad |a'_x(\zeta')| < \alpha$$

and the new neighborhoods and maps satisfy the properties of Definition 5.3.

(b) *Construction of* $h \in \mathcal{H}$. Define the function $a : V \to \mathbf{R}$ as

$$a(v) \stackrel{\text{def}}{=} \sup_{x \in \partial\Omega} |\bar{a}'_x(v)|.$$

Note that $\bar{a}'_x(0) = 0$ implies $a(0) = 0$. By uniform boundedness, the mapping a is well defined everywhere in V and

$$\forall x \in \partial\Omega, \quad 0 \le |\bar{a}'_x(v)| \le a(v) \le \alpha \le r_\lambda/2.$$

Moreover a is continuous: for any v_1 and v_2

$$\begin{aligned} |\bar{a}'_x(v_2)| &\le |\bar{a}'_x(v_2) - \bar{a}'_x(v_1)| + |\bar{a}'_x(v_1)| \\ \sup_{x \in \partial\Omega} |\bar{a}'_x(v_2)| &\le \sup_{x \in \partial\Omega} |\bar{a}'_x(v_2) - \bar{a}'_x(v_1)| + \sup_{x \in \partial\Omega} |\bar{a}'_x(v_1)| \\ a(v_2) &\le \sup_{x \in \partial\Omega} |\bar{a}'_x(v_2) - \bar{a}'_x(v_1)| + a(v_1). \end{aligned} \tag{5.22}$$

By equicontinuity of $\bar{a}'_x$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &\forall x \in \partial\Omega, \quad \forall v_1, v_2 \in V, \quad |v_2 - v_1| < \delta \quad \Rightarrow |\bar{a}'_x(v_2) - \bar{a}'_x(v_1)| < \varepsilon. \\ &\Rightarrow \forall v_1, v_2 \in V, \quad |v_2 - v_1| < \delta \quad \Rightarrow a(v_2) \le \varepsilon + a(v_1) \end{aligned}$$

and by interchanging the role of v_1 and v_2 in Eq. (5.22)

$$\begin{aligned} &\forall v_1, v_2 \in V, \quad |v_2 - v_1| < \delta \quad \Rightarrow a(v_1) \le \varepsilon + a(v_2) \\ &\Rightarrow \forall v_1, v_2 \in V, \quad |v_2 - v_1| < \delta \quad \Rightarrow |a(v_2) - a(v_1)| \le \varepsilon \end{aligned}$$

and a is uniformly continuous on V. Therefore the function $\zeta \mapsto a(\theta\zeta)$ is continuous on $\{\zeta \in H : |\zeta| = 1\}$ and bounded by $\alpha \le r_\lambda/2$. Hence the function

$$h(\theta) \stackrel{\text{def}}{=} \max_{\zeta \in H,\, |\zeta|=1} a(\theta\zeta)$$

is well defined and non-negative in $[0, \rho[$, and $h(0) = 0$. Moreover, by continuity of a in 0, h is continuous in 0: for all $\varepsilon > 0$, there exists $0 < \delta \le \rho$, such that

$$\begin{aligned} &|v - 0| < \delta \Rightarrow |a(v)| = |a(v) - a(0)| < \varepsilon \\ &\Rightarrow \forall \theta,\, 0 \le \theta < \delta, \quad \forall |\zeta| = 1, \quad |a(\theta\zeta)| = |a(\theta\zeta) - a(0)| < \varepsilon \\ &\Rightarrow \forall \theta,\, 0 \le \theta < \delta, \quad |h(\theta) - h(0)| = |h(\theta)| = \max_{\zeta \in H,\, |\zeta|=1} |a(\theta\zeta)| < \varepsilon. \end{aligned}$$

Thus $h \in \mathcal{H}$ and for all $x \in \partial\Omega$ and $\zeta \in V$, $0 \le |\bar{a}'_x(\zeta)| \le a(\zeta) \le h(|\zeta|) \le \alpha$.

(c) *Inequality* (5.20). Consider the region

$$C(\alpha, h, \rho) \stackrel{\text{def}}{=} \left\{ \zeta' + \zeta_N e_N : \begin{array}{l} \zeta' \in B_H(0, \rho) \text{ and} \\ \limsup h(|\zeta'|) < \zeta_N < \alpha \end{array} \right\}. \tag{5.23}$$

To each $x \in \partial\Omega$ and $\zeta \in C(\alpha, h, \rho)$ associate the point

$$y_\zeta \stackrel{\text{def}}{=} x + A_x(\zeta' + \zeta_N e_N) \in \mathcal{U}'(x) \cap A_x^+ = \mathcal{U}'(x) \cap \operatorname{int}\Omega \subset \operatorname{int}\Omega$$
$$A_x'^+ \stackrel{\text{def}}{=} \{x + A_x(\zeta' + \zeta_N e_N) : \zeta' \in V \quad \text{and} \quad \zeta_N > \bar{a}_x'(\zeta')\}, \tag{5.24}$$

because $P_H A_x^{-1}(y_\zeta - x) = \zeta' \in B_H(0, \rho)$, $h \geq 0$, and hence $0 < \zeta_N < \alpha$.
Fix ζ' and ξ' in $V = B_H(0, \rho)$ such that $\xi' \neq \zeta'$ and $|\xi' - \zeta'| < \rho$. By construction

$$y_{\zeta'} = x + A_x(\zeta' + \bar{a}_x'(\zeta')e_N) \in \partial\Omega \cap \mathcal{U}'(x)$$
$$y_{\xi'} = x + A_x(\xi' + \bar{a}_x'(\xi')e_N) \in \partial\Omega \cap \mathcal{U}'(x)$$

and $y_{\zeta'} \neq y_{\xi'}$. By the property of Eq. (5.24), $y_{\zeta'} + A_x C(\alpha, h, \rho) \subset \operatorname{int}\Omega$ and, because $y_{\xi'} \in \partial\Omega$, $y_{\xi'} \notin y_{\zeta'} + A_x C(\alpha, h, \rho)$. This means that $y_{\xi'} - y_{\zeta'} \notin A_x C(\alpha, h, \rho)$ or equivalently $\xi' - \zeta' + (\bar{a}_x'(\xi') - \bar{a}_x'(\zeta'))e_N \notin C(\alpha, h, \rho)$. Therefore, for all ζ' and ξ' in $B_H(0, \rho)$ such that $|\xi' - \zeta'| < \rho$, either

$$\limsup h(|\xi' - \zeta'|) \geq (y_{\xi'} - y_{\zeta'}) \cdot A_x e_N = \bar{a}_x'(\xi') - \bar{a}_x'(\zeta')$$
$$\text{or } 0 < \alpha \leq (y_{\xi'} - y_{\zeta'}) \cdot A_x e_N = \bar{a}_x'(\xi') - \bar{a}_x'(\zeta').$$

But the last inequality would contradict the continuity of a_x'. Therefore

$$\bar{a}_x'(\xi') - \bar{a}_x'(\zeta') \leq \limsup h(|\xi' - \zeta'|).$$

A similar inequality is obtained by interchanging the role of ζ' and ξ' in the above proof. This establishes Eq. (5.20) for all $\zeta', \ \xi' \in B_H(0, \rho)$ such that $|\xi' - \zeta'| < \rho$.

(d) *Uniform cusp property*. Consider the region

$$C(\alpha, h, \rho/2) \stackrel{\text{def}}{=} \left\{ \zeta' + \zeta_N e_N : \begin{array}{l} \zeta' \in B_H(0, \rho/2) \text{ and} \\ \limsup h(|\zeta'|) < \zeta_N < \alpha \end{array} \right\}. \tag{5.25}$$

Note that by continuity of h in 0 and the fact that $h(0) = 0$, $C(\alpha, h, \rho/2) \neq \varnothing$. The uniform cusp property of Definition 5.2 (3) is verified with the parameters $(r', \lambda', \rho', h') = (\rho/2, \alpha, \rho/2, h)$.

2. Similar to the proof of Theorem 7.4 (ii) in Chapter 2 of Reference 1. ∎

5.4 Sufficient Condition on the Local Graphs for Compactness

We have seen in the proof of Theorem 5.6 (1) that for a set Ω with compact boundary verifying the segment property, the uniform cusp property is verified for some parameters $(r^\Omega, \lambda^\Omega, \rho^\Omega, h^\Omega)$, which depend only on Ω. Moreover, the neighborhoods $V^\Omega_{H(x)}$ and $\mathcal{U}^\Omega(x)$ can be chosen as follows:

$$V^\Omega_{H(x)} = A_x^\Omega V^\Omega, \quad \mathcal{U}^\Omega(x) = x + A_x^\Omega U^\Omega \tag{5.26}$$

for some bounded open neighborhoods V^Ω of 0 in H and U^Ω of 0 in $\mathbf{R}^{\mathrm{N}}$ and some matrix $A^\Omega(x) \in \mathrm{O(N)}$. Moreover, the family of mappings

$$\mathcal{A}^\Omega \stackrel{\text{def}}{=} \left\{ \bar{a}_x^\Omega = a_x^\Omega \circ A_x^\Omega : V^\Omega \to \mathbf{R} : \forall x \in \partial\Omega \right\} \tag{5.27}$$

is equibounded and equicontinuous.

This suggests formulation of a sufficient condition for compactness on the local graphs of a family of domains contained in a holdall D.

THEOREM 5.7

Fix $\rho > 0$ and a bounded open neighborhood U of 0 such that

$$U \subset \left\{\zeta \in \mathbf{R}^{\mathrm{N}} \,:\, P_H(\zeta) \in B_H(0,\rho)\right\}, \quad V \stackrel{\text{def}}{=} B_H(0,\rho). \tag{5.28}$$

Given a bounded open nonempty subset D of $\mathbf{R}^{\mathrm{N}}$, consider a family $L(D,\rho,U)$ of subsets Ω of $\bar{D}$ with the following properties: for each $\Omega \in L(D,\rho,U)$ and

$$\forall x \in \partial\Omega, \quad \mathcal{U}^{\Omega}(x) \stackrel{\text{def}}{=} x + A_x^{\Omega} U, \tag{5.29}$$

there exist $A_x^{\Omega} \in \mathrm{O(N)}$ and a C^0-mapping $\bar{a}_x^{\Omega} : V \to \mathbf{R}$ such that

$$\begin{aligned} \mathcal{U}^{\Omega}(x) \cap \partial\Omega &= \left\{x + A_x(\zeta' + \zeta_N e_N) : \zeta' \in V,\ \zeta_N = \bar{a}_x^{\Omega}(\zeta')\right\} \\ \mathcal{U}^{\Omega}(x) \cap \operatorname{int}\Omega &= \mathcal{U}^{\Omega}(x) \cap \left\{x + A_x(\zeta' + \zeta_N e_N) : \zeta' \in V,\ \zeta_N > \bar{a}_x^{\Omega}(\zeta')\right\}. \end{aligned} \tag{5.30}$$

1. *Assume that there exists $h \in \mathcal{H}$ such that*

   $$\forall \Omega \in L(D,\rho,U),\ \forall y \in V, \quad \bar{a}_x^{\Omega}(y) \le \limsup h(|y|) \tag{5.31}$$

   and let $r > 0$ be such that $B(0,4r) \subset U$. Then each Ω of $L(D,\rho,U)$ satisfies the uniform cusp property for the parameters $(r^{\Omega}, \lambda^{\Omega}, \rho^{\Omega}, h^{\Omega}) = (r, r, 2r, h)$, which are independent of Ω. Hence the family

   $$B(D,\rho,U) \stackrel{\text{def}}{=} \{b_{\Omega} \,:\, \forall \Omega \in L(D,\rho,U) \text{ that verifies Eq. (5.31)}\}$$

   is compact in $C(\overline{D})$ and $W^{1,p}(D)$, $1 \le p < \infty$.

2. *Assume that the family of mappings*

   $$\mathcal{A} \stackrel{\text{def}}{=} \left\{\bar{a}_x^{\Omega} : V \to \mathbf{R} \,:\, \forall \Omega \in L(D,\rho,U) \quad \text{and} \quad \forall x \in \partial\Omega\right\} \tag{5.32}$$

   is equicontinuous. Then there exists a nonnegative $h \in \mathcal{H}$ such that

   $$\forall \Omega \in L(D,\rho,U),\ \forall y \in V, \quad \left|\bar{a}_x^{\Omega}(y)\right| \le \limsup h(|y|). \tag{5.33}$$

   Hence condition of Eq. (5.31) is verified and the conclusions of part (1) are true.

PROOF

1. The sets Ω are clearly local epigraphs of the C^0-mappings $\{\bar{a}_x^{\Omega} : \forall x \in \partial\Omega\}$ in the sense of Definition 5.3.

   (a) *A first inequality*. By assumption of Eq. (5.28), $B(0,4r) \subset U$ implies $4r \le \rho$. Consider the region

   $$C(r,h,2r,) \stackrel{\text{def}}{=} \left\{\zeta' + \zeta_N e_N : \zeta' \in B_H(0,2r) \text{ and } \limsup h(|\zeta'|) < \zeta_N < r\right\}.$$

   Fix $\Omega \in L(D,\rho,U)$. For convenience we now drop the superscript Ω. For instance, we write $\mathcal{U}(x)$ and A_x instead of $\mathcal{U}^{\Omega}(x)$ and A_x^{Ω}. For any $\zeta \in C(r,h,2r,)$, consider the point $y_\zeta = x + A_x(\zeta' + \zeta_N e_N)$. It is readily seen that $|y_\zeta - x| = |\zeta| < \sqrt{(2r)^2 + r^2} < 3r$, that $y_\zeta \in \mathcal{U}(x)$, and that $y_\zeta \in \mathcal{U}(x) \cap A_x^{+}$ because, by assumption (5.31), $\bar{a}_x(\zeta') \le$

$\limsup h(|\zeta'|) < \zeta_N$ for all $\zeta' \in V = B_H(0,\rho)$ $(4r \le \rho)$. Hence $y_\zeta \in \mathcal{U}(x) \cap A_x^+ = \mathcal{U}(x) \cap \operatorname{int}\Omega \subset \operatorname{int}\Omega$ and

$$\forall x \in \partial\Omega, \quad x + A_x C(r,h,2r,) \subset \operatorname{int}\Omega. \tag{5.34}$$

Given ζ' and ξ' in $B_H(0,2r)$, consider the two points

$$y_{\zeta'} \stackrel{\text{def}}{=} x + A_x(\zeta' + \bar{a}_x(\zeta')e_N) \in A_x^0 = \mathcal{U}(x) \cap \partial\Omega,$$
$$y_{\xi'} \stackrel{\text{def}}{=} x + A_x(\xi' + \bar{a}_x(\xi')e_N) \in A_x^0 = \mathcal{U}(x) \cap \partial\Omega.$$

Because $y_{\xi'}$ and $y_{\zeta'}$ belong to $\mathcal{U}(x) \cap \partial\Omega$, $y_{\xi'}$ does not belong to $y_{\zeta'} + A_x C(r,h,2r,) \subset \operatorname{int}\Omega$ or equivalently $y_{\xi'} - y_{\zeta'} \notin A_x C(r,h,2r,)$. Hence

$$\xi' - \zeta' + (\bar{a}_x(\xi') - \bar{a}_x(\zeta'))e_N = A_x^{-1}(y_{\xi'} - y_{\zeta'}) \notin C(r,h,2r,).$$

For $|\xi' - \zeta'| < 2r$, this is equivalent to say that one of the following inequalities is verified:

$$\limsup h(|\xi' - \zeta'|) \ge A_x e_N \cdot (y_{\xi'} - y_{\zeta'}) = \bar{a}_x(\xi') - \bar{a}_x(\zeta')$$
$$\text{or} \quad r \le A_x e_N \cdot (y_{\xi'} - y_{\zeta'}) = \bar{a}_x(\xi') - \bar{a}_x(\zeta')$$

by definition of $\bar{a}_x$ and that $\xi' - \zeta' \in B_H(0,4r) \subset B_H(0,\rho) = V$. But the second inequality contradicts the continuity of $\bar{a}_x$. By interchanging the roles of ξ' and ζ', we get a second inequality that yields

$$\begin{aligned} &\forall \xi', \zeta' \in B_H(0,2r), \quad |\xi' - \zeta'| < 2r, \\ &|\bar{a}_x(\xi') - \bar{a}_x(\zeta')| \le \limsup h\,(|\xi' - \zeta'|). \end{aligned} \tag{5.35}$$

(b) *The uniform cusp property*. Consider the region

$$C(r,h,r,) \stackrel{\text{def}}{=} \left\{\zeta' + \zeta_N e_N : \zeta' \in B_H(0,r) \text{ and } \limsup h(|\zeta'|) < \zeta_N < r\right\}.$$

Clearly $C(r,h,r,) \ne \varnothing$ because $h(0) = 0$ and h is continuous in 0. The uniform cusp property is verified with the parameters $(r', \lambda', h', \rho') = (r, r, h, r)$.

2. Because the family $\mathcal{A}$ is equibounded, the mapping

$$a(y) \stackrel{\text{def}}{=} \sup_{\Omega \in L(D,\rho,U),\, x \in \partial\Omega} \left|\bar{a}_x^\Omega(y)\right| \tag{5.36}$$

is well defined on V, $\forall \Omega \in L(D,\rho,U)$, $\forall x \in \partial\Omega$, $0 \le |\bar{a}_x^\Omega(y)| \le a(y) \le c$, and $\bar{a}_x^\Omega(0) = 0$ implies $a(0) = 0$. It remains to show that it is continuous

$$\left|\bar{a}_x^\Omega(z)\right| \le \left|\bar{a}_x^\Omega(z) - \bar{a}_x^\Omega(y)\right| + \left|\bar{a}_x^\Omega(y)\right|$$
$$\sup_{\substack{\Omega \in L(D,\rho,U)\\ \forall x \in \partial\Omega}} \left|\bar{a}_x^\Omega(z)\right| \le \sup_{\substack{\Omega \in L(D,\rho,U)\\ \forall x \in \partial\Omega}} \left|\bar{a}_x^\Omega(z) - \bar{a}_x^\Omega(y)\right| + \sup_{\substack{\Omega \in L(D,\rho,U)\\ \forall x \in \partial\Omega}} \left|\bar{a}_x^\Omega(y)\right|.$$

By equicontinuity for $y \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \Omega \in L(D,\rho,U),\ \forall x \in \partial\Omega, \quad \forall z \in V, \quad |z - y| < \delta \quad \Rightarrow \left|\bar{a}_x^\Omega(z) - \bar{a}_x^\Omega(y)\right| < \varepsilon.$$

Hence

$$\forall z \in V, \quad |z - y| < \delta \quad \Rightarrow a(z) \le \varepsilon + a(y)$$
$$\Rightarrow \forall z \in V, \quad |z - y| < \delta, \quad \Rightarrow |a(z) - a(y)| \le \varepsilon,$$

and a is continuous on V. By the same technique as at the end of the proof of Theorem 5.6 (1), the function

$$h(\theta) \stackrel{\text{def}}{=} \max_{\zeta' \in H,\, |\zeta'|=1} a(\theta\zeta')$$

is well defined and belongs to $\mathcal{H}$. Finally, by continuity of $\bar{a}_x^\Omega$ and a,

$$\forall \Omega \in L(D, \rho, U),\ \forall x \in \partial\Omega,\ \forall \zeta' \in V, \quad 0 \leq \left|\bar{a}_x^\Omega(\zeta')\right| \leq a(\zeta') \leq \limsup h(|\zeta'|)$$

and the condition (5.31) of part (1) is verified. □

REMARK 5.1 Condition (2). is a streamlined version of the sufficient condition given in References 3 and 2. Condition (1). relaxes the equicontinuity condition to the simpler dominating condition: existence of a (cusp) function h continuous in 0, which dominates all the local C^0-mappings on the side of the epigraph.

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Chapter 6

Gårding's Inequality on Manifolds with Boundary

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Abstract Gårding's inequality for pseudodifferential operators on manifolds with boundary is discussed. The pseudodifferential operators are restricted to operators that are differential in normal direction. Through the technique of symbol smoothing, the main results are shown to hold for symbols with limited smoothness. As an application, general Carleman estimates with boundary terms for elliptic operators as well as for operators with real principal symbols are proved.

6.1 Introduction and Main Result

Gårding's inequality is one of the classical results of pseudodifferential calculus. Given an open set $X \subset \mathbf{R}^n$ and a symbol $b \in S^{2m}(X \times \mathbf{R}^n)$ such that $\Re b(x, \xi) \geq C|\xi|^2$ for some positive constant C and $|\xi|$ large, there exist two positive constants c and d such that

$$\Re(b(x, D)u, u) \geq c\|u\|_m^2 - d\|u\|^2 \tag{6.1}$$

for every $u \in C_0^\infty(X)$ (see Reference 8, p. 55). Here $(\cdot, \cdot)$ denotes the L_2 inner product, $\|\cdot\|$ is the L_2 norm, and $\|\cdot\|_m$ is the norm in the Sobolev space H^m. This estimate has proved to be useful in many problems concerning partial differential equations, in particular in the theory of elliptic operators. There exist also stronger versions of the estimate in Eq. (6.1) such as the sharp Gårding inequality and the Fefferman-Phong inequality (see Reference 3, Chapter 18, and Reference 1).

In this note we like to discuss Gårding's inequality for manifolds with boundary. In other words, we ask the question, what happens to the inequality in Eq. (6.1) if the function u is not compactly supported in X? This problem has received surprisingly little attention in the past 40 years. Sakamoto developed a Gårding inequality with boundary terms in her work on hyperbolic problems (see

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Reference 4, Section 1.2), and Tataru proved a similar result when proving Carleman estimates with boundary terms (see Reference 6, Section 3). Here we will give a more systematic treatment of this problem.

In the following discussion we will assume that $X \subset \mathbf{R}^n$ is an open, connected set with a smooth boundary ∂X. Local coordinates and a partition of unity will allow us to focus on the special case $X = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_n \geq 0\}$ with $\partial X = \{x \in R^n : x_n = 0\}$. Accordingly, we will set $x' = (x_1, x_2, \dots x_{n-1})$, and we split the Fourier variable also into tangential and normal variables (i.e., $\xi = (\xi', \xi_n)$). Because all the arguments given in this paper are of local nature, the main results are also valid on Riemannian manifolds with boundary.

In the sequel, we will also need the Sobolev spaces $H^s = H^s(\mathbf{R}^n)$, $H^s(X)$, and $H^s(\partial X)$. Given $s \in \mathbf{R}$, the space H^s is the space of all distributions such that

$$\|u\|_s = \left((2\pi)^{-n} \int \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty \tag{6.2}$$

where $\hat{u}$ denotes the Fourier transform of u and $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. We say that $v \in H^s(X)$ if there exists a function $u \in H^s$ such that $v = u$ on X. The space $H^s(X)$ is equipped with the quotient norm

$$\|v\|_{(s)} = \inf \left\{ \|u\|_s : u \in H^s,\ u = v \text{ on } X \right\}.$$

Furthermore, the space $H^s(\mathbf{R}^{n-1}) = H^s(\partial X)$ is the Hilbert space with the norm

$$|u|_s = \left((2\pi)^{-n+1} \int \langle \xi' \rangle^{2s} |\hat{u}(\xi')|^2 d\xi' \right)^{1/2}.$$

We will also need to work with some anisotropic Sobolev spaces. Given two real numbers k and s we say that $u \in H^{k,s}$ if

$$\|u\|^2_{k,s} = (2\pi)^{-n} \int \langle \xi \rangle^{2k} \langle \xi' \rangle^{2s} |\hat{u}(\xi)|^2 d\xi$$

is finite. Of course, $\| \cdot \|_{k,s}$ is a norm in $H^{k,s}$. The space $H^{k,s}(X)$ is defined as a quotient space, and its norm is denoted by $\| \cdot \|_{(k,s)}$.

We will make use of the symbol classes used in pseudodifferential calculus. More specifically, we say $a \in S^k(X \times \mathbf{R}^n)$ if $a \in C^\infty(X \times \mathbf{R}^n)$ and if for all multiindices α and β there exist positive constants $C = C(\alpha, \beta)$ such that

$$\left| D^\alpha_\xi D^\beta_x a(x, \xi) \right| \leq C(\alpha, \beta) \langle \xi \rangle^{k - |\alpha|} \quad \text{for all } x \in X \quad \text{and} \quad \xi \in \mathbf{R}^n. \tag{6.3}$$

We will also make use of symbols with limited smoothness in x. The set $C^r S^k(X \times \mathbf{R}^n)$ denotes the set of C^r functions on X with values in $C^\infty(\mathbf{R}^n)$ such that there exist positive constants $C(\alpha, \beta)$ such that inequality of Eq. (6.3) holds for all multiindices α and $|\beta| \leq r$. For later purposes we will also introduce symbols that are polynomial in the ξ_n variable. We say $a(x, \xi) \in C^r S^{m,k}(X \times \mathbf{R}^n)$ if

$$a(x, \xi) = \sum_{j=0}^{m} a_j(x, \xi') \xi_n^j \quad \text{and} \quad a_j(x, \xi') \in C^r S^{m+k-j}(X \times \mathbf{R}^{n-1}).$$

In the case of compactly support functions u, the L_2 inner product $(a(x, D)u, u)$ is commonly used in the formulation of Gårding's inequality. This is very convenient because the correspondence between the symbol $a(x, \xi)$ and the operator $a(x, D)$ is one-to-one. When we consider functions supported up to the boundary, we need to modify this setup because we cannot use adjoint operators as freely. We have to distinguish differentiation in normal direction vs. differentiation in tangential direction. For this purpose we introduce quadratic forms involving pseudodifferential operators that are polynomial in normal direction.

DEFINITION 6.1 *Given $u \in H^m(X)$, we say that*

$$B(u,u)_X = \sum_{l=1}^{\nu} (p^l(x,D)u, q^l(x,D)u)_X = \sum_{l=1}^{\nu} \int_X p^l(x,D)u\overline{q^l(x,D)u} \tag{6.4}$$

is a quadratic form of order $(\mu, m, 2k)$ with C^r coefficients provided $p^l(x,\xi), q^l(x,\xi) \in C^r S^{m,k}(X \times \mathbf{R}^n)$ and $\overline{q^l(x,\xi)}p^l(x,\xi) \in C^r S^{\mu,2k}(X \times \mathbf{R}^n)$ for $l = 1, 2, \ldots \nu$. The symbol of a quadratic form is

$$b(x,\xi) = \sum_{l=1}^{\nu} \overline{q^l(x,\xi)}p^l(x,\xi).$$

We see that the correspondence between quadratic forms and symbols is not one-to-one. Different quadratic forms can have the same symbol. For example, the quadratic forms $(\nabla u, \nabla u)_X$ and $(-\Delta u, u)_X$ have both the symbol $|\xi|^2$. Our main result is the following.

THEOREM 6.1
Assume $b(x,\xi) \in C^1 S^{2m,0}(X \times \mathbf{R}^n)$ satisfies

$$\Re b(x,\xi) \geq C\langle\xi\rangle^{2m} \quad \text{for all } (x,\xi) \in X \times \mathbf{R}^n \tag{6.5}$$

where C is a positive constant. Then, for every quadratic form $B(u,u)_X$ of order $(2m, m, 0)$ with symbol $b(x,\xi)$ and for all $\varepsilon > 0$ there exists a constant $c = c(\varepsilon)$ such that

$$\Re B(u,u)_X \geq (C-\varepsilon)\|u\|^2_{(m)} - c\left[\|u\|^2_{(m,-1/2)} + \sum_{j=0}^{m-1} \left|D_n^j u\right|^2_{m-j-1/2}\right] \tag{6.6}$$

for all $u \in H^m(X)$.

One can also establish Gårding's inequality for anisotropic symbols. As an illustration, we state the following theorem.

THEOREM 6.2
Assume $b(x,\xi) \in C^1 S^{2m,-2}(X \times \mathbf{R}^n)$ satisfies

$$\Re b(x,\xi) \geq C\frac{\langle\xi\rangle^{2m}}{\langle\xi'\rangle^2} \quad \text{for all } (x,\xi) \in X \times \mathbf{R}^n$$

for some positive constant C. Then, for every quadratic form $B(u,u)_X$ of order $(2m, m, -2)$ with symbol $b(x,\xi)$ and for all $\varepsilon > 0$, there exists a constant $c = c(\varepsilon)$ such that

$$\Re B(u,u)_X \geq (C-\varepsilon)\|u\|^2_{(m,-1)} - c\left[\|u\|^2_{(m,-3/2)} + \sum_{j=0}^{m-1} \left|D_n^j u\right|^2_{m-j-3/2}\right] \tag{6.7}$$

for all $u \in H^m(X)$.

REMARK 6.1 We like to point out that in the case of compactly supported functions, the inequality of Eq. (6.6) is also true for $\varepsilon = 0$. This inequality is known as the sharp Gårding inequality. As of now, it is not known whether Eq. (6.6) is valid with $\varepsilon = 0$.

Now we discuss Gårding's inequality for quadratic forms with a parameter. Instead of working with $\langle\xi\rangle = \sqrt{1+\xi^2}$, we will now use $\langle\xi\rangle_\tau = \sqrt{\tau^2+\xi^2}$ where τ is a real number. This way, the norm of Eq. (6.2) is modified to a weighted norm on H^s and denoted by $\|\cdot\|_{s,\tau}$. All the other norms introduced above will be modified in the same way. Similarly, we say $a(x,\xi,\tau) \in S^k_\tau(X \times \mathbf{R}^n)$ if the inequality of Eq. (6.3) holds with $\langle\xi\rangle$ replaced by $\langle\xi\rangle_\tau$.

COROLLARY 6.1
Assume $b(x,\xi,\tau) \in C^1 S^{2m,0}_\tau(X \times \mathbf{R}^n)$ satisfies

$$\Re b(x,\xi,\tau) \geq C\langle\xi\rangle^{2m}_\tau \quad \textit{for all } (x,\xi) \in X \times \mathbf{R}^n$$

for all $\tau \geq 1$. Then, for every quadratic form $B(u,u)_X$ of order $(2m,m,0)$ with symbol $b(x,\xi,\tau)$ and for all $\varepsilon > 0$, there exist constants c and τ_0 (depending on ε) such that for $\tau \geq \tau_0$ we have

$$\Re B(u,u)_X \geq (C-\varepsilon)\|u\|^2_{(m,\tau)} - c\sum_{j=0}^{m-1}\left|D^j_n u\right|^2_{m-j-1/2,\tau} \tag{6.8}$$

for all $u \in H^m(X)$.

COROLLARY 6.2
Assume $b(x,\xi,\tau) \in C^1 S^{2m,-2}_\tau(X \times \mathbf{R}^n)$ satisfies

$$\Re b(x,\xi,\tau) \geq C\frac{\langle\xi\rangle^{2m}_\tau}{\langle\xi'\rangle^2_\tau} \quad \textit{for all } (x,\xi) \in X \times \mathbf{R}^n$$

for all $\tau \geq 1$. Then, for every quadratic form $B(u,u)_X$ of order $(2m,m,-2)$ with symbol $b(x,\xi,\tau)$ and for all $\varepsilon > 0$, there exist two constants c and τ_0 (depending on ε) such that for $\tau \geq \tau_0$ we have

$$\Re B(u,u)_X \geq (C-\varepsilon)\|u\|^2_{(m,-1,\tau)} - c\sum_{j=0}^{m-1}\left|D^j_n u\right|^2_{m-j-3/2,\tau} \tag{6.9}$$

for all $u \in H^m(X)$.

6.2 Proofs

We begin with two lemmas of technical nature. The first lemma provides an estimate for certain smooth quadratic forms of type $(2m,m,-2l)$ that will be useful when discussing quadratic forms with zero symbol. The second lemma discusses the factorization of a polynomial with coefficients of specified regularity. In the following discussion, we will use Hörmander's class of symbols $S^m_{\rho,\delta}$ (see Reference 3, Section 18.1). We say $a \in S^m_{\rho,\delta}(X \times \mathbf{R}^n)$ if

$$\left|D^\alpha_\xi D^\beta_x a(x,\xi)\right| \leq C(\alpha,\beta)\langle\xi\rangle^{k-\rho|\alpha|+\delta|\beta|} \quad \text{for all } x \in X \quad \text{and} \quad \xi \in \mathbf{R}^n.$$

For the calculus of pseudodifferential operators we refer to the books by Hörmander [3] and Taylor [8].

LEMMA 6.1
Let l be an integer and assume $a_{jk}(x,\xi') \in S^{2m-j-k-2l}_{1,\delta}(X \times \mathbf{R}^{n-1})$ for $j,k = 0,1,\ldots m$ such that

$$\sum_{j+k=s} a_{jk}(x,\xi') \in S^{2m-s-2l-(1-\delta)}_{1,\delta}(X \times \mathbf{R}^{n-1}) \quad \textit{for } s = 0,1,\ldots 2m \tag{6.10}$$

for some $0 \le \delta \le 1/2$. *Then there exists a positive constant* C *such that*

$$\left| \sum_{j,k=0}^{m} \left(a_{jk}(x, D') D_n^j u, D_n^k u \right)_X \right| \le C \left[\sum_{j=0}^{m-1} \left| D_n^j u \right|^2_{m-j-l-1/2} + \|u\|^2_{(m,-l-(1-\delta)/2)} \right] \tag{6.11}$$

for all $u \in C^\infty(\overline{X})$.

PROOF The assumption in Eq. (6.10) implies that

$$a_{jm}(x, \xi') = -\sum_{s=1}^{m-j} a_{j+s\, m-s}(x, \xi') + r_{jm}(x, \xi') \tag{6.12}$$

where $r_{jm}(x, \xi') \in S^{m-j-2l-(1-\delta)}_{1,\delta}(X \times \mathbf{R}^n)$ for $j = 0, 1, \dots m$ and

$$a_{0k}(x, \xi') = -\sum_{s=1}^{k} a_{s\, k-s}(x, \xi') + r_{0k}(x, \xi') \tag{6.13}$$

where $r_{0k}(x, \xi') \in S^{2m-k-2l-(1-\delta)}_{1,\delta}(X \times \mathbf{R}^n)$ for $k = 0, 1, \dots m$. Hence

$$\begin{aligned}\sum_{j,k=0}^{m} a_{jk}(x, D') D_n^j u \overline{D_n^k u} = & \sum_{j,k=0}^{m-1} \sum_{s=1}^{\min\{k+1,m-j\}} D_n \left[a_{j+s\, k-s+1}(x, D') D_n^j u \overline{D_n^k u} \right] \\ & - \sum_{j,k=0}^{m-1} \sum_{s=1}^{\min\{k+1,m-j\}} D_n a_{j+s\, k-s+1}(x, D') D_n^j u \overline{D_n^k u} \\ & + \sum_{j=0}^{m} r_{jm}(x, D') D_n^j u \overline{D_n^m u} + \sum_{k=0}^{m-1} r_{0k}(x, D') u \overline{D_n^k u}. \end{aligned} \tag{6.14}$$

This formula can be verified using Equations (6.12) and (6.13). After integrating over X, we observe that the integration in normal direction in the first term on the right-hand side leads to an integral over ∂X. Then we estimate the right-hand side terms by the continuity properties of pseudodifferential operators in Sobolev spaces. ∎

LEMMA 6.2
Let k *be an integer satisfying* $0 \le k \le m$. *Assume that* $b(x, \xi) \in C^r S^{2m,-2k}(X \times \mathbf{R}^n)$ *such that*

$$b(x, \xi) \ge C \frac{\langle \xi \rangle^{2m}}{\langle \xi' \rangle^{2k}} \quad \textit{for all } \xi \in \mathbf{R}^n, x \in X. \tag{6.15}$$

For all $\varepsilon > 0$, *there exists a symbol* $c(x, \xi) \in C^r S^{m,-k}(X \times \mathbf{R}^n)$ *such that*

$$b(x, \xi) - (C - \varepsilon) \frac{\langle \xi \rangle^{2m}}{\langle \xi' \rangle^{2k}} = \overline{c(x, \xi)} c(x, \xi) \tag{6.16}$$

PROOF The symbol $b(x, \xi) - (C - \varepsilon)\langle \xi \rangle^{2m} / \langle \xi' \rangle^{2k}$ is a polynomial of degree $2m$ in ξ_n, and it is positive for $\xi_n \in \mathbf{R}$.

Consider for a moment a polynomial with real coefficients $f(z)$ of degree $2m$ that is positive for $z \in \mathbf{R}$. Then $f(z) = c(z)\overline{c(z)}$ where $c(z)$ is a polynomial of degree m that has roots with only a positive imaginary part.

We can apply this factorization to $b(x,\xi)-(C-\varepsilon)\langle\xi\rangle^{2m}/\langle\xi'\rangle^{2k}$ at every point $(x,\xi')\in X\times\mathbf{R}^{n-1}$. However, we have to show that this factorization is smooth. Set

$$f(x,\xi',\zeta_n)=b(x,\xi',\zeta_n)-(C-\varepsilon)\frac{\left(1+|\xi'|^2+\zeta_n^2\right)^m}{(1+|\xi'|^2)^k}$$

and denote the lead coefficient by $f_{2m}(x,\xi')$. Because of the assumption in Eq. (6.15), we know that f_{2m} is always positive. At each point $(x,\xi')\in X\times\mathbf{R}^{n-1}$, we have the factorization

$$f(x,\xi',\zeta_n)=\overline{c(x,\xi',\zeta_n)}c(x,\xi',\zeta_n),$$

and we need to show that the coefficients of c are in the space $C^r(X;C^\infty(\mathbf{R}^{n-1}))$ (i.e., the coefficients are C^r in x and C^∞ in ξ'). Assume for a moment that $(x,\xi')\in\mathcal{B}$ where $\mathcal{B}$ is a bounded set in $X\times\mathbf{R}^{n-1}$. Then the zeros of $f(x,\xi',\zeta_n)$ with respect to ζ_n will be bounded as well (i.e., no zero will be outside a circle with radius R in the complex plane). We denote the zeros in the upper half plane by $\zeta_n^{(j)}$ for $j=1,2,\ldots m$ and let Γ be a closed curve in the complex plane consisting of the line segment $[-R,R]$ on the real line and the semicircle of radius R in the upper half plane. Now

$$c(x,\xi',\zeta_n)=\sqrt{f_{2m}}\prod_{j=1}^{m}\left(\zeta_n-\zeta_n^{(j)}\right)=\exp\left\{\frac{1}{2\pi i}\int_\Gamma\frac{\partial_s f(x,\xi',s)}{f(x,\xi',s)}\log(\zeta_n-s)ds\right\}$$

where the last equality is only true for $\Im\zeta_n<0$ because this will guarantee that the logarithm is an analytic function for s inside of Γ. This formula follows from the Residue Theorem. Using Lagrange interpolation we conclude that the polynomial $c(x,\xi',\zeta_n)$ has coefficients that are as regular as the ones of the polynomial $f(x,\xi',\zeta)$ for $(x,\xi')\in\mathcal{B}$. Because $\mathcal{B}$ is an arbitrary bounded set, the coefficients of $c(x,\xi',\zeta_n)$ must be in $C^r(X;C^\infty(\mathbf{R}^{n-1}))$.

Finally, because $\overline{c(x,\xi)}c(x,\xi)\in C^rS^{2m,-2k}(X\times\mathbf{R}^n)$, we obtain $c(x,\xi)\in C^rS^{m,-k}(X\times\mathbf{R}^n)$. $\square$

6.2.1 Proof of Theorem 6.1

At first we consider quadratic forms with C^∞ coefficients. Let $B(u,u)_X$ be a quadratic form with symbol $b(x,\xi)$. Then the symbol of the quadratic form $\Re B(u,u)_X$ is $\Re b(x,\xi)$. Using Lemma 6.2 with $k=0$, we have

$$\Re b(x,\xi)-(C-\varepsilon)\langle\xi\rangle^{2m}-\overline{c(x,\xi)}c(x,\xi)=0\quad\text{for }(x,\xi)\in X\times\mathbf{R}^n.$$

In other words, the quadratic form

$$A(u,u)_X=\Re B(u,u)_X-(C-\varepsilon)\|u\|_{(m)}^2-\|c(x,D)u\|_{(0)}^2 \tag{6.17}$$

has a zero symbol. Here we used the fact that

$$\langle\xi\rangle^{2m}=(\langle\xi'\rangle+i\xi_n)^m(\langle\xi'\rangle-i\xi_n)^m$$

and that $\|(\langle D'\rangle-iD_n)^m\|_{(0)}=\|u\|_{(m)}$ (see Reference 3, Theorem B.2.4). We know that

$$\begin{aligned}
A(u,u)_X&=\sum_{l=1}^{\nu}(p^l(x,D)u,q^l(x,D)u)_X\\
&=\sum_{l=1}^{\nu}\sum_{j,k=0}^{m}\left(p_j^l(x,D')D_n^ju,q_k^l(x,D')D_n^ku\right)_X\\
&=\sum_{l=1}^{\nu}\sum_{j,k=0}^{m}\left(q_k^l(x,D')^*\circ p_j^l(x,D')D_n^ju,D_n^ku\right)_X
\end{aligned}$$

where $q_k^l(x, D')^*$ is the adjoint of the operator $q_k^l(x, D')$. The symbol of the composition of the operators $q_k^l(x, D')^*$ and the operator $p_j(x, D')$ has an asymptotic expansion with the first term $\overline{q_k^l(x, \xi')} p_j^l(x, \xi')$. The remainder of the expansion is in the symbol space $S^{2m-j-k-1}(X \times \mathbf{R}^{n-1})$ (see Reference 3, Theorem 18.1.7, 18.1.8). Hence

$$\sum_{l=1}^{\nu} \sum_{j+k=s} \operatorname{sym}\left[q_k^l(x, D')^* \circ p_j^l(x, D')\right] \in S^{2m-j-k-1}(X \times \mathbf{R}^{n-1}) \quad \text{for } s = 0, 1, \ldots 2m \tag{6.18}$$

because

$$\sum_{l=1}^{\nu} \sum_{j+k=s} \overline{q_k^l(x, \xi')} p_j^l(x, \xi') = 0 \quad \text{for } s = 0, 1, \ldots 2m.$$

Now we can apply Lemma 6.1 with the choice

$$a_{jk}(x, \xi') = \sum_{l=1}^{\nu} \operatorname{sym}\left[q_k^l(x, D')^* \circ p_j^l(x, D')\right]$$

and $l = 0$, $\delta = 0$ and obtain

$$|A(u, u)_X| \leq C \left[\sum_{j=0}^{m-1} \left| D_n^j u \right|_{m-j-1/2}^2 + \|u\|_{(m,-1/2)}^2 \right].$$

This proves Theorem 6.1 in the case of C^∞ coefficients when we make use of the formula in Eq. (6.17).

Now we consider quadratic forms with C^1 coefficients. The proof is the same except for the last argument because there is no nice asymptotic expansion neither for the symbol of the adjoint nor for the symbol of the composition of two operators. We will use a technique know as symbol smoothing (see Reference 9, Section 1.3). As before, we obtain the quadratic form $A(u, u)_X$ of order $(2m, m, 0)$ with zero symbol. Given the symbols $p_j^l(x, \xi') \in C^1 S^{m-j}(X \times \mathbf{R}^{n-1})$, one has

$$p_j^l(x, \xi') = p_j^{l,\#}(x, \xi') + p_j^{l,b}(x, \xi') \tag{6.19}$$

where $p_j^{l,\#}(x, \xi') \in S_{1,\delta}^{m-j}(X \times \mathbf{R}^{n-1})$ and $p_j^{l,b}(x, \xi') \in C^1 S_{1,\delta}^{m-j-\delta}(X \times \mathbf{R}^{n-1})$ for some $\delta \in (0, 1]$. The point of this decomposition is that the symbol gets split into a smooth symbol and a symbol of lower order. The same decomposition will be used on the symbols $q_k^l(x, \xi')$. Hence

$$A(u, u)_X = A^\#(u, u)_X + A^b(u, u)_X \tag{6.20}$$

where $A^\#$ has smooth coefficients and A^b has C^1 coefficients and is of order $(2m, m, -\delta)$. From this decomposition and the fact that $a(x, \xi) = \sum_{l=1}^{\nu} \overline{q^l(x, \xi)} p^l(x, \xi) = 0$, it follows that

$$\sum_{l=1}^{\nu} \sum_{j+k=s} \overline{q_k^{l,\#}(x, \xi')} p_j^{l,\#}(x, \xi') \in S_{1,\delta}^{2m-s-\delta}(X \times \mathbf{R}^{n-1}) \quad \text{for } s = 0, 1, 2, \ldots 2m.$$

The first term in the asymptotic expansion of $q_k^{l,\#}(x, D')^* \circ p_j^{l,\#}(x, D')$ is $\overline{q_k^{l,\#}(x, \xi')} p_j^{l,\#}(x, \xi')$. The remainder of the asymptotic expansion is in $S_{1,\delta}^{2m-j-k-(1-\delta)}(X \times \mathbf{R}^{n-1})$ (see Reference 3, Section 18.1). Set

$$a_{jk}(x, \xi') = \sum_{l=1}^{\nu} \operatorname{sym}\left[q_k^{l,\#}(x, D')^* \circ p_j^{l,\#}(x, D')\right]$$

and apply Lemma 6.1 with $\delta = 1/2$ and $l = 0$. Thus,

$$|A^{\#}(u,u)_X| \le C\left[\sum_{j=0}^{m-1} \left|D_n^j u\right|^2_{m-j-1/2} + \|u\|^2_{(m,-1/4)}\right].$$

On the other hand, the formula in Eq. (6.20) results in

$$|A(u,u)_X| \le C\left[\sum_{j=0}^{m-1} \left|D_n^j u\right|^2_{m-j-1/2} + \|u\|^2_{(m,-1/4)}\right].$$

The proof is now completed as in the smooth case. Finally, we use the interpolation inequality

$$\|u\|^2_{(m,-1/4)} \le c\|u\|_{(m)}\|u\|_{(m,-1/2)}.$$

6.2.2 Proof of Theorem 6.2

The proof of Theorem 6.2 goes along the same lines as the proof of Theorem 6.1. The only difference is that the order of the tangential operators in the quadratic form is different. Hence, the formula in Eq. (6.11) has to be applied with $l = 1$ and $\delta = 0$ in the smooth case and with $\delta = 1/2$ in the case of C^1 coefficients. Lemma 6.2 has to be applied in the case $k = 1$. This time the interpolation inequality is

$$\|u\|^2_{(m,-5/4)} \le c\|u\|_{(m,-1)}\|u\|_{(m,-3/2)}.$$

6.2.3 Proof of the Corollaries

All proofs can be adjusted to symbols with parameters in a straightforward manner. Finally, we use the inequalities

$$\|u\|^2_{(m,-1/2,\tau)} \le \frac{1}{\tau}\|u\|^2_{(m,\tau)} \quad \text{and} \quad \|u\|^2_{(m,-3/2,\tau)} \le \frac{1}{\tau}\|u\|^2_{(m,-1,\tau)}$$

to absorb the norms of lower order into the left-hand side for large values of τ.

6.3 Application: Carleman Estimates

Carleman estimates were developed to prove uniqueness of ill-posed Cauchy problems. These estimates have been used in inverse problems and boundary control problems as well. Tataru proved general Carleman estimates with boundary traces (see Reference 6, Proposition 4.1). Tataru states this result for operators with real principal part. We will state general Carleman estimates also for elliptic operators and give a rather simple proof relying on Gårding's inequality.

We begin with the definition of a strongly pseudoconvex function [6].

DEFINITION 6.2 *Let $\phi \in C^2(\overline{X})$ and assume that P is either elliptic or has a real principal symbol p. The function ϕ is strongly pseudoconvex with respect to P at $x_0 \in X$ if*

$$\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}(x_0,\xi,\tau) > 0 \text{ on } \left\{\xi \in \mathbf{R}^n, \tau \ge 0, (\xi,\tau) \ne 0 : p_\phi(x_0,\xi,\tau) = 0\right\}.$$

Here $p_\phi(x,\xi,\tau) = p(x,\xi + i\tau\phi'(x))$ and $\{\cdot,\cdot\}$ denotes the Poisson bracket.

REMARK 6.2 In the case $\tau = 0$, the convexity condition has to be interpreted for the limit $\tau \to 0$. It is well known that

$$\lim_{\tau\to 0} \frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}(x_0, \xi, \tau) = \{p, \{p, \phi\}\}(x_0, \xi)$$

for operators with a real principal symbol. In the elliptic case, the condition at $\tau = 0$ is void because $p(x_0, \xi) \neq 0$ for $\xi \neq 0$.

Now we can state the main result of this section.

THEOREM 6.3
Let $P(x, D)$ be a linear differential operator of order m with C^2 coefficients in the principal symbol p and bounded coefficients otherwise. Moreover, assume that ∂X is noncharacteristic with respect to P, and let $\phi \in C^m(\overline{X})$ be a strongly pseudoconvex function with respect to P on X.

1. *If p has real coefficients then there exist two constants c and τ_0 such that*

$$\tau\|e^{\tau\phi}u\|^2_{(m-1,\tau)} \leq c\left[\|e^{\tau\phi}P(x, D)u\|^2_{(0)} + \tau\sum_{j=0}^{m-1}\left|e^{\tau\phi}D_n^j u\right|^2_{m-j-1,\tau}\right] \tag{6.21}$$

for all $u \in H^m(X)$ and $\tau \geq \tau_0$.

2. *If p is elliptic, then there exist two constants c and τ_0 such that*

$$\frac{1}{\tau}\|e^{\tau\phi}u\|^2_{(m,\tau)} \leq c\left[\|e^{\tau\phi}P(x, D)u\|^2_{(0)} + \sum_{j=0}^{m-1}\left|e^{\tau\phi}D_n^j u\right|^2_{m-j-1/2,\tau}\right] \tag{6.22}$$

for all $u \in H^m(X)$ and $\tau \geq \tau_0$.

REMARK 6.3 Carleman estimates were proved for principally normal operators for compactly supported functions (see Reference 3, Chapter 28) in the smooth case and very recently by Tataru (see Reference 7, Theorem 9) in the case of C^2 coefficients. Hörmander's proof relies on the Fefferman-Phong inequality, whereas Tataru uses only the sharp Gårding inequality. Here we focus on elliptic operators and operators with real principal symbol because we do not have a sharp Gårding inequality with boundary terms (see Remark 6.1).

Before we prove the theorem, we need one more result regarding quadratic forms. The following proposition generalizes a result by Sakamoto (see Reference 5, p.127) to nonsmooth symbols (see also Reference 6, Lemma 3.6 for a related result).

PROPOSITION 6.1
Let $p \in C^2S^{m,0}(X \times \mathbf{R}^n)$ and $q \in C^2S^{m-1,0}(X \times \mathbf{R}^n)$ be real symbols. Then there exists a quadratic form F of order $(2m-1, m, -1)$ with symbol

$$f(x, \xi) = \sum_{j=1}^{n}\partial_{x_j}[\partial_{\xi_j}p(x, \xi)q(x, \xi) - p(x, \xi)\partial_{\xi_j}q(x, \xi)] \tag{6.23}$$

such that

$$|2\Im(p(x, D)u, q(x, D)u)_X - F(u, u)_X| \leq c\left[\sum_{j=0}^{m-1}\left|D_n^j u\right|^2_{m-1-j} + \|u\|^2_{(m,-5/4)}\right]$$

for all $u \in H^m(X)$.

PROOF We consider at first the case of smooth symbols. Using the asymptotic expansion of the symbol of the operator $q_k(x, D')^* p_j(x, D')$ (see Reference 3, Theorem 18.1.7, 18.1.8), we compute

$$\begin{aligned}
&2\Im[p(x, D)u, q(u, D)u]_X \\
&\quad = \frac{1}{i}\{(p(x, D)u, q(x, D)u)_X - (q(x, D)u, p(x, D)u)_X\} \\
&\quad = \frac{1}{i}\sum_{j=0}^{m}\sum_{k=0}^{m-1}\left\{\left(p_j(x, D')D_n^j u, q_k(x, D')D_n^k u\right)_X - \left(q_k(x, D')D_n^k u, p_j(x, D')D_n^j u\right)_X\right\} \\
&\quad = \frac{1}{i}\sum_{j=0}^{m}\sum_{k=0}^{m-1}\left\{\left((q_k^* \circ p_j)D_n^j u, D_n^k u\right)_X - \left((p_j^* \circ q_k)D_n^k u, D_n^j u\right)_X\right\} \\
&\quad = \frac{1}{i}\sum_{j,k}\int_X (q_k p_j)D_n^j u\overline{D_n^k u} - (q_k p_j)D_n^k u\overline{D_n^j u} \\
&\qquad + \frac{1}{i}\sum_{j,k}\left\{\frac{1}{i}\left(\partial_{x'}(\partial_{\xi'}q_k p_j)D_n^j u, D_n^k u\right)_X - \frac{1}{i}\left(\partial_{x'}(\partial_{\xi'}p_j q_k)D_n^j u, D_n^k u\right)_X\right\} + R(u, u)_X
\end{aligned}$$

where R is a quadratic form of order $(2m-1, m, -2)$. Now we have a closer look at the first term on the right-hand side. For $j > k$, we have

$$\begin{aligned}
&(q_k p_j)(x, D')D_n^j u\overline{D_n^k u} - (q_k p_j)(x, D')D_n^k u\overline{D_n^j u} \\
&\quad = D_n\left[(q_k p_j)D_n^{j-1}u\overline{D_n^k u} + (q_k p_j)D_n^{j-2}u\overline{D_n^{k+1}u} + \cdots + (q_k p_j)D_n^k u\overline{D_n^{j-1}u}\right] \\
&\qquad - \left[D_n(q_k p_j)D_n^{j-1}u\overline{D_n^k u} + D_n(q_k p_j)D_n^{j-2}u\overline{D_n^{k+1}u} + \cdots + D_n(q_k p_j)D_n^k u\overline{D_n^{j-1}u}\right]
\end{aligned}$$

where the sums consist of $j-k$ terms. Hence, after integration in the normal direction, we obtain

$$\begin{aligned}
&2\Im(p(x, D)u, q(x, D)u)_X \\
&\quad = -\sum_{j>k}\int_{\partial X}\left[(q_k p_j)D_n^{j-1}u\overline{D_n^k u} + (q_k p_j)D_n^{j-2}u\overline{D_n^{k+1}u} + \cdots + (q_k p_j)D_n^k u\overline{D_n^{j-1}u}\right] \\
&\qquad + \sum_{j<k}\int_{\partial X}\left[(q_k p_j)D_n^j u\overline{D_n^{k-1}u} + (q_k p_j)D_n^{j+1}u\overline{D_n^{k-2}u} + \cdots + (q_k p_j)D_n^{k-1}u\overline{D_n^j u}\right] \\
&\qquad + \sum_{j>k}\left(\partial_{x_n}(q_k p_j)D_n^{j-1}u, D_n^k u\right)_X + \cdots + \left(\partial_{x_n}(q_k p_j)D_n^k u, D_n^{j-1}u\right)_X \\
&\qquad - \sum_{j<k}\left(\partial_{x_n}(q_k p_j)D_n^j u, D_n^{k-1}u\right)_X + \cdots + \left(\partial_{x_n}(q_k p_j)D_n^{k-1}u, D_n^j u\right)_X \\
&\qquad - \sum_{j,k}\left\{\left(\partial_{x'}(\partial_{\xi'}q_k p_j)D_n^j u, D_n^k u\right)_X - \left(\partial_{x'}(\partial_{\xi'}p_j q_k)D_n^j u, D_n^k u\right)_X\right\} + R(u, u)_X.
\end{aligned}$$

One can verify that the last three lines represent a quadratic form with the symbol given in Eq. (6.23) modulo a quadratic form of the order $(2m-1, m, -2)$. Hence, the proposition is proved in the case of smooth symbols.

Now we consider symbols with regularity C^2. As in the proof of Theorem 6.1, we will use the technique of symbol smoothing. Given $p_j(x, \xi') \in C^2S^{m-j}(X \times \mathbf{R}^{n-1})$ and $q_k(x, \xi') \in C^2S^{m-k-1}(X \times \mathbf{R}^{n-1})$, we have the decompositions

$$\begin{aligned}
p_j &= p_j^\# + p_j^b \quad \text{where } p_j^\# \in S_{1,\delta}^{m-j} \quad \text{and} \quad p_j^b \in C^2S_{1,\delta}^{m-j-2\delta} \\
q_k &= q_k^\# + q_k^b \quad \text{where } q_k^\# \in S_{1,\delta}^{m-k-1} \quad \text{and} \quad q_k^b \in C^2S_{1,\delta}^{m-k-1-2\delta}
\end{aligned}$$

for some $\delta \in (0, 1]$. Using these decompositions, we have

$$2\Im(pu, qu)_X = G^\#(u, u)_X + G^b(u, u)_X \quad \text{and} \quad F(u, u)_X = F^\#(u, u)_X + F^b(u, u)_X$$

where $G^\#$ is a smooth quadratic form of the order $(2m - 1, m, 0)$ and G^b is a quadratic form of the order $(2m - 1, m, -2\delta)$. The form $G^\#$ will be treated as in the smooth case where we use the asymptotic expansions of the symbols of the operators $q_k^{\#,*} \circ p_j^\#$ and $p_j^{\#,*} \circ q_k^\#$. As in the smooth case, we will need only the first two terms, and thanks to Reference 9, Proposition 1.3D, we guarantee that the remainder is a quadratic form of the order $(2m-1, m, -2)$. Furthermore, $F^b(u, u)_X$ is a quadratic form of the order $(2m - 1, m, -1 - \delta)$ and thus $|F^b(u, u)_X| \leq C\|u\|^2_{(m,-1-\delta/2)}$. Choosing $\delta = 3/4$, we obtain the desired estimate. ∎

REMARK 6.4 In the case of differential operators one can use the theory of differential quadratic forms (see Reference 2, Section 8.2) to prove the same result even in the case of C^1 coefficients.

6.3.1 Proof of Theorem 6.3 in the Case of a Real Principal Symbol

Introducing local coordinates and a partition of unity, it will be sufficient to assume $U \subset X$. Moreover, Carleman estimates for compactly supported functions are well understood, and we assume that $\partial U \cap \partial X \neq \emptyset$. This allows us also to assume that the set U is small enough such that

$$p(x, 0, \xi_n) \neq 0 \quad \text{for } x \in U \quad \text{and} \quad \xi_n \neq 0 \tag{6.24}$$

because the boundary $\partial U \cap \partial X$ is noncharacteristic with respect to P.

Given a strongly pseudoconvex function ϕ, one can derive the inequality

$$c^{-1}\langle\xi\rangle_\tau^{2m} \leq \langle\xi'\rangle_\tau^2 \frac{1}{\tau}\partial_x[\partial_\xi \Re p_\phi \Im p_\phi - \Re p_\phi \partial_\xi \Im p_\phi] + c|p_\phi|^2 \tag{6.25}$$

for $x \in U$ and $(\xi, \tau) \in \mathbf{R}^n \times [0, \infty)$. This follows from the fact that the symbol p_ϕ is homogeneous in (ξ, τ) of the order m and Formula (6.24). Here we observe that $\Re p_\phi \in C^2 S_\tau^{m,0}$ and $\Im p_\phi/\tau \in C^2 S_\tau^{m-1,0}$ because the symbol p is real. Dividing by $\langle\xi'\rangle_\tau^2$ allows us to apply Corollary 6.2 and obtain

$$c^{-1}\|v\|^2_{(m,-1,\tau)} \leq c\|P_\phi v\|^2_{(0,-1,\tau)} + F(v, v)_X + d\sum_{j=0}^{m-1} \left|D_n^j v\right|^2_{m-j-3/2,\tau}$$

for $\tau \geq \tau_0$ (with a larger constant c). The quadratic form F is treated by using Proposition 6.1 with the adjustment for the large parameter τ. Thus,

$$\|v\|^2_{(m,-1,\tau)} \leq C\left[\|P_\phi v\|^2_{(0,-1,\tau)} + \frac{2}{\tau}\Im(\Re P_\phi v, \Im P_\phi v)_X + \sum_{j=0}^{m-1}\left|D_n^j v\right|^2_{m-j-1,\tau} + \|v\|^2_{(m,-5/4,\tau)}\right],$$

and using the inequalities

$$\|P_\phi v\|^2_{(0,-1,\tau)} \leq \frac{1}{\tau^2}\|P_\phi v\|^2_{(0)} \quad \text{and} \quad \|v\|^2_{(m,-5/4,\tau)} \leq \frac{1}{\sqrt{\tau}}\|v\|^2_{(m,-1,\tau)}$$

we get

$$\|v\|^2_{(m,-1,\tau)} \leq C\left[\frac{1}{\tau^2}\|P_\phi v\|^2_{(0)} + \frac{2}{\tau}\Im(\Re P_\phi v, \Im P_\phi v)_X + \sum_{j=0}^{m-1}\left|D_n^j v\right|^2_{m-j-1,\tau}\right].$$

A short calculation gives $2\Im(\Re P_\phi v, \Im P_\phi v)_X = \|P_\phi v\|^2_{(0)} - \|\Re P_\phi v\|^2_{(0)} - \|\Im P_\phi v\|^2_{(0)}$ and hence

$$\|v\|^2_{(m,-1,\tau)} \leq C\left[\frac{1}{\tau}\|P_\phi v\|^2_{(0)} + \sum_{j=0}^{m-1}\left|D_n^j v\right|^2_{m-j-1,\tau}\right]. \tag{6.26}$$

Now we observe that there exist constants c_1 and c_2 such that

$$\|P_\phi(x, D, \tau)v\|^2_{(0)} = \|P(x, D + i\tau\phi'(x))v\|^2_{(0)} + c_1\|v\|^2_{(m-1,\tau)}$$

and

$$\sum_{j=0}^{m-1}\left|D_n^j v\right|^2_{m-j-1,\tau} \leq c_2 \sum_{j=0}^{m-1}\left|(D_n + i\tau\partial_n\phi)^j v\right|^2_{m-j-1,\tau},$$

which modifies the inequality of Eq. (6.26) to

$$\tau\|v\|^2_{(m,-1,\tau)} \leq C\left\{\|P[x, D + i\tau\phi'(x)]v\|^2_{(0)} + \tau\sum_{j=0}^{m-1}\left|[D_n + i\tau\partial_n\phi(x)]^j v\right|^2_{m-j-1,\tau}\right\}.$$

Finally, let $v = e^{\tau\phi}u$ and inequality of Eq. (6.21) follows.

6.3.2 Proof of Theorem 6.3 in the Elliptic Case

The proof is very similar to the previous case. However, we need a couple of modifications. Instead of inequality of Eq. (6.25), we will use

$$c^{-1}\langle\xi\rangle^{2m}_\tau \leq \tau\partial_x[\partial_\xi\Re p_\phi \Im p_\phi - \Re p_\phi \partial_\xi \Im p_\phi] + c|p_\phi|^2. \tag{6.27}$$

This time we will apply Corollary 6.1. However, we have $\Re p_\phi, \Im p_\phi \in C^2 S^{m,0}_\tau$, and hence we need a small modification of Proposition 6.1. This time we have to use the estimate

$$F(v, v)_X \leq 2\Im(\Re P_\phi v, \Im P_\phi v)_X + C\left[\sum_{j=0}^{m-1}\left|D_n^j u\right|^2_{m-1/2-j,\tau} + \|u\|^2_{(m,-1/4,\tau)}\right].$$

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Chapter 7

An Inverse Problem for the Dynamical Lame System with Two Sets of Local Boundary Data

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7.1 Introduction

We are interested in the classical elasticity system

$$\mathbf{A}_e\mathbf{u} = 0 \quad \text{in } Q, \tag{7.1}$$

where

$$\mathbf{A}_e\mathbf{u} = \rho(x)\partial_t^2\mathbf{u} - \mu(x)(\Delta\mathbf{u} + \nabla div\,\mathbf{u}) - \nabla(\lambda(x)div\,\mathbf{u}) - \sum_{j=1}^{3}\nabla\mu\cdot(\nabla u_j + \partial_j\mathbf{u})\mathbf{e}_j$$

for the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ depending on $(x, t) \in Q$, $Q = \Omega \times (-T, T)$. Here Ω is a bounded domain in $\mathbf{R}^3$ with the C^4-boundary. We will assume that the density ρ is in $C^6(\bar{\Omega})$, the Lame parameters μ, λ are in $C^7(\bar{\Omega})$, and

$$\rho(x) > 0, \quad \mu(x) > 0, \quad \lambda(x) > 0 \quad \text{for all } x \in \bar{\Omega}.$$

From the energy integrals methods [5], it is known that for any initial data $\mathbf{u}_0 \in H_{(1)}(\Omega), \mathbf{u}_1 \in H_{(0)}(\Omega)$ and the lateral Dirichlet data $\mathbf{g} \in C([-T, T]; H_{(1)}(\partial\Omega)$ there is a unique (weak) solution $\mathbf{u}(; \rho, \lambda, \mu; (\mathbf{u}_0, \mathbf{u}_1, \mathbf{g}))$ to the system of Eq. (7.1) such that

$$\mathbf{u} = \mathbf{u}_0, \quad \partial_t\mathbf{u} = \mathbf{u}_1 \text{ on } \Omega \times \{0\} \tag{7.2}$$

and

$$\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega \times (-T, T). \tag{7.3}$$

We will use the following.

7.1.1 Notation

$Q_\omega = (-T, T) \times \omega$; C are generic constants depending only on $\Omega, \mathcal{E}, \mathbf{G}, \varphi, \varepsilon_0$, we will mention any additional dependence when needed; bold symbols denote vectors and matrices; $\mathbf{G} = (\mathbf{u}_0, \mathbf{u}_1, \mathbf{g})$ is the initial and boundary conditions for a solution $\mathbf{u}(; \lambda, \mu, \rho; \mathbf{G})$ to the elasticity system of Eq. (7.1); $\partial_j = \frac{\partial}{\partial x_j}$, $\partial_t = \frac{\partial}{\partial t}$, $\nabla \mathbf{u} = (\partial_j u_k)$; $\mathbf{I}_3$ is the identity matrix in $\mathbf{R}^3$; $H_{(k)}(\Omega)$ is the Sobolev space with the norm $\| \|_{(k)}(\Omega)$.

The main subject of this paper is the problem of determining $\rho = \rho(x)$, $\lambda = \lambda(x)$, and $\mu = \mu(x)$ from observed data of $\boldsymbol{u}$ in a subdomain Q_ω, $\omega \subset \Omega$. Finding ρ, λ, μ inside the elastic body from measurements near a part of the boundary $\partial\Omega$ is important in the geophysics and in the material sciences.

In spite of the importance of the inverse problem for the Lamé system (and because of its difficulty), there are only a few papers concerning the mathematical analysis. For the case of many boundary measurements (the given lateral Dirichlet to Neumann map) we refer to Rachele [16] and Yakhno [18], and for the stationary Lamé system we refer to Eskin and Ralston [4] and Nakamura and Uhlmann [15]. Observe that even with all possible boundary measurements, uniqueness theorems for all three elastic patameters are obtained under very restrictive conditions (like complicated conditions in Reference 16 and smallness assumptions in References 4 and 15). Determination of ρ from few boundary measurements was considered by Ikehata, Nakamura, and Yamamoto [7] and Isakov [11], and global uniqueness and stability of recovery of all three elastic parameters was shown in the recent paper of Imanuvilov, Isakov, and Yamamoto [8] when ω is a boundary layer and T is sufficiently large. Uniqueness of the continuation is obtained by Eller, Isakov, Nakamura, and Tataru [3].

In this paper, by adjusting the proof in Reference 8 we obtain uniqueness and conditional stability results when ω is a part of the boundary layer and T is arbitrary. We use the technique originated by Bukhgeim and Klibanov [1]. To utilize it we need a weighted L^2-estimate called a Carleman estimate (e.g., Hörmander [6], Isakov [12], and Tataru [17]). For works on inverse problems for scalar hyperbolic equations using Carleman estimates, we refer to Bukhgeim, Cheng, Isakov, and Yamamoto [2]; Imanuvilov and Yamamoto [9, 10]; Isakov [12]; Isakov and Yamamoto [13]; and Klibanov [14]. The Lamé system of Eq. (7.1) is strongly coupled, and for a long time there was no proper Carleman estimate for this system. Such an estimate was obtained recently by Imanuvilov, Isakov, and Yamamoto [8]. This estimate enables to solve the inverse problem without looking for the second-order derivatives of λ and μ, so we can reduce the numbers of observations.

7.2 Main Result

Let ω be an open subset of Ω. Let $d = inf|x|$ and $D = sup|x|$ over $x \in \omega$. By translating Ω we can assume that

$$D^2 < 2d^2. \tag{7.4}$$

Let

$$Q(\varepsilon) = Q \cap \{\varepsilon < |x|^2 - \theta^2 t^2 - d^2\}, \quad \Omega(\varepsilon) = \Omega \cap \{\varepsilon < |x|^2 - d^2\}. \tag{7.5}$$

We will introduce the conditions

$$-\theta_0 < \frac{x \cdot \nabla a}{a} \text{ on } \Omega(0) \text{ for some } \theta_0 < 1 \tag{7.6}$$

and

$$a^2\theta^2 + 2|\nabla a|\theta d < 1 - \theta_0 \text{ on } \Omega(0) \text{ for some } \theta > 0, \tag{7.7}$$

and the class of the elastic coefficients

$$\mathcal{E} = \left\{ (\lambda, \mu, \rho) : |\lambda|_{C^6(\Omega)} + |\mu|_{C^7(\Omega)} + |\rho|_{C^7(\Omega)} \le M, \varepsilon_0 < \lambda + \mu \text{ on } \bar{\Omega}, \right.$$

$$\left. a = \left(\frac{\rho}{\mu}\right)^{\frac{1}{2}} \quad \text{and} \quad a = \left(\frac{\rho}{2\mu + \lambda}\right)^{\frac{1}{2}} \text{ satisfy Eqs. (7.6) and (7.7))} \right\}. \tag{7.8}$$

We will use two sets of the initial and lateral boundary data $\mathbf{u}_0 \in H_{(8)}(\Omega)$, $\mathbf{u}_1 \in H_{(7)}(\Omega)$, $\mathbf{g} \in C^7([0, T]; H_{(1)}(\Omega)$ satisfying the standard compatibility conditions at $\partial\Omega \times \{0\}$ (of order 7). Differentiating Eq. (7.1) and the lateral boundary conditions of Eq. (7.3) with respect to t, finding the initial conditions for time derivatives from the initial conditions of Eq. (7.2) and the Eq. (7.1), and using energy estimates for the dynamical elasticity system and Sobolev imbedding theorems, we can conclude (see Reference 8) that

$$\left|\partial_t^6 \mathbf{u}(; \rho, \lambda, \mu, \mathbf{G})\right|_0(Q) + \left|\partial^\alpha \partial_t^k \mathbf{u}(; \rho, \lambda, \mu, \mathbf{G})\right|_0(Q) < C \quad \text{when } |\alpha| \le 2, k = 0, \ldots, 4 \tag{7.9}$$

for all elastic parameters in $\mathcal{E}$ and the initial and boundary data of the described regularity.

Denote by $\mathbf{D}$ the 12×7-matrix

$$\begin{pmatrix} \mu\Delta\mathbf{u}_0(;1) + (\lambda+\mu)\nabla(div\,\mathbf{u}_0(;1)) & (div\,\mathbf{u}_0(;1))\mathbf{I}_3 & \nabla\mathbf{u}_0(;1) + (\nabla\mathbf{u}_0(;1))^T \\ \mu\Delta\mathbf{u}_1(;1) + (\lambda+\mu)\nabla(div\,\mathbf{u}_1(;1)) & (div\,\mathbf{u}_1(;1))\mathbf{I}_3 & \nabla\mathbf{u}_1(;1) + (\nabla\mathbf{u}_1(;1))^T \\ \mu\Delta\mathbf{u}_0(;2) + (\lambda+\mu)\nabla(div\,\mathbf{u}_0(;2)) & (div\,\mathbf{u}_0(;2))\mathbf{I}_3 & \nabla\mathbf{u}_0(;2) + (\nabla\mathbf{u}_0(;2))^T \\ \mu\Delta\mathbf{u}_1(;2) + (\lambda+\mu)\nabla(div\,\mathbf{u}_1(;2)) & (div\,\mathbf{u}_1(;2))\mathbf{I}_3 & \nabla\mathbf{u}_1(;2) + (\nabla\mathbf{u}_1(;2))^T \end{pmatrix}.$$

THEOREM 7.1

Let ω satisfy the condition of Eq. (7.4) *and*

$$\overline{Q(0)} \subset Q \cup (-T, T) \times \overline{\omega}. \tag{7.10}$$

Let us assume that (ρ, λ, μ), $(\rho_1, \lambda_1, \mu_1)$ are in $\mathcal{E}$ and that

$$\lambda_1 = \lambda, \mu_1 = \mu \text{ on } \omega. \tag{7.11}$$

Finally, we assume that there exists $\epsilon_0 > 0$ such that

$$\begin{gathered}\text{at any point of } \overline{\Omega} \text{ absolute value of one of} \\ 7 \times 7 \text{ minors of the matrix } \mathbf{D} \text{ is not less than the number } \varepsilon_0.\end{gathered} \tag{7.12}$$

Then there are constants $C = C(\varepsilon)$ and $\gamma = \gamma(\varepsilon) \in (0, 1)$ such that

$$\begin{aligned}&\|\lambda_1 - \lambda\|_{(0)}[\Omega(\varepsilon)] + \|\mu_1 - \mu\|_{(0)}[\Omega(\varepsilon)] + \|\rho_1 - \rho\|_{(0)}[\Omega(\varepsilon)] \\ &\quad \le C \sum_{j=1,2} \|\mathbf{u}[;\lambda_1, \mu_1, \rho_1; \mathbf{G}(;j)] - \mathbf{u}[;\lambda, \mu, \rho; \mathbf{G}(;j)]\|_{(5)}^{\gamma}(Q_\omega).\end{aligned} \tag{7.13}$$

7.2.1 Remarks

We give some examples showing how to apply Theorem 7.1 to some interesting domains provided the elastic coefficients satisfy the monotonicity type conditions of Eqs. (7.6) and (7.7).

First we assume that the ball $|x| < d_1$ is in Ω and D_1 is $sup|x|$ over $x \in \Omega$, let ω be a boundary layer $\Omega \cap \{d_1 < |x|\}$. In Eq. (7.5) we let $d = d_1$. Then $D = D_1$ and the condition of Eq. (7.10) is satisfied when $D^2 - \theta^2 t^2 < d^2$ or when $D_1^2 - d_1^2 < \theta^2 T^2$. In particular, if Ω is a ball of radius D_1 centered at the origin, that given any $T > 0$ we can identify elastic parameters in the boundary layer $D_1^2 - \theta^2 T^2 < |x|^2$.

Now we will consider data of the inverse problem localized in the space. Let $\Omega \subset \{-H < x_n\}$. Let $\Gamma = \partial\Omega \cap \{-H < x_n < -H + h < 0\} \subset \{|x'| < r\}$, where $x' = (x_1, \ldots, x_{n-1}, 0)$. Let ω be $\omega_0 \cap \Omega$, where ω_0 is a neighborhood of Γ. We choose $d^2 = (H - h)^2 + r^2$. Then the condition of Eq. (7.4) is satisfied when $4h < H$, and the condition of Eq. (7.10) is satisfied when $H^2 + r^2 - \theta^2 T^2 < (H - h)^2 + r^2$ or when $2Hh < \theta^2 T^2 + h^2$. Again, for given T it can be achieved for small h. Theorem 7.1 then guarantees uniqueness (and stability) of reconstruction of elastic parameters in the subdomain $\Omega \cap \{x_n < -[(H - h)^2 + r^2]^{\frac{1}{2}}\}$. A more detailed analysis of the uniqueness domain $\Omega(0)$ is given in Reference 12, Section 3.4.

The condition of Eq. (7.12) is somehow restrictive, but it guarantees the independency of the data $[\mathbf{u}_0(;1), \mathbf{u}_1(;1)]$ and $[\mathbf{u}_0(;2), \mathbf{u}_1(;2)]$. For example, as shown in Reference 8 it is satisfied for

$$\mathbf{u}_0(x;1) = \begin{pmatrix} x_1 x_2 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{u}_1(x;1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\mathbf{u}_0(x;2) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\mathbf{u}_1(x;2) = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix},$$

provided $(\rho, \lambda, \mu) \in \mathcal{E}$.

7.3 Proof of Main Theorem

In this proof we will use the following results.

THEOREM 7.2

Assume that the function φ is strongly pseudo-convex with respect to the differential operators $\rho\partial_t^2 - \mu\Delta$, $\rho\partial_t^2 - (2\mu + \lambda)\Delta$ on $\bar{Q}$ for any $(\rho, \lambda, \mu) \in \mathcal{E}$.

Then there is constant $C_1 > 0$ such that for $\tau > C_1$

$$\tau \int_Q e^{2\tau\varphi}(|\mathbf{u}|^2 + |div\,\mathbf{u}|^2 + |curl\,\mathbf{u}|^2) \leq C_1 \int_Q e^{2\tau\varphi}|\mathbf{A}_e\mathbf{u}|^2 \tag{7.14}$$

for all $\mathbf{u} \in H_0^2(Q)$.

A proof of this theorem is given in Reference 8.

LEMMA 7.1
Let φ be given by Eq. (7.18). *Then there exists C such that when $\tau > C$*

$$\int_{\Omega} \tau |w|^2 e^{2\tau\varphi(0,\cdot)} \leq C \int_{\Omega} \sum_{j=1,2,3} |\partial_j w|^2 e^{2\tau\varphi(0,\cdot)} \tag{7.15}$$

for all $w \in H_0^{(1)}(\Omega)$.

This result follows from general theorems on Carleman estimates [6], and in Reference 8 there is a short direct proof.

PROOF OF THEOREM 7.1 We will modify the method of Reference 8.
We set

$$\mathbf{u}(;j) = \mathbf{u}[;\lambda,\mu,\rho;\mathbf{G}(;j)], \qquad \mathbf{u}(;1,j) = \mathbf{u}[;\lambda_1,\mu_1,\rho_1;\mathbf{G}(;j)]$$

and introduce the differences $\mathbf{v}(;j) = \mathbf{u}(;1,j) - \mathbf{u}(;j)$, $F_1 = \rho - \rho_1$, $F_2 = \lambda - \lambda_1$, $F_3 = \mu - \mu_1$. Then

$$\mathbf{A}_e \mathbf{v}(;j) = \mathcal{A}[\mathbf{u}(;1)]\mathbf{F} \quad \text{in } Q, \tag{7.16}$$

where

$$\mathcal{A}(\mathbf{u})\mathbf{F} = -F_1\partial_1^2\mathbf{u} + (F_2+F_3)\nabla(div\,\mathbf{u}) + F_3\Delta\mathbf{u} + div\,\mathbf{u}(F_4,F_5,F_6) + (\nabla\mathbf{u} + \nabla\mathbf{u}^T)(F_7,F_8,F_9)$$

and $(F_4,F_5,F_6) = \nabla F_2$, $(F_7,F_8,F_9) = \nabla F_3$. Because the functions $\mathbf{u}(;1,j)$, $\mathbf{u}(;j)$ have the same initial data,

$$\mathbf{v}(;j) = \partial_t \mathbf{v}(;j) = 0 \quad \text{on } \Omega \times \{0\}. \tag{7.17}$$

∎

We will use

$$\varphi(x,t) = e^{\sigma(|x|^2 - \theta^2 t^2 - d^2)}. \tag{7.18}$$

As in Reference 12, by using the conditions of Eqs. (7.6) and (7.7) one can show that φ is strongly pseudo-convex on $\bar{Q}$ for some (large) σ for any triple of elastic parameters in $\mathcal{E}$.

To use the Carleman estimate of Theorem 7.2, we introduce a cut-off function $\chi \in C_0^\infty(Q)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on $Q(\frac{\varepsilon}{2}) \setminus Q_\omega$ and $\chi = 0$ on $Q \setminus Q(0)$. Differentiating the Eq. (7.16) with respect to t and using time independence of the coefficients of the elasticity system, we obtain

$$\mathbf{A}_e\mathbf{V}(;j) = \mathcal{A}[\mathbf{U}(;j)]\mathbf{F} \quad \text{on } Q,$$

$$\mathbf{V}(;j) = \left[\partial_t^2\mathbf{v}(;j), \partial_t^3\mathbf{v}(;j), \partial_t^4\mathbf{v}(;j)\right], \qquad \mathbf{U}(;j) = \left[\partial_t^2\mathbf{u}(;j), \partial_t^3\mathbf{u}(;j), \partial_t^4\mathbf{u}(;j)\right].$$

Using the Leibnitz formula, we will have

$$\mathbf{A}_e[\chi\mathbf{V}(;j)] = \chi\mathcal{A}[\mathbf{U}(;)]\mathbf{F} + \mathbf{A}(;1)\mathbf{V}(;j),$$

where $\mathbf{A}(;1)$ is a matrix partial differential operator of first order with bounded measurable coefficients depending on χ. Because of the choice of χ, $\mathbf{A}(;1)\mathbf{V}=0$ on $Q(\frac{\varepsilon}{2})\setminus Q_\omega$. By Theorem 7.2,

$$\begin{aligned}\int_{Q(0)}\tau|\chi\mathbf{V}(;j)|^2e^{2\tau\varphi} &\le C\left(\int_Q(\chi^2|\mathcal{A}(\mathbf{U}(;j))\mathbf{F}|^2+|\mathbf{A}(;1)\mathbf{V}(;j)|^2\right)e^{2\tau\varphi}\\ &\le C\left(\int_Q|\mathbf{F}|^2e^{2\tau\varphi}+e^{2\tau\varepsilon_1}+e^{2\tau\Phi}V^2\right),\quad \text{where } \Phi=sup_Q\varphi,\ \varepsilon_1=e^{\frac{\sigma\varepsilon}{2}}.\end{aligned} \tag{7.19}$$

To obtain the last inequality, we break the integration domain for $|\mathbf{A}(;1)\mathbf{V}(;j)|^2$ into $Q(0)\setminus Q(\frac{\varepsilon}{2})$ (where $\varphi\le\varepsilon_1$) and $Q(\frac{\varepsilon}{2})$ and use that $\mathbf{A}(;1)\mathbf{V}(;j)=0$ on $Q(\frac{\varepsilon}{2})\setminus Q_\omega$. On $Q(0)\setminus Q(\frac{\varepsilon}{2})$ we utilize the bound Eq. (7.9), and to get the bound on Q_ω we let

$$V^2=\sum_{j=1}^{2}\|\mathbf{v}(;j)\|^2_{H^5(Q_\omega)}. \tag{7.20}$$

We have

$$\begin{aligned}&\int_\Omega\left|\chi\partial_t^k\mathbf{v}(0,x;j)\right|^2e^{2\tau\varphi(0,x)}dx=-\int_0^T\partial_t\left(\int_\Omega\left|\partial_t^k\mathbf{v}(t,x;j)\right|^2\chi^2e^{2\tau\varphi(t,x)}dx\right)dt\\ &\le\int_Q2\chi^2\left(\left|\partial_t^{k+1}\mathbf{v}(;j)\right|\left|\partial_t^k\mathbf{v}(;j)\right|+\tau\left|\partial_t\varphi\right|\left|\partial_t^k\mathbf{v}(;j)|^2\right)e^{2\tau\varphi}+2\int_{Q(0)\setminus Q(\frac{\varepsilon}{2})}\left|\partial_t^k\mathbf{v}(;j)\right|^2\chi|\partial_t\chi|e^{2\tau\varphi}\\ &\le C\left[\int_Q\tau|\chi\mathbf{V}(;j)|^2e^{2\tau\varphi}+\int_{Q\setminus Q(\frac{\varepsilon}{2})}|\mathbf{V}(;j)|^2e^{2\tau\varphi}\right].\end{aligned}$$

Using that $\chi=1$ on $\Omega(\frac{\varepsilon}{2})\setminus\omega$ and that $\varphi<\varepsilon_1$ on $Q\setminus Q(\frac{\varepsilon}{2})$ from these inequalities and from Eq. (7.19), we yield

$$\int_{\Omega(\varepsilon)}\left|\partial_t^k\mathbf{v}[0,;j)]\right|^2e^{2\tau\varphi(0,)}\le C\left(\int_Q|\mathbf{F}|^2e^{2\tau\varphi}+e^{2\tau\varepsilon_1}+e^{2\tau\Phi}V^2\right). \tag{7.21}$$

On the other hand, from Eq. (7.16) at $t=0$ and from Eq. (7.17), we obtain

$$\rho\partial_t^2\mathbf{v}(0,;j)=\mathbf{a}_{1j}F_1+\mathbf{b}_{1j}F_2+\mathbf{c}_{1j}F_3+\mathbf{B}_{1j}(F_4,F_5,F_6)+\mathbf{C}_{1j}(F_7,F_8,F_9), \tag{7.22}$$

where

$$\begin{aligned}&\mathbf{a}_{1j}=-\partial_1^2\mathbf{u}(0,;j),\qquad \mathbf{b}_{1j}=\nabla[div\mathbf{u}(0,;j)],\qquad \mathbf{c}_{1j}=\Delta\mathbf{u}(;j)+\nabla div\mathbf{u}(0,;j),\\ &\mathbf{B}_{1j}=div\mathbf{u}(0,;j)\mathbf{I}_3,\qquad \mathbf{C}_{1j}=[\nabla\mathbf{u}+(\nabla\mathbf{u})^T](;j).\end{aligned}$$

Differentiating Eq. (7.16) with respect to t and letting $t=0$, we similarly have

$$\rho\partial_t^3\mathbf{v}(0,;j)=\mathbf{a}_{2j}F_1+\mathbf{b}_{2j}F_2+\mathbf{c}_{2j}F_3+\mathbf{B}_{2j}(F_4,F_5,F_6)+\mathbf{C}_{2j}(F_7,F_8,F_9), \tag{7.23}$$

where $\mathbf{a}_{2j},\ldots,\mathbf{C}_{2j}$ are defined similarly to $\mathbf{a}_{1j},\ldots,\mathbf{C}_{1j}$ by replacing $\mathbf{u}$ by $\partial_t\mathbf{u}$. From the system of Eq. (7.1) at $t=0$ and this system differentiated with respect to t and taken at $t=0$, we have

$$\mathbf{a}_{kj}=-\frac{\mu_1}{\rho_1}\Delta\mathbf{u}_k(;1)-\frac{\lambda_1+\mu_1}{\rho_1}\nabla[div\,\mathbf{u}_k(;1)]-div\,[\mathbf{u}_k(;1)]\frac{\nabla\lambda_1}{\rho_1}-(\nabla\mathbf{u}_k+(\nabla\mathbf{u}_k)^T)(;j)\frac{\nabla\mu_1}{\rho_1} \tag{7.24}$$

where $j,k=1,2$.

From the relations of Eqs. (7.22) and (7.23), we have

$$\begin{pmatrix} \mathbf{a}_{11} & \mathbf{B}_{11} & \mathbf{C}_{11} \\ \mathbf{a}_{21} & \mathbf{B}_{21} & \mathbf{C}_{21} \\ \mathbf{a}_{12} & \mathbf{B}_{12} & \mathbf{C}_{12} \\ \mathbf{a}_{22} & \mathbf{B}_{22} & \mathbf{C}_{22} \end{pmatrix} \begin{pmatrix} F_1 \\ F_4 \\ . \\ F_9 \end{pmatrix} = \rho \begin{pmatrix} \partial_t^2 \mathbf{v}(0, ; 1) \\ \partial_t^3 \mathbf{v}(0, ; 1) \\ \partial_t^2 \mathbf{v}(0, ; 2) \\ \partial_t^3 \mathbf{v}(0, ; 2) \end{pmatrix} - \begin{pmatrix} \mathbf{b}_{11} & \mathbf{c}_{11} \\ \mathbf{b}_{21} & \mathbf{c}_{21} \\ \mathbf{b}_{12} & \mathbf{c}_{12} \\ \mathbf{b}_{22} & \mathbf{c}_{22} \end{pmatrix} \begin{pmatrix} F_2 \\ F_3 \end{pmatrix} \text{ on } \Omega. \quad (7.25)$$

Let us consider the matrix in the left side of Eq. (7.25). Adding to the first column of this matrix the second column multiplied by $\frac{\partial_1 \lambda_1}{\rho_1}$, the third column multiplied by $\frac{\partial_2 \lambda_1}{\rho_1}$, the fourth column multiplied by $\frac{\partial_3 \lambda_1}{\rho_1}$ and using Eq. (7.24), we will eliminate from the first column the terms with $div(\mathbf{u})$ without changing 7×7 minors of the whole matrix. Similarly, multiplying the fifth, sixth, and seventh columns by $\frac{\partial_1 \mu_1}{\rho_1}$, $\frac{\partial_2 \mu_1}{\rho_1}$, and by $\frac{\partial_3 \mu_1}{\rho_1}$, we will eliminate the terms with $(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$. After these eliminations we arrive at the matrix $\mathbf{D}$. So, the matrix in the left side of Eq. (7.25) satisfies the condition of Eq. (7.12) and hence

$$\sum_{i=1}^{9} |F_i|^2 \leq C\Big[\sum \left|\partial_t^k \mathbf{v}(0, ; j)\right|^2 + |F_2|^2 + |F_3|^2\Big]$$

where the sum is over $j = 1, 2, k = 2, 3$.

Now from the inequality of Eq. (7.21), we obtain

$$\int_{\Omega(\frac{\varepsilon}{2})} \sum_{i=1}^{9} |F_i|^2 e^{2\tau\varphi(0,)} \leq C\left(\int_{\Omega(\frac{\varepsilon}{2})} \left(F_2^2 + F_3^2\right) e^{2\tau\varphi(0,)} + \int_Q |\mathbf{F}|^2 e^{2\tau\varphi} + e^{2\tau\varepsilon_1} + e^{2\tau\Phi} V^2\right). \quad (7.26)$$

Applying Lemma 7.1 to χF_2, χF_3, we conclude that

$$\int_{\Omega} \chi^2 \left(F_2^2 + F_3^2\right) e^{2\tau\varphi(0,)} \leq \frac{C}{\tau}\left(\int_{\Omega} \chi^2 \left(F_4^2 + \cdots + F_9^2\right) e^{2\tau\varphi(0,)} + \int_{\Omega \setminus \Omega(\frac{\varepsilon}{2})} \left(F_2^2 + F_3^2\right) e^{2\tau\varphi(0,)}\right),$$

where we used that $\nabla(\chi F_2) = \chi(F_4, F_5, F_6) + \nabla\chi F_2$, the similar formula for $\nabla(\chi F_3)$ and the equality $\nabla\chi = 0$ on $\Omega(\frac{\varepsilon}{2}) \setminus \omega$ and the equalities $F_2 = F_3 = 0$ on ω because of the condition of Eq. (7.11). Splitting Ω into $\Omega(\frac{\varepsilon}{2})$ and its complement and arguing as in the derivation of Eq. (7.19), we get

$$\int_{\Omega(\frac{\varepsilon}{2})} \left(F_2^2 + F_3^2\right) e^{2\tau\varphi(0,)} \leq \frac{C}{\tau}\left(\int_{\Omega(\frac{\varepsilon}{2})} \left(F_4^2 + \cdots + F_9^2\right) e^{2\tau\varphi(0,)} + e^{2\tau\varepsilon_1}\right),$$

so choosing τ large we can eliminate F_2, F_3 from the right side of Eq. (7.26) and obtain

$$\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(0,)} \leq C\left(\int_Q |\mathbf{F}|^2 e^{2\tau\varphi} + e^{2\tau\varepsilon_1} + e^{2\tau\Phi} V^2\right).$$

To handle the integral over Q in the right side, we split the integration domain into $Q(\frac{\varepsilon}{2})$ and its complement $Q \setminus Q(\frac{\varepsilon}{2})$. Using that $t\partial_t\varphi \leq 0$, we have

$$\int_{Q(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x) e^{2\tau\varphi(t,x)} dt dx \leq \int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x) e^{2\tau\varphi(0,x)} \left(\int_{-T}^{T} e^{2\tau[\varphi(t,x) - \varphi(0,x)]} dt\right) dx.$$

Because of our choice of the function φ, we have $\varphi(t, x) - \varphi(0, x) < 0$ when $t \neq 0$. Hence, by the Lebesque Theorem the inner integral with respect to t is convergent to 0 when τ goes to infinity. Choosing τ large, we can eliminate the integral over $Q(\frac{\varepsilon}{2})$ and using that $\varphi < \varepsilon_1$ on its complement and that $|\mathbf{F}| < C$ because of the definiton of F and Eq. (7.9) to obtain

$$\int_{\Omega(\varepsilon)} |\mathbf{F}|^2 e^{2\tau\varphi(0,x)} dx \leq C(e^{2\tau\varepsilon_1} + e^{2\tau\Phi} V^2).$$

Dividing both parts of this inequality by $e^{2\tau\varepsilon_2}$ and using that $\varepsilon_2 < \varphi(0,)$ on $\Omega(\varepsilon)$, we finally arrive at

$$\int_{\Omega(\varepsilon)} |\mathbf{F}|^2 \leq C\left(e^{-2\tau(\varepsilon_2-\varepsilon_1)} + e^{2\tau(\Phi-\varepsilon_2)}V^2\right), \quad \varepsilon_2 = e^{\sigma\varepsilon}. \tag{7.27}$$

Observe that $\varepsilon_1 < \varepsilon_2$.

To prove Eq. (7.13), it suffices to assume that $V < \frac{1}{C}$. Then $\tau = \frac{-logV}{\Phi - \varepsilon_1} > C$, and we can use it in Eq. (7.27). Because of this choice,

$$e^{-2\tau(\varepsilon_2-\varepsilon_1)} = e^{2\tau(\Phi-\varepsilon_2)}V^2 = V^{2\frac{\varepsilon_2-\varepsilon_1}{\Phi-\varepsilon_1}}$$

and from Eq. (7.27) we obtain Eq. (7.13) with $\gamma = \frac{\varepsilon_2-\varepsilon_1}{\Phi-\varepsilon_1}$.

The proof is complete.

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Chapter 8

On Singular Perturbations in Problems of Exact Controllability of Second-Order Control Systems

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Abstract This paper is concerned with exact controllability of abstract, second-order conservative linear control systems with a possibly unbounded control operator, a canonical example of which is a linear hyperbolic partial differential equation in a bounded region, controlled at the boundary of that region, whose energy in the uncontrolled dynamics is conserved. It is well known that for convenient control spaces, solutions of such systems exhibit poor regularity below the level of finite energy. We then consider a singular perturbation, or regularization, of the dynamics by introducing weak dissipation with a small parameter ε into the dynamics. In typical examples, the dissipation is through boundary impedance. Solutions of the perturbed system exhibit finite energy for all times and enjoy trace regularity properties not possessed by solutions of the original control system. For each control system, we consider the problem of norm minimal control of that system from the rest state to a designated target state. In each case the optimal control and optimal trajectory may be obtained through an appropriate optimality system. Our main result is that the optimality system associated with the unperturbed dynamics may be obtained in the limit as $\varepsilon \to 0$ of the optimality system associated with the perturbed dynamics. An immediate corollary of our arguments is the so-called Russell's Principle, which states that if the uncontrolled perturbed dynamics are uniformly asymptotically stable, the controlled unperturbed system is exactly controllable.

8.1 Introduction

Let V and H be Hilbert spaces such that $V \hookrightarrow H$. The norm and scalar product in H are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. We identify H with its dual space, denote by V' the dual space of V with respect to H, and by A the Riesz isomorphism of V onto V'. The scalar product in the duality between elements $\phi' \in V'$ and $\phi \in V$ will be denoted by $\langle \phi', \phi \rangle_V$. Scalar products and duality pairings between elements in other spaces will be similarly denoted with subscripts.

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Let U be another Hilbert space and B a bounded linear operator from U into V'. We consider the control system

$$\begin{aligned} \frac{d^2w}{dt^2} + Aw &= Bf \quad \text{in } V' \\ w(0) &= \frac{dw}{dt}(0) = 0, \end{aligned} \tag{8.1}$$

as well as its perturbation

$$\begin{aligned} \frac{d^2w^\varepsilon}{dt^2} + Aw^\varepsilon + \varepsilon BB'\frac{dw^\varepsilon}{dt} &= Bf \quad \text{in } V' \\ w^\varepsilon(0) &= \frac{dw^\varepsilon}{dt}(0) = 0, \end{aligned} \tag{8.2}$$

where $\varepsilon > 0$ and $B' \in \mathcal{L}(V, U)$ is the operator dual to B (identifying U with its dual space). In Eq. (8.1) and Eq. (8.2)

$$f \in L^2(0, T; U) := \mathcal{U}, \quad \text{the control space.} \tag{8.3}$$

When Eq. (8.3) holds, each of the systems in Eq. (8.1) and Eq. (8.2) has a unique solution. It is proved in Reference 4, Chapter 7, that the solution of Eq. (8.2) has regularity

$$\left(\frac{dw^\varepsilon}{dt}, w^\varepsilon\right) \in C([0, T]; \mathcal{H}), \quad B'w^\varepsilon \in H^1(0, T; U), \quad \text{where } \mathcal{H} := H \times V. \tag{8.4}$$

On the other hand, as a rule the regularity of the solution of Eq. (8.1) is strictly weaker than that of Eq. (8.4), and all that can be said in general is that

$$\left(w, \frac{dw}{dt}\right) \in C([0, T]; \mathcal{H}'), \quad \mathcal{H}' = H \times V'. \tag{8.5}$$

Thus, Eq. (8.2) is a *singular perturbation* of Eq. (8.1).

Now let $T > 0$ be given. Define the *control-to-state mappings* L_T and L_T^ε for Eq. (8.1) and Eq. (8.2), respectively, by

$$\begin{aligned} L_T \in \mathcal{L}(\mathcal{U}, V' \times H), \quad L_T f &= \left[\frac{dw}{dt}(T), -w(T)\right], \\ L_T^\varepsilon \in \mathcal{L}(\mathcal{U}, \mathcal{H}), \quad L_T^\varepsilon f &= \left[\frac{dw^\varepsilon}{dt}(T), -w^\varepsilon(T)\right]. \end{aligned} \tag{8.6}$$

The systems of Eq. (8.1) and Eq. (8.2), are *exactly controllable to $\mathcal{H}$ at time T* if $\mathrm{Rg}\,(L_T) \supset \mathcal{H}$, and $\mathrm{Rg}\,(L_T^\varepsilon) = \mathcal{H}$, respectively. If, for some $\varepsilon = \varepsilon_0$, Eq. (8.2) is exactly controllable to $\mathcal{H}$ at time T, by rewriting Eq. (8.2) it is easy to see that the same is true for Eq. (8.1). Suppose that Eq. (8.2) is exactly controllable to $\mathcal{H}$ at time T for every $\varepsilon > 0$ and it is true for some $\varepsilon_0 > 0$. Let $(v_1, w_1) \in \mathcal{H}$ be a designated target state, and consider the optimal control problems

$$\inf_{f \in \mathcal{U}} \|f\|_{\mathcal{U}} \text{ subject to Eq. (8.1)} \quad \text{and} \quad w(T) = w_1, \quad \frac{dw}{dt}(T) = v_1; \tag{8.7}$$

$$\inf_{f \in \mathcal{U}} \|f\|_{\mathcal{U}} \text{ subject to Eq. (8.2)} \quad \text{and} \quad w^\varepsilon(T) = w_1, \quad \frac{dw^\varepsilon}{dt}(T) = v_1. \tag{8.8}$$

Associated with each optimal control problem is a certain *optimality system* (see Section 8.2). The question to be addressed in this paper is whether, and in what manner, the solution of the optimality system associated with Eq. (8.7) can be approximated by the (more regular) solution of the optimality system associated with Eq. (8.8). In particular, we show that as $\varepsilon \to 0$ the optimal controls $f^{\varepsilon}_{\text{opt}}$ for the problems in Eq. (8.8) converge strongly in $\mathcal{U}$ to the optimal control f_{opt} for the problem in Eq. (8.7), and that the optimal trajectories $(w^{\varepsilon}, dw^{\varepsilon}/dt)$ corresponding to $f^{\varepsilon}_{\text{opt}}$ converge in $C([0, T]; \mathcal{H}')$ to the optimal trajectory $(w, dw/dt)$ corresponding to f_{opt}. A key element in the proof is a result which states that the mapping $\varepsilon \mapsto \|(L^{\varepsilon}_T)'\|$ is nonincreasing, where $(L^{\varepsilon}_T)' \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ is the operator dual to L^{ε}_T. As an immediate consequence we obtain *uniform observability estimates* (in ε) for the system adjoint to Eq. (8.2) for $0 < \varepsilon \leq \varepsilon_0$. Another immediate consequence is the so-called Russell's Principle (see Remark 8.1), which asserts that Eq. (8.1) is exactly controllable to $\mathcal{H}$ in some time T if solutions of

$$\frac{d^2w}{dt^2} + Aw + \varepsilon BB'\frac{dw}{dt} = 0$$

are uniformly asymptotically stable in $\mathcal{H}$ for some $\varepsilon > 0$.

We remark that results analogous to, but weaker than, those presented here were previously obtained in Reference 1 in the context of a singular perturbation of Maxwell's system of electromagnetism. The idea of using Eq. (8.2) as a "regularizer" in the approximation of various quantities associated with Eq. (8.1) may also be found in the work of Hendrickson and Lasiecka [2, 3]. In Reference 2, for example, the authors consider the problem of finite dimensional approximation of the solution of the algebraic Riccati equation associated with the problem of optimal feedback stabilization of the system in Eq. (8.1). The regularization in Eq. (8.2) is introduced in order to assure numerical stability in the finite element approximation scheme for the solution of the Riccati equation associated with the optimal feedback control for the system in Eq. (8.2).

8.2 Optimality Systems

Set

$$D_A = \{\phi \in V : A\phi \in H\}, \quad \|\phi\|_{D_A} := \|A\phi\|.$$

Then $A|_{D_A}$ is a positive, self-adjoint operator in H such that $\text{Dom}\,(A^{1/2}) = V$. Furthermore, A extends to an isomorphism $H \mapsto D'_A$ (the dual space of D_A with respect to H) through the definition

$$\langle A\phi, \psi\rangle_{D_A} = \langle \phi, A\psi\rangle, \quad \forall\phi \in H, \quad \psi \in D_A.$$

Consider the system

$$\frac{d^2\phi}{dt^2} + A\phi = 0, \quad \phi(T) = \phi_0, \quad \frac{d\phi}{dt}(T) = \phi_1 \tag{8.9}$$

or the equivalent first-order system

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \phi\end{pmatrix} = \begin{pmatrix}0 & -A\\ I & 0\end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix} := \mathcal{A}\begin{pmatrix}\theta\\ \phi\end{pmatrix}, \quad \begin{bmatrix}\theta(T)\\ \phi(T)\end{bmatrix} = \begin{pmatrix}\phi_1\\ \phi_0\end{pmatrix}.$$

It is standard that $\mathcal{A}$, with $D_{\mathcal{A}} = V \times D_A$, is the generator of a unitary group $S(t)$, $t \in \mathbb{R}$, on $\mathcal{H}$. Thus, for $(\phi_1, \phi_0) \in \mathcal{H}$, Eq. (8.9) has a unique solution with regularity

$$\left(\frac{d\phi}{dt}, \phi\right) \in C([0, \infty); \mathcal{H}), \quad \frac{d^2\phi}{dt^2} \in C([0, \infty); V').$$

On the other hand, if $(\phi_0, \phi_1) \in \mathcal{H}'$, Eq. (8.9) has a unique solution with regularity

$$\left(\phi, \frac{d\phi}{dt}\right) \in C([0, \infty); \mathcal{H}'), \quad \frac{d^2\phi}{dt^2} \in C([0, \infty); D'_A),$$

as may be seen by noting that

$$\phi = \frac{d\psi}{dt}, \quad \frac{d\phi}{dt} = -A\psi \tag{8.10}$$

where ψ is the solution of

$$\frac{d^2\psi}{dt^2} + A\psi = 0, \quad \psi(T) = -A^{-1}\phi_1, \quad \frac{d\psi}{dt}(T) = \phi_0.$$

In this case the mapping $(\phi_0, \phi_1) \mapsto [\phi(t), (d\phi/dt)(t)]$ is a unitary group $S'(t),\ t \in \mathbb{R}$, on $\mathcal{H}'$ whose generator is

$$\mathcal{A}' = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad D_{\mathcal{A}'} = V \times H.$$

The operators $\mathcal{A}'$ and $S'(t)$ are those dual to $\mathcal{A}$ and $S(t)$, respectively. It follows that, for any $(w_0, v_0) \in \mathcal{H}'$ and $f \in \mathcal{U}$, the unique mild solution of the problem

$$\frac{d^2w}{dt^2} + Aw = Bf, \quad w(0) = w_0, \quad \frac{dw}{dt}(0) = v_0$$

is given by

$$\begin{bmatrix} w(t) \\ \dfrac{dw}{dt}(t) \end{bmatrix} = S'(t)\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \int_0^t S'(t-s)\begin{bmatrix} 0 \\ Bf(s) \end{bmatrix} ds \tag{8.11}$$

and that the mapping $((w_0, v_0), f) \mapsto (w, dw/dt)$ is continuous $\mathcal{H}' \times \mathcal{U} \mapsto C([0, T]; \mathcal{H}')$.

Consider now the control-to-state mapping L_T defined in Eq. (8.6). If $(\phi_0, \phi_1) \in V \times H$, a direct calculation gives

$$\langle L_T f, (\phi_0, \phi_1)\rangle_{V\times H} = \langle f, B'\phi\rangle_{\mathcal{U}} \tag{8.12}$$

where ϕ is the solution of Eq. (8.9). Thus, $L'_T(\phi_0, \phi_1) = B'\phi$ if $(\phi_0, \phi_1) \in V \times H$, where L'_T denotes the operator dual to L_T. Assume that Eq. (8.9) is *observable*, that is, that $\mathcal{N}(L'_T) = \{(0, 0)\}$, and let $\mathcal{K}$ denote the completion of $V \times H$ in the *norm*

$$\|(\phi_0, \phi_1)\|_{\mathcal{K}} := \|B'\phi\|_{\mathcal{U}},$$

where ϕ is the solution of Eq. (8.9). Because $B' \in \mathcal{L}(V, U)$, we have $V \times H \hookrightarrow \mathcal{K}$. Clearly Eq. (8.12) remains valid for $(\phi_0, \phi_1) \in \mathcal{K}$ and so

$$L'_T(\phi_0, \phi_1) = B'\phi, \quad \forall(\phi_0, \phi_1) \in \mathcal{K}. \tag{8.13}$$

Thus, $L'_T \in \mathcal{L}(\mathcal{K}, \mathcal{U})$ so that $L_T \in \mathcal{L}(\mathcal{U}, \mathcal{K}')$, where $\mathcal{K}'$ denotes the dual space of $\mathcal{K}$. Because $\mathcal{N}(L'_T) = \{(0, 0)\}$ we have $\mathrm{Rg}\,(L_T) = \mathcal{K}'$. Thus, Eq. (8.1) is exactly controllable to $\mathcal{H}$ if $\mathcal{H} \hookrightarrow \mathcal{K}'$ or, equivalently, if $\mathcal{K} \hookrightarrow \mathcal{H}'$. The latter is the same as *observability estimate*

$$\|(\phi_0, \phi_1)\|_{\mathcal{H}'} \le C_T\|B'\phi\|_{\mathcal{U}}, \quad \forall(\phi_0, \phi_1) \in \mathcal{K}. \tag{8.14}$$

When Eq. (8.14) holds, the solution of Eq. (8.7) may be constructed by the Hilbert uniqueness method introduced in Reference 5 (see also Reference 6). In this method, the optimal control is given by

$$f_{\text{opt}} = B'\phi \tag{8.15}$$

where ϕ is the solution of Eq. (8.9) in which (ϕ_0, ϕ_1) is the unique pair in $\mathcal{K}$ such that

$$\langle (v_1, -w_1), (\phi_0, \phi_1) \rangle_{\mathcal{K}} = \|B'\phi\|^2_{\mathcal{U}}. \tag{8.16}$$

To prove the existence of a unique pair satisfying Eq. (8.16), let $(\phi_0, \phi_1) \in \mathcal{K}$ be arbitrary, set $f = B'\phi \in \mathcal{U}$, and define the mapping $\Lambda : \mathcal{K} \mapsto \mathcal{K}'$ by

$$\Lambda(\phi_0, \phi_1) = \left[\frac{dw}{dt}(T), -w(T)\right] = L_T L'_T(\phi_0, \phi_1).$$

Thus,

$$\langle \Lambda(\phi_0, \phi_1), (\phi_0, \phi_1) \rangle_{\mathcal{K}} = \|B'\phi\|^2_{\mathcal{U}}$$

so that Λ is the canonical isomorphism $\mathcal{K} \mapsto \mathcal{K}'$. Thus, for any $(v_1, w_1) \in \mathcal{K}'$, there is a unique pair $(\phi_0, \phi_1) \in \mathcal{K}$ such that $\Lambda(\phi_0, \phi_1) = (v_1, -w_1)$. Obviously, $(\phi_0, \phi_1) = (L_T L'_T)^{-1}(v_1, -w_1)$, so that $f_{\text{opt}} = L'_T\,(L_T L'_T)^{-1}(v_1, -w_1)$. The *optimality system associated with the optimal control problem* Eq. (8.7) is the coupled system

$$\frac{d^2 w}{dt^2} + Aw = BB'\phi, \quad \frac{d^2\phi}{dt^2} + A\phi = 0$$

(OS)

$$w(0) = \frac{dw}{dt}(0) = 0, \quad \phi(T) = \phi_0, \quad \frac{d\phi}{dt}(T) = \phi_1$$
$$\langle (v_1, -w_1), (\phi_0, \phi_1) \rangle_{\mathcal{K}} = \|B'\phi\|^2_{\mathcal{U}}.$$

The *a priori* regularity of the solution of (OS) is

$$\left(w, \frac{dw}{dt}\right) \in C([0, T]; \mathcal{H}'), \quad \left(\phi, \frac{d\phi}{dt}\right) \in C([0, T]; \mathcal{H}'), \quad B'\phi \in \mathcal{U}.$$

Let us note that in view of the representation Eq. (8.10), the observability estimate in Eq. (8.14) is equivalent to the observability estimate

$$\|(\psi_1, \psi_0)\|_{\mathcal{H}} \le C_T \left\| B' \frac{d\psi}{dt} \right\|_{\mathcal{U}}, \quad \forall (\psi_1, \psi_0) \in \widehat{\mathcal{K}}, \tag{8.17}$$

where $\widehat{\mathcal{K}}$ is the completion of $V \times D_A$ in the norm

$$\|(\psi_1, \psi_0)\|_{\widehat{\mathcal{K}}} = \left\| B' \frac{d\psi}{dt} \right\|_{\mathcal{U}}$$

and where ψ is the solution on $(0, T)$ of

$$\frac{d^2\psi}{dt^2} + A\psi = 0, \quad \psi(T) = \psi_0, \quad \frac{d\psi}{dt}(T) = \psi_1. \tag{8.18}$$

Similarly, for $\varepsilon > 0$ given, the system in Eq. (8.2) is exactly controllable to $\mathcal{H}$ at time T if and only if

$$\|(\phi_0, \phi_1)\|_{\mathcal{H}'} \le C^\varepsilon_T \|B'\phi^\varepsilon\|_{\mathcal{U}} = C^\varepsilon_T \left\| L^\varepsilon_T(\phi_0, \phi_1) \right\|_{\mathcal{H}}, \quad \forall (\phi_0, \phi_1) \in \mathcal{H}', \tag{8.19}$$

where ϕ^ε is the solution of

$$\frac{d^2\phi^\varepsilon}{dt^2} + A\phi^\varepsilon - \varepsilon B d_t(B'\phi^\varepsilon) = 0, \quad \phi^\varepsilon(T) = \phi_0, \quad \frac{d\phi^\varepsilon}{dt}(T) = \phi_1. \tag{8.20}$$

The operator Bd_t in Eq. (8.20) is the mapping $\mathcal{U} \mapsto [H^1(0,T;V)]'$ defined by

$$\langle Bd_t h, \chi\rangle = -\left\langle h, B'\frac{d\chi}{dt}\right\rangle_{\mathcal{U}}, \quad \forall h \in \mathcal{U},\ \chi \in H^1(0,T;V). \tag{8.21}$$

Equation (8.20) has a unique solution with regularity (see the Appendix)

$$\left(\phi^\varepsilon, \frac{d\phi^\varepsilon}{dt}\right) \in C([0,T];\mathcal{H}'), \quad B'\phi^\varepsilon \in \mathcal{U}.$$

Further the solution of Eq. (8.20) may be represented by (see Lemma 8.3)

$$\phi^\varepsilon = \frac{d\psi^\varepsilon}{dt}, \quad \frac{d\phi^\varepsilon}{dt} = -A\psi^\varepsilon, \tag{8.22}$$

where ψ^ε is the solution of

$$\frac{d^2\psi^\varepsilon}{dt^2} + A\psi^\varepsilon - \varepsilon BB'\frac{d\psi^\varepsilon}{dt} = 0, \quad \psi^\varepsilon(T) = -A^{-1}\phi_1, \quad \frac{d\psi^\varepsilon}{dt}(T) = \phi_0. \tag{8.23}$$

When the observability estimate of Eq. (8.19) holds, the solution of Eq. (8.8) is given by

$$f^\varepsilon_{\text{opt}} = B'\phi^\varepsilon \tag{8.24}$$

where ϕ^ε is the solution of

$$\frac{d^2\phi^\varepsilon}{dt^2} + A\phi^\varepsilon - \varepsilon B d_t(B'\phi^\varepsilon) = 0, \quad \phi^\varepsilon(T) = \phi_0^\varepsilon, \quad \frac{d\phi^\varepsilon}{dt}(T) = \phi_1^\varepsilon \tag{8.25}$$

in which $(\phi_0^\varepsilon, \phi_1^\varepsilon)$ is the unique pair in $\mathcal{H}'$ satisfying

$$\left\langle \left(\phi_0^\varepsilon, \phi_1^\varepsilon\right), (v_1, -w_1)\right\rangle_{\mathcal{H}} = \|B'\phi^\varepsilon\|^2_{\mathcal{U}}. \tag{8.26}$$

The *optimality system for the problem* Eq. (8.8) is the coupled system

$$\frac{d^2w^\varepsilon}{dt^2} + Aw^\varepsilon + \varepsilon BB'\frac{dw^\varepsilon}{dt} = BB'\phi^\varepsilon$$
$$\frac{d^2\phi^\varepsilon}{dt^2} + A\phi^\varepsilon - \varepsilon Bd_t(B'\phi^\varepsilon) = 0$$

$(\text{OS})_\varepsilon$

$$w^\varepsilon(0) = \frac{dw^\varepsilon}{dt}(0) = 0, \quad \phi^\varepsilon(T) = \phi_0^\varepsilon, \ \frac{d\phi^\varepsilon}{dt}(T) = \phi_1^\varepsilon$$
$$\left\langle \left(\phi_0^\varepsilon, \phi_1^\varepsilon\right), (v_1, -w_1)\right\rangle_{\mathcal{H}} = \|B'\phi^\varepsilon\|^2_{\mathcal{U}}.$$

Its solution has *a priori* regularity

$$\left(\frac{dw^\varepsilon}{dt}, w^\varepsilon\right) \in C([0,T];\mathcal{H}), \quad B'\frac{dw^\varepsilon}{dt} \in \mathcal{U},$$
$$\left(\phi^\varepsilon, \frac{d\phi^\varepsilon}{dt}\right) \in C([0,T];\mathcal{H}'), \quad B'\phi^\varepsilon \in \mathcal{U}.$$

Finally, we note that in view of the representation in Eq. (8.22), the estimate in Eq. (8.19) is equivalent to the observability estimate

$$\|(\psi_1, \psi_0)\|_{\mathcal{H}} \le C_T^\varepsilon \left\| B' \frac{d\psi^\varepsilon}{dt} \right\|_{\mathcal{U}}, \quad \forall (\psi_1, \psi_0) \in \mathcal{H}, \tag{8.27}$$

where ψ^ε is the solution of

$$\frac{d^2\psi^\varepsilon}{dt^2} + A\psi^\varepsilon - \varepsilon BB' \frac{d\psi^\varepsilon}{dt} = 0, \quad \psi^\varepsilon(T) = \psi_0, \quad \frac{d\psi^\varepsilon}{dt}(T) = \psi_1. \tag{8.28}$$

8.3 Main Results

Assume that the observability estimate in Eq. (8.19) holds for some $\varepsilon_0 > 0$. We show in Corollary 8.1 below that the observability estimates in Eq. (8.14) and Eq. (8.19), for $\varepsilon \in (0, \varepsilon_0]$, likewise hold with the same constant $C_T := C_T^{\varepsilon_0}$. Let $w,\ \phi,\ (\phi_0, \phi_1)$ be the solution of the optimality system (OS), and let $w^\varepsilon,\ \phi^\varepsilon,\ (\phi_0^\varepsilon, \phi_1^\varepsilon)$ be the solution of the optimality system $(\mathrm{OS})_\varepsilon$. Our main result describes how $w^\varepsilon,\ \phi^\varepsilon,\ (\phi_0^\varepsilon, \phi_1^\varepsilon)$ converge to $w,\ \phi,\ (\phi_0, \phi_1)$.

THEOREM 8.1
As $\varepsilon \to 0$, we have

$$\begin{gathered}
B'\phi^\varepsilon \to B'\phi \quad \textit{strongly in } \mathcal{U} \\
\left(\phi_0^\varepsilon, \phi_1^\varepsilon\right) \to (\phi_0, \phi_1) \quad \textit{weakly* in } \mathcal{H}' \\
\left(\phi^\varepsilon, \frac{d\phi^\varepsilon}{dt}\right) \to \left(\phi, \frac{d\phi}{dt}\right) \quad \textit{weakly* in } L^\infty(0, T; \mathcal{H}') \\
\left\| \left[\phi^\varepsilon(t), \frac{d\phi^\varepsilon}{dt}(t)\right] \right\|_{\mathcal{H}'}^2 - \|\left(\phi_0^\varepsilon, \phi_1^\varepsilon\right)\|_{\mathcal{H}'}^2 = O(\varepsilon),\ t \in [0, T] \\
\varepsilon B' \frac{dw^\varepsilon}{dt} \to 0 \quad \textit{strongly in } \mathcal{U} \\
\left(w^\varepsilon, \frac{dw^\varepsilon}{dt}\right) \to \left(w, \frac{dw}{dt}\right) \quad \textit{in } C([0, T]; \mathcal{H}').
\end{gathered} \tag{8.29}$$

The key to the proof of Theorem 8.1 is the following result.

LEMMA 8.1
Let ϕ^ε be the solution of Eq. (8.20) *and ϕ the solution of Eq.* (8.9). *For each $(\phi_0, \phi_1) \in \mathcal{H}'$ and for $0 < \varepsilon_1 < \varepsilon_2$ we have*

$$\|B'\phi^{\varepsilon_1}\|_{\mathcal{U}} \ge \|B'\phi^{\varepsilon_2}\|_{\mathcal{U}}. \tag{8.30}$$

The estimate in Eq. (8.30) *also holds if $\varepsilon_1 = 0$ (with $\phi^0 := \phi$) for each $(\phi_0, \phi_1) \in \mathcal{K}$. In this case, we also have*

$$\begin{gathered}
B'\phi^\varepsilon \to B'\phi \quad \textit{strongly in } \mathcal{U}, \\
\left\| \left(\phi^\varepsilon, \frac{d\phi^\varepsilon}{dt}\right) - \left(\phi, \frac{d\phi}{dt}\right) \right\|_{L^\infty(0,T;\mathcal{H}')} = o(\sqrt{\varepsilon}) \ \textit{ as } \varepsilon \to 0.
\end{gathered}$$

COROLLARY 8.1
If the observability estimate in Eq. (8.19) *holds for some* $\varepsilon_0 > 0$*, then it holds for all* $\varepsilon \in (0, \varepsilon_0]$ *and there is a constant* $C_T > 0$ *independent of* ε *such that*

$$\|(\phi_0, \phi_1)\|_{\mathcal{H}'} \leq C_T \|B'\phi^\varepsilon\|_{\mathcal{U}}, \quad \forall(\phi_0, \phi_1) \in \mathcal{H}'. \tag{8.31}$$

Further the observability estimate of Eq. (8.14) *holds with this same constant.*

In other words, the observability estimates of Eq. (8.19) hold *uniformly* with respect to ε on each interval $(0, \varepsilon_0)$.

PROOF OF LEMMA 8.1 By using the representation of Eq. (8.22) for the solution ϕ^ε and Eq. (8.10) for the solution ϕ, it is seen that Eq. (8.30) is equivalent to showing that

$$\left\|B'\frac{d\psi^{\varepsilon_1}}{dt}\right\|_{\mathcal{U}} \geq \left\|B'\frac{d\psi^{\varepsilon_2}}{dt}\right\|_{\mathcal{U}} \tag{8.32}$$

for each $(\psi_1, \psi_0) \in \mathcal{H}$ and for $0 < \varepsilon_1 < \varepsilon_2$. Set $\Psi = \psi^{\varepsilon_2} - \psi^{\varepsilon_1}$. Then Ψ satisfies

$$\begin{gathered}\frac{d^2\Psi}{dt^2} + A\Psi - \varepsilon_2 BB'\frac{d\Psi}{dt} = (\varepsilon_2 - \varepsilon_1)BB'\frac{d\psi^{\varepsilon_1}}{dt} \\ \Psi(T) = \frac{d\Psi}{dt}(T) = 0.\end{gathered} \tag{8.33}$$

From Eq. (8.53) below, applied to the solution of Eq. (8.33), we obtain

$$\begin{aligned}&\left\|\left[\frac{d\Psi}{dt}(0), \Psi(0)\right]\right\|_{\mathcal{H}}^2 + \varepsilon_2\left\|B'\frac{d\Psi}{dt}\right\|_{\mathcal{U}}^2 + \frac{1}{\varepsilon_2}\left\|(\varepsilon_2 - \varepsilon_1)B'\frac{d\psi^{\varepsilon_1}}{dt} - \varepsilon_2 B'\frac{d\Psi}{dt}\right\|_{\mathcal{U}}^2 \\ &\quad = \frac{(\varepsilon_2 - \varepsilon_1)^2}{\varepsilon_2}\left\|B'\frac{d\psi^{\varepsilon_1}}{dt}\right\|_{\mathcal{U}}^2.\end{aligned} \tag{8.34}$$

One has

$$(\varepsilon_2 - \varepsilon_1)B'\frac{d\psi^{\varepsilon_1}}{dt} - \varepsilon_2 B'\frac{d\Psi}{dt} = \varepsilon_2 B'\frac{d\psi^{\varepsilon_2}}{dt} - \varepsilon_1 B'\frac{d\psi^{\varepsilon_1}}{dt},$$

hence,

$$\left\|B'\frac{d\psi^{\varepsilon_2}}{dt}\right\|_{\mathcal{U}} \leq \frac{1}{\varepsilon_2}\left\|(\varepsilon_2 - \varepsilon_1)B'\frac{d\psi^{\varepsilon_1}}{dt} - \varepsilon_2 B'\frac{d\Psi}{dt}\right\|_{\mathcal{U}} + \frac{\varepsilon_1}{\varepsilon_2}\left\|B'\frac{d\psi^{\varepsilon_1}}{dt}\right\|_{\mathcal{U}}. \tag{8.35}$$

It follows from Eq. (8.34) and Eq. (8.35) that

$$\left\|B'\frac{d\psi^{\varepsilon_2}}{dt}\right\|_{\mathcal{U}} \leq \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}\left\|B'\frac{d\psi^{\varepsilon_1}}{dt}\right\|_{\mathcal{U}} + \frac{\varepsilon_1}{\varepsilon_2}\left\|B'\frac{d\psi^{\varepsilon_1}}{dt}\right\|_{\mathcal{U}} = \left\|B'\frac{d\psi^{\varepsilon_1}}{dt}\right\|_{\mathcal{U}},$$

which proves Eq. (8.10).

Now suppose that $(\phi_0, \phi_1) \in \mathcal{K}$ or, equivalently, that $(\psi_0, \psi_1) \in \widehat{\mathcal{K}}$, set $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon$ and write Ψ^ε in place of Ψ. From (A2) we have

$$\left\|\left[\frac{d\Psi^\varepsilon}{dt}(t), \Psi^\varepsilon(t)\right]\right\|_{\mathcal{H}}^2 + \varepsilon\int_t^T \left[\left\|B'\frac{d\Psi^\varepsilon}{dt}\right\|_U^2 + \left\|B'\frac{d\psi^\varepsilon}{dt}\right\|_U^2\right]dt = \varepsilon\int_t^T \left\|B'\frac{d\psi}{dt}\right\|_U^2 dt. \quad (8.36)$$

Returning to the original variables ϕ and ϕ^ε, Eq. (8.36) is the same as

$$\left\|\left[\Phi^\varepsilon(t), \frac{d\Phi^\varepsilon}{dt}(t)\right]\right\|_{\mathcal{H}'}^2 + \varepsilon\int_t^T \left(\|B'\Phi^\varepsilon\|_U^2 + \|B'\phi^\varepsilon\|_U^2\right) dt = \varepsilon\int_t^T \|B'\phi\|_U^2\, dt \quad (8.37)$$

where $\Phi^\varepsilon = \phi^\varepsilon - \phi$. It follows immediately that $(\phi^\varepsilon, d\phi^\varepsilon/dt) \to (\phi, d\phi/dt)$ in $C([0,T]; \mathcal{H}')$. Further $B'\phi^\varepsilon$ is bounded in $\mathcal{U}$ so that on a sequence $\varepsilon = \varepsilon_n \to 0$, we have $B'\phi^\varepsilon \to g$ weakly in $\mathcal{U}$. Because also $(\phi^\varepsilon, d\phi^\varepsilon/dt) \to (\phi, d\phi/dt)$ in $C([0,T]; \mathcal{H}')$, necessarily $g = B'\phi$, and one has weak convergence $B'\phi^\varepsilon \to B'\phi$ as $\varepsilon \to 0$. Therefore, $\|B'\phi\|_{\mathcal{U}} \le \lim_{\varepsilon\to 0} \|B'\phi^\varepsilon\|_{\mathcal{U}}$. But from Eq. (8.30) we also have the opposite inequality. Therefore, $\|B'\phi\|_{\mathcal{U}} = \lim_{\varepsilon\to 0} \|B'\phi^\varepsilon\|_{\mathcal{U}}$. It follows that $B'\phi^\varepsilon \to B'\phi$ strongly in $\mathcal{U}$, and then, from Eq. (8.37), that

$$\left\|\left[\Phi^\varepsilon(t), \frac{d\Phi^\varepsilon}{dt}(t)\right]\right\|_{\mathcal{H}'} = o(\sqrt{\varepsilon}).$$

□

REMARK 8.1 The observability estimate in Eq. (8.27) is equivalent to the *stability estimate*

$$\left\|\left[\frac{dw^\varepsilon}{dt}(T), w^\varepsilon(T)\right]\right\|_{\mathcal{H}}^2 \le C_T^\varepsilon \int_0^T \left\|B'\frac{dw^\varepsilon}{dt}(t)\right\|_U^2 dt, \quad (8.38)$$

for every $(v_0, w_0) \in \mathcal{H}$, where w^ε is the solution of

$$\frac{d^2w^\varepsilon}{dt^2} + Aw^\varepsilon + \varepsilon BB'\frac{dw^\varepsilon}{dt} = 0, \quad w^\varepsilon(0) = w_0, \quad \frac{dw^\varepsilon}{dt}(0) = v_0. \quad (8.39)$$

The estimate of Eq. (8.38) is in turn equivalent to the uniform exponential stability of the solution of Eq. (8.39). The time T in Eq. (8.38) is then the smallest value for which $\|S_\varepsilon(T)\| < 1$, where $S_\varepsilon(t),\ t \ge 0$, is the contraction semigroup in $\mathcal{H}$ associated with the solution of Eq. (8.39). It follows from Lemma 8.1 that if Eq. (8.39) is uniformly asymptotically stable for some $\varepsilon > 0$, then Eq. (8.1) is exactly controllable to $\mathcal{H}$ in time T. This observation, informally known as "Russell's Principle" [7], although well known, is usually argued by solving a forward problem, then a backward problem, followed by an application of the contraction mapping principle. Our point is that Russell's Principle is an immediate consequence of Lemma 8.1.

PROOF OF THEOREM 8.1 Let $\varepsilon_0 > 0$ be fixed. For $0 < \varepsilon \le \varepsilon_0$, we have from Corollary 8.1

$$\|(\phi_0^\varepsilon, \phi_1^\varepsilon)\|_{\mathcal{H}'} \le C_T \|B'\phi^\varepsilon\|_{\mathcal{U}}. \quad (8.40)$$

By Eq. (8.26) we also have, for any $\alpha > 0$,

$$\|B'\phi^\varepsilon\|_{\mathcal{U}}^2 = \left\langle\left(\phi_0^\varepsilon, \phi_1^\varepsilon\right), (v_1, -w_1)\right\rangle_{\mathcal{H}} \le \frac{1}{2\alpha}\|(v_1, -w_1)\|_{\mathcal{H}}^2 + \frac{\alpha}{2}\left\|\left(\phi_0^\varepsilon, \phi_1^\varepsilon\right)\right\|_{\mathcal{H}'}^2. \quad (8.41)$$

It follows from Eq. (8.40) and Eq. (8.41) that

$$\left\|\left(\phi_0^\varepsilon, \phi_1^\varepsilon\right)\right\|_{\mathcal{H}'} \le C, \quad \|B'\phi^\varepsilon\|_{\mathcal{U}} \le C, \quad 0 < \varepsilon \le \varepsilon_0,$$

for some constant C independent of ε. From Eq. (8.56) below we immediately obtain

$$\left\|\left[\phi^\varepsilon(t),\, \frac{d\phi^\varepsilon}{dt}(t)\right]\right\|^2_{\mathcal{H}'} - \left\|\left(\phi_0^\varepsilon, \phi_1^\varepsilon\right)\right\|^2_{\mathcal{H}'} = O(\varepsilon) \text{ as } \varepsilon\to 0.$$

In addition, on a sequence $\varepsilon = \varepsilon_n$ tending to zero, we have

$$\begin{gathered}\left(\phi^\varepsilon, \frac{d\phi^\varepsilon}{dt}\right) \to \left(\tilde\phi, \frac{d\tilde\phi}{dt}\right) \quad \text{weakly* in } L^\infty(0, T; \mathcal{H}') \\ (\phi_0^\varepsilon, \phi_1^\varepsilon) \to (\tilde\phi_0, \tilde\phi_1) \quad \text{weakly* in } \mathcal{H}' \\ B'\phi^\varepsilon \to \tilde g \quad \text{weakly in } \mathcal{U},\end{gathered} \tag{8.42}$$

where $\tilde\phi$, $(\tilde\phi_0, \tilde\phi_1)$ satisfy

$$\frac{d^2\tilde\phi}{dt^2} + A\tilde\phi = 0, \quad \tilde\phi(T) = \tilde\phi_0, \quad \frac{d\tilde\phi}{dt}(T) = \tilde\phi_1$$

and where $g = B'\tilde\phi$.

We next show that

$$\|B'\tilde\phi\|_{\mathcal{U}} = \|B'\phi\|_{\mathcal{U}}. \tag{8.43}$$

To prove this we first note that $B'\phi^\varepsilon - \varepsilon B'(dw^\varepsilon/dt)$ is an admissible control for the optimal control problem of Eq. (8.7). Because $B'\phi$ is the optimal control for that problem, we have

$$\|B'\phi\|_{\mathcal{U}} \le \left\|B'\phi^\varepsilon - \varepsilon B'\frac{dw^\varepsilon}{dt}\right\|_{\mathcal{U}} \le \|B'\phi^\varepsilon\|_{\mathcal{U}}$$

where we have used the energy identity of Eq. (8.53). Therefore,

$$\|B'\phi\|^2_{\mathcal{U}} \le \lim_{\varepsilon_n\to 0} \|B'\phi^{\varepsilon_n}\|^2_{\mathcal{U}} = \langle (v_1, -w_1), \quad (\tilde\phi_0, \tilde\phi_1)\rangle_{\mathcal{K}} = \langle B'\phi, B'\tilde\phi\rangle_{\mathcal{U}} \tag{8.44}$$

where we have used Eq. (8.12). On the other hand, since $B'\phi^{\varepsilon_n} \to B'\tilde\phi$ weakly in $\mathcal{U}$, we also have

$$\|B'\tilde\phi\|^2_{\mathcal{U}} \le \liminf_{\varepsilon_n\to 0} \|B'\phi^{\varepsilon_n}\|^2_{\mathcal{U}} = \langle B'\phi, B'\tilde\phi\rangle_{\mathcal{U}} \tag{8.45}$$

Equation (8.43) follows from Eq. (8.44) and Eq. (8.45), which also yield

$$\langle (v_1, -w_1), (\tilde\phi_0, \tilde\phi_1)\rangle_{\mathcal{K}} = \|B'\tilde\phi\|^2_{\mathcal{U}}. \tag{8.46}$$

It now follows from Eq. (8.16), Eq. (8.43), and Eq. (8.46) that $(\tilde\phi_0, \tilde\phi_1) = (\phi_0, \phi_1)$, and therefore $\tilde\phi \equiv \phi$, so that the convergence in Eq. (8.42) takes place as $\varepsilon \to 0$. We may further conclude that

$$\|B'\phi\|_{\mathcal{U}} = \lim_{\varepsilon\to 0} \|B'\phi^\varepsilon\|_{\mathcal{U}},$$

hence, $B'\phi^\varepsilon \to B'\phi$ strongly in $\mathcal{U}$ as $\varepsilon\to 0$.

To complete the proof of Theorem 8.1, we next show that

$$\varepsilon B'\frac{dw^\varepsilon}{dt} \to \quad \text{strongly in } \mathcal{U} \tag{8.47}$$

and

$$\left(w^\varepsilon, \frac{dw^\varepsilon}{dt}\right) \to \left(w, \frac{dw}{dt}\right) \text{ in } C([0, T]; \mathcal{H}'). \tag{8.48}$$

In fact, Eq. (8.47) and $B'\phi^\varepsilon \to B'\phi$ imply Eq. (8.48) because, for the solution of Eq. (8.1), the mapping $\mathcal{U} \mapsto C([0,T];\mathcal{H}')$ is continuous.

To prove Eq. (8.47), we decompose the control space according to

$$\mathcal{U} = \mathrm{Ker}\,(L_T) \oplus \mathrm{Rg}\,(L_T') := \mathcal{U}_0 \oplus \mathcal{U}_0^\perp.$$

The following result is classical. For the sake of completeness, we include its simple proof. ∎

LEMMA 8.2

Let $(v_1, -w_1) \in \mathcal{K}'$ be given. Controls $f \in \mathcal{U}_0^\perp$ such that $L_T f = (v_1, -w_1)$ are unique. Furthermore, if $f \in \mathcal{U}_0^\perp$ satisfies $L_T f = (v_1, -w_1)$, then f is the control of minimum norm among all controls $h \in \mathcal{U}$ such that $L_T h = (v_1, -w_1)$.

PROOF Let $f,\ g \in \mathcal{U}_0^\perp$ be such that $L_T f = L_T g$. Then $f - g \in \mathcal{U}_0 \cap \mathcal{U}_0^\perp$, hence $f - g = 0$. Now let h be any control in $\mathcal{U}$ such that $L_T h = (v_1, -w_1)$, and write $h = f + f^\perp$ where $f \in \mathcal{U}_0$ and $f^\perp \in \mathcal{U}_0^\perp$. Because $L_T f = 0$, necessarily $L_T f^\perp = (v_1, -w_1)$ and $\|h\|_{\mathcal{U}}^2 = \|f\|_{\mathcal{U}}^2 + \|f^\perp\|_{\mathcal{U}}^2 \geq \|f^\perp\|_{\mathcal{U}}^2$. Q.E.D.

We write

$$B'\phi^\varepsilon - \varepsilon B' \frac{dw^\varepsilon}{dt} = f_\varepsilon + f_\varepsilon^\perp, \quad f_\varepsilon \in \mathcal{U}_0,\ f_\varepsilon^\perp \in \mathcal{U}_0^\perp.$$

Because $B'\phi^\varepsilon - \varepsilon B'(dw^\varepsilon/dt) \in \mathcal{U}$ and $L_T\left(B'\phi^\varepsilon - \varepsilon B'(dw^\varepsilon/dt)\right) = (v_1, -w_1)$, necessarily $L_T f_\varepsilon^\perp = (v_1, -w_1)$. But also $L_T(B'\phi) = (v_1, -w_1)$ and so, according to Lemma 8.2, we must have $f_\varepsilon^\perp = B'\phi$ for all $\varepsilon \in (0, \varepsilon_0]$. Thus,

$$B'\phi^\varepsilon - \varepsilon B' \frac{dw^\varepsilon}{dt} = f_\varepsilon + B'\phi, \quad \varepsilon \in (0, \varepsilon_0]. \tag{8.49}$$

We next apply the energy identity of Eq. (8.53) to w^ε and obtain

$$\varepsilon \|(v_1, w_1)\|_{\mathcal{H}}^2 + \varepsilon^2 \left\| B' \frac{dw^\varepsilon}{dt} \right\|_{\mathcal{U}}^2 + \left\| B'\phi^\varepsilon - \varepsilon \frac{dw^\varepsilon}{dt} \right\|_{\mathcal{U}}^2 = \|B'\phi^\varepsilon\|_{\mathcal{U}}^2. \tag{8.50}$$

By using Eq. (8.49), we may write Eq. (8.50) as

$$\varepsilon \|(v_1, w_1)\|_{\mathcal{H}}^2 + \varepsilon^2 \left\| B' \frac{dw^\varepsilon}{dt} \right\|_{\mathcal{U}}^2 + \|B'\phi\|_{\mathcal{U}}^2 + \|f_\varepsilon\|_{\mathcal{U}}^2 = \|B'\phi^\varepsilon\|_{\mathcal{U}}^2. \tag{8.51}$$

Because $\|B'\phi^\varepsilon\|_{\mathcal{U}} \to \|B'\phi\|_{\mathcal{U}}$, we immediately obtain Eq. (8.47) from Eq. (8.51). ∎

Appendix

Consider Eq. $(8.2)_1$ with the right-hand side $Bf + F$, where $F \in L^1(0, T; H)$ and with initial data (w_0, v_0) or its equivalent first-order formulation

$$\frac{d}{dt}\begin{pmatrix} v^\varepsilon \\ w^\varepsilon \end{pmatrix} = \mathcal{A}_\varepsilon \begin{pmatrix} v^\varepsilon \\ w^\varepsilon \end{pmatrix} + \begin{pmatrix} Bf \\ 0 \end{pmatrix} + \begin{pmatrix} F \\ 0 \end{pmatrix}, \quad \begin{pmatrix} v^\varepsilon \\ w^\varepsilon \end{pmatrix}(0) = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}, \tag{8.52}$$

where

$$\mathcal{A}_\varepsilon = \begin{pmatrix} -\varepsilon BB' & -A \\ I & 0 \end{pmatrix}.$$

From Reference 4, Chapter 7, it is known that for $\varepsilon > 0$ the operator $\mathcal{A}_\varepsilon$, with

$$D(\mathcal{A}_\varepsilon) = \{(w_1, w_0) \in \mathcal{H} : v_0 \in V, \quad Aw_0 + \varepsilon BB'v_0 \in H\},$$

is the generator of a C_0-semigroup, $S_\varepsilon(t),\ t \geq 0$, of contractions on $\mathcal{H}$. Furthermore, for $(\phi_1, \phi_0) \in \mathcal{H}$, $f \in \mathcal{U}$ and $F \in L^1(0, T; H)$ the unique mild solution of Eq. (8.52) satisfies $B'v^\varepsilon \in \mathcal{U}$, and the map $[(\phi_1, \phi_0), f, F] \mapsto [(v^\varepsilon, w^\varepsilon), B'v^\varepsilon]$ is continuous $\mathcal{H} \times \mathcal{U} \times L^1(0, T; H) \mapsto C([0, T]; \mathcal{H}) \times \mathcal{U}$. When $F = 0$, one has the energy identity

$$\begin{aligned} &\|[v^\varepsilon(t), w^\varepsilon(t)]\|^2_{\mathcal{H}} + \int_0^t [\varepsilon \|B'v^\varepsilon\|^2_U + \frac{1}{\varepsilon}\|f - \varepsilon B'v^\varepsilon\|^2_U]\,dt \\ &\quad = \|(v_0, w_0)\|^2_{\mathcal{H}} + \frac{1}{\varepsilon}\int_0^t \|f\|^2_U\,dt, \quad 0 \leq t \leq T. \end{aligned} \tag{8.53}$$

Let $\mathcal{A}'_\varepsilon$ denote the dual operator of $\mathcal{A}_\varepsilon$. It is the unbounded operator in $\mathcal{H}'$ given by

$$\mathcal{A}'_\varepsilon = \begin{pmatrix} -\varepsilon BB' & I \\ -A & 0 \end{pmatrix},$$
$$D(\mathcal{A}'_\varepsilon) = \{(\phi_0, \phi_1) \in \mathcal{H}' : \phi_0 \in V, \quad \phi_1 - \varepsilon BB'\phi_0 \in H\}.$$

The operator $\mathcal{A}'_\varepsilon$ is the generator of the semigroup $S'_\varepsilon(t)$ dual to $S_\varepsilon(t)$. Therefore, if $\Phi_0 \in \mathcal{H}'$ then $\Phi^\varepsilon(t) = S'_\varepsilon(T - t)\Phi_0$ is the unique mild solution of

$$\frac{d\Phi^\varepsilon}{dt} = -\mathcal{A}'\Phi^\varepsilon, \quad \Phi^\varepsilon(T) = \Phi_0. \tag{8.54}$$

By writing $\Phi^\varepsilon = \begin{pmatrix} \phi^\varepsilon \\ \theta^\varepsilon \end{pmatrix}$, $\Phi_0 = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$, Eq. (8.54) is the same as

$$\begin{aligned} &\frac{d\phi^\varepsilon}{dt} = \varepsilon BB'\phi^\varepsilon - \theta^\varepsilon, \quad \frac{d\theta^\varepsilon}{dt} = A\phi^\varepsilon, \\ &\phi^\varepsilon(T) = \phi_0, \quad \theta^\varepsilon(T) = \phi_1. \end{aligned} \tag{8.55}$$

The following lemma is simple to verify.

LEMMA 8.3
Let $(\phi_0, \phi_1) \in \mathcal{H}'$. Then $(\phi^\varepsilon, \theta^\varepsilon)$ is the solution of Eq. (8.55) *if and only if*

$$\phi^\varepsilon = \frac{d\psi^\varepsilon}{dt}, \quad \theta^\varepsilon = A\psi^\varepsilon$$

where ψ^ε is the solution of

$$\frac{d^2\psi^\varepsilon}{dt^2} + A\psi^\varepsilon - \varepsilon BB'\frac{d\psi^\varepsilon}{dt} = 0$$
$$\psi^\varepsilon(T) = A^{-1}\phi_1, \quad \frac{d\psi^\varepsilon}{dt}(T) = \phi_0.$$

As a consequence of Lemma 8.3, we see that the unique mild solution of Eq. (8.55) satisfies $B'\phi^\varepsilon \in \mathcal{U}$. In addition, it follows from Eq. (8.53) that

$$\|[\phi^\varepsilon(t), \theta^\varepsilon(t)]\|^2_{\mathcal{H}'} + 2\varepsilon \int_t^T \|B'\phi^\varepsilon\|^2_U \, dt = \|(\phi_0, \phi_1)\|^2_{\mathcal{H}'}. \tag{8.56}$$

LEMMA 8.4

Let $(\phi_0, \phi_1) \in \mathcal{H}'$ *and* $(\phi^\varepsilon, \theta^\varepsilon)$ *be the solution of Eq.* (8.55). *Then* ϕ^ε, $[\phi^\varepsilon(0), \theta^\varepsilon(0)]$ *and* $B'\phi^\varepsilon$ *are the unique elements in* $L^\infty(0, T; H)$, $\mathcal{H}'$ *and* $\mathcal{U}$, *respectively, satisfying*

$$\begin{aligned}&\langle[\phi^\varepsilon(0), \theta^\varepsilon(0)], (v_0, w_0)\rangle_{\mathcal{H}} + \langle B'\phi^\varepsilon, f\rangle_{\mathcal{U}} + \int_0^T \langle \phi^\varepsilon(t), F(t)\rangle \, dt \\ &\quad = \langle(\phi_0, \phi_1), [v^\varepsilon(T), w^\varepsilon(T)]\rangle_{\mathcal{H}}, \quad \forall (v_0, w_0) \in \mathcal{H}, \ f \in \mathcal{U}, \ F \in L^1(0, T; H),\end{aligned} \tag{8.57}$$

where $(v^\varepsilon, w^\varepsilon)$ *is the solution of Eq.* (8.55).

The variational Eq. (8.57) amounts to a definition of the solution of Eq. (8.55) by the transposition method. The proof of Lemma 8.4 is by a straightforward computation and is omitted. We remark that existence of a unique triple $\phi^\varepsilon \in L^\infty(0, T; H)$, $(\phi_1^\varepsilon, \theta_1^\varepsilon) \in \mathcal{H}'$, and $g^\varepsilon \in \mathcal{U}$ satisfying

$$\begin{aligned}&\left\langle\left(\phi_1^\varepsilon, \theta_1^\varepsilon\right), (v_0, w_0)\right\rangle_{\mathcal{H}} + \langle g^\varepsilon, f\rangle_{\mathcal{U}} + \int_0^T \langle \phi^\varepsilon(t), F(t)\rangle \, dt \\ &\quad = \langle(\phi_0, \phi_1), [v^\varepsilon(T), w^\varepsilon(T)]\rangle_{\mathcal{H}}, \quad \forall (v_0, w_0) \in \mathcal{H}, \ f \in \mathcal{U}, \ F \in L^1(0, T; H)\end{aligned} \tag{8.58}$$

is a consequence of the fact that the mapping $[(v_0, w_0), f, F) \mapsto (v^\varepsilon(T), w^\varepsilon(T)]$ is continuous from $\mathcal{X} := \mathcal{H} \times \mathcal{U} \times L^1(0, T; H)$ into $\mathcal{H}$; thus, the right-hand side of Eq. (8.58) is a continuous linear functional on $\mathcal{X}$.

Now consider the problem of Eq. (8.20) where the operator Bd_t is defined in Eq. (8.21). If we formally calculate the inner product in H of Eq. (8.20) with w^ε, where $(v^\varepsilon, w^\varepsilon)$ is the solution of Eq. (8.52), and then integrate over $(0, T)$, we obtain

$$\begin{aligned}&\left\langle \left[\phi^\varepsilon(0), -\frac{d\phi^\varepsilon}{dt}(0)\right], (v_0, w_0)\right\rangle_{\mathcal{H}} + \langle B'\phi^\varepsilon, f\rangle_{\mathcal{U}} + \int_0^T \langle \phi^\varepsilon(t), F(t)\rangle \, dt \\ &\quad = \langle(\phi_0, -\phi_1), [v^\varepsilon(T), w^\varepsilon(T)]\rangle_{\mathcal{H}}, \quad \forall (v_0, w_0) \in \mathcal{H}, \ f \in \mathcal{U}, \ F \in L^1(0, T; H).\end{aligned} \tag{8.59}$$

The variational Eq. (8.59) is the same as Eq. (8.57) if we identify $d\phi^\varepsilon/dt$ with $-\theta^\varepsilon$. This justifies the *definition* that for $(\phi_0, \phi_1) \in \mathcal{H}'$, the solution of Eq. (8.20) is $(\phi^\varepsilon, d\phi^\varepsilon/dt) := (\phi^\varepsilon, -\theta^\varepsilon)$, where $(\phi^\varepsilon, \theta^\varepsilon)$ is the unique mild solution of

$$\frac{d\phi^\varepsilon}{dt} = \varepsilon BB'\phi^\varepsilon - \theta^\varepsilon, \quad \frac{d\theta^\varepsilon}{dt} = A\phi^\varepsilon,$$
$$\phi^\varepsilon(T) = \phi_0, \quad \theta^\varepsilon(T) = -\phi_1,$$

in other words, by definition

$$\begin{bmatrix} \phi^\varepsilon(t) \\ -d\phi^\varepsilon/dt(t) \end{bmatrix} = S'_\varepsilon(T - t)\begin{pmatrix} \phi_0 \\ -\phi_1 \end{pmatrix}.$$

According to Lemma 8.3, we may also express $(\phi^\varepsilon, d\phi^\varepsilon/dt)$ by Eq. (8.22), and Eq. (8.23) above.

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Chapter 9

Domain Decomposition in Optimal Control Problems for Partial Differential Equations Revisited

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Abstract We consider some elementary model problems that are taken to be representative of more important models on complex spatial structures. We discuss domain decomposition techniques from the point of view of optimal control in that coupling conditions are viewed as controllability

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constraints. This leads to the notion of virtual controls, which has been introduced by J.L. Lions. We pursue an augmented Lagrangian point of view. By this method the iterative coupling turns into a sequence of PDE control problems. We also provide extensions of the methods to elliptic problems on networked domains. This contribution is in honor of J.E. Lagnese, with whom the author collaborated over the past 15 years. Most of the results of this paper have been obtained in this collaboration.

9.1 Introduction, Notations, and Review of Basic Methods

9.1.1 Introduction

This paper is dedicated to J.E. Lagnese, with whom the author collaborated over the past 15 years. The work done during that period concentrated on problems of controllability and stabilizability, and problems of optimal control for linear and nonlinear hyperbolic problems or more generally for problems of second order in time. A particular emphasis has been on problems defined on networked domains. Networked domains become more and more important in dealing with complex dynamical systems, such as flexible mechatronic structures; fluid-structure interactions; or infrastructures composed of networked pipes, reservoirs, rivers, etc. A major part of the joint work with Lagnese has been devoted to the mathematical modeling and its analytical properties. More recently, however, we have focused on domain decomposition methods for such optimal control problems. The reason is that numerical treatment typically becomes prohibitive if one considers the entire system at once. The systems are composed of elliptic, parabolic, or hyperbolic equations coupled at joints and interfaces; the most natural decomposition is in terms of substructuring (i.e., nonoverlapping domain decompositions). The complex networked system thereby decomposes into substructures that, all by themselves, have a physical meaning. The underlying idea is that, upon decomposition, specialized and validated industrial software on the local level can be used in parallel in order to cope with the entire (global) problem. In an iterative decomposition process one has the possibility of controlling the degree of exactness, in particular using *a posteriori* error estimates. Although second-order methods are known to be fast and accurate when operating in the neighborhood of the solution, they are not particularly robust and may fall far away from the solution. Good guesses of the true solution are not easy to obtain given the complexity of the process. Therefore, first-order methods, in particular gradient-based methods, appear favorable. Moreover, such methods can be viewed as preconditioners for higher-order methods.

Domain decomposition methods have been developed for a large variety of problems in mathematical physics and engineering, on the level of both partial differential equations and the corresponding discretizations. The main work in domain decomposition concentrates on the latter. However, from the point of view of mesh independence and by its own right, it is very important to also have the theory available in the infinite dimensional setting.

Another point is that domain decomposition methods as such do not provide a decomposition of the gradients or the corresponding optimality systems. The main problem is whether the decomposed system itself can be regarded as an optimality system corresponding to an optimal control problem for the substructure.

The present article combines and reviews ideas and results that have been obtained in this direction in collaboration with Lagnese. For the sake of brevity, the class of problems is restricted to elliptic problems. The methods can be and have been obtained for general dynamic problems of hyperbolic or Petrowski type. We refer the reader to the monograph by the author and Lagnese (see Reference 18).

9.1.2 Basic Ideas and Methods

In order to fix ideas, we begin our discussion with a standard elliptic problem. In particular, for easier reference, we consider the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{9.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with piecewise Lipschitz boundary $\partial\Omega$ and its classical generalizations. Let $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $D_{\mathcal{A}} = H^2(\Omega) \cap V$, and $f \in V^*$ (the dual of V). (See, e.g., Grisvard [8] for standard notions and properties of Sobolev spaces.) The problem in Eq. (9.1) is a special case of a class of problems that can be expressed in variational form as follows:

$$a(u, v) = \langle f, v \rangle_V \ \forall\, v \in V, \tag{9.2}$$

where $\langle f, v \rangle_V$ denotes the duality pairing between elements $f \in V^*$ and $v \in V$. By applying the Lax-Milgram lemma to Eq. (9.2) we obtain, for any given $f \in V^*$, a unique solution $u \in V$ to Eq. (9.2).

We now decompose the domain Ω into disjoint subdomains Ω_i, $i = 1, \dots, I$, $I \geq 2$. To this end we introduce the notation

$$\begin{aligned} \bar{\Omega} &= \bigcup_{i=1}^{I} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \\ \Gamma_{ij} &:= \partial\Omega_i \cap \partial\Omega_j = \Gamma_{ji}, \quad \Gamma_i := \partial\Omega_i \cap \partial\Omega \\ \gamma_i &= \bigcup_{j \in \mathcal{I}_i} \Gamma_{ij}, \quad \Gamma := \bigcup_{1 \leq i \neq j \leq I} \Gamma_{ij}, \\ \mathcal{I} &= \{1, \dots, I\}, \quad \mathcal{I}_i := \{j \in \mathcal{I} : \Gamma_{ij} \neq \emptyset\}. \end{aligned} \tag{9.3}$$

With this notation, Γ denotes to the total transmission boundary, γ_i denotes the transmission boundary for Ω_i, and $\mathcal{I}_i$ is the set of indices of adjacent domains for Ω_i. Let us introduce the following spaces associated with the splitting of Eq. (9.3):

$$\begin{aligned} V_i &= \{u \in H^1(\Omega_i) : u = 0 \ \text{ on } \ \Gamma_i\} \\ U_{ij} &= L^2(\Gamma_{ij}) = U_{ji} = L^2(\Gamma_{ji}). \end{aligned} \tag{9.4}$$

Then we may consider the operators $r_i : V \to V_i, r_i u := u|_{\Omega_i}$, and $r_{ji} : V_i \to L^2(\Gamma_{ij}), r_{ji} u_i := u_i|_{\Gamma_{ij}}$. Obviously, r_i, r_{ij} are all linear and bounded. Let $\mathcal{A}_i : V_i \to V_i^*$ be the canonical Riesz isomorphism. The splitting of Eq. (9.3) corresponds to the splitting of the "global" operator $\mathcal{A}$. With this basic notation at hand, we may now review various classical domain decomposition techniques. The material to follow is, however, selective because of our special interest in the relation to controllability questions. We refer the reader to Reference 32 and Reference 12 for a general treatment.

For the sake of simplicity we go back to Eq. (9.1) and consider two subdomains only. The problem in Eq. (9.1) is equivalent to

$$\begin{aligned} -\Delta u_i &= f_i \quad \text{in } \Omega_i \\ u_i &= 0 \quad \text{on } \Gamma_i \\ u_i &= u_j \quad \text{on } \Gamma_{ij} = \Gamma \\ \frac{\partial u_i}{\partial \nu_i} + \frac{\partial u_j}{\partial \nu_j} &= 0 \quad \text{on } \Gamma_{ij} = \Gamma, \end{aligned} \tag{9.5}$$

$i = 1, 2$. In Eq. (9.5) the third and fourth equation constitute so-called transmission conditions. The decomposition of those transmission conditions is at the heart of nonoverlapping domain decomposition methods (DDMs).

One may either directly apply an iterative procedure to Eq. (9.5) by some alternating step between the problems on Ω_i or formulate the transmission conditions of Eq. $(9.5)_3$ and Eq. $(9.5)_4$ in an operator format as follows. We first concentrate on methods that start with $(9.5)_3$. The following procedure parallels the one in Reference 32, Chapter 1.2. We include the presentation for the sake of self consistency. We enforce $(9.5)_3$ by setting $r_{ji}u_i = u_i|_\Gamma = \lambda = u_j|_\Gamma = r_{ij}u_j$:

$$\begin{aligned} -\Delta u_i &= f_i \quad \text{in } \Omega_i \\ u_i &= 0 \quad \text{on } \Gamma_i \\ u_i &= \lambda \quad \text{on } \Gamma, \end{aligned} \tag{9.6}$$

$i = 1, 2$. The problems in Eq. (9.6) can be solved in parallel yielding the unique solutions u_i. Thus, Eq. $(9.5)_4$, the other transmission condition, may be seen as an equality constraint as follows. To this end we decompose $u_i = w_i + z_i$ such that w_i is given by the Dirichlet map D_i and z_i solves the "driven" problem with homogeneous boundary conditions. In order to express Eq. $(9.5)_4$ in operator notation, we need to consider Neumann traces T_i. We also denote by $\mathcal{P}_i := T_i D_i$ the Steklov-Poincaré interface operator for Ω_i and by $\mathcal{P} := \mathcal{P}_1 + \mathcal{P}_2$ the total Steklov-Poincaré operator. Moreover, let g_i denote the Neumann traces of z_i. Then

$$\begin{aligned} \mathcal{P}_1\lambda + \mathcal{P}_2\lambda &= g_1 + g_2 \\ \mathcal{P}\lambda &= g. \end{aligned} \tag{9.7}$$

Solving Eq. (9.5) is equivalent to solving the Steklov-Poincaré Eq. (9.7) on Γ. We proceed to derive a variational format of this equation which then has the potential of being generalized. Let R_i be any extension operator from Γ into V_i (with dense range in V_i) and $\mathcal{R}$ their concatenation. We have $r_{ji}\mathcal{R}_i\mu = \mu$, for all i, j, and hence $r_{ji}\mathcal{R}_i\mu = r_{ij}\mathcal{R}_j\mu$. Equation (9.5) is equivalent to

$$\begin{aligned} &a_i(u_i, v_i) = (f_i, v_i) \quad \forall\, v_i \in V_i^0 \\ &r_{21}u_1 = r_{12}u_2 \\ &a_1(u_1, \mathcal{R}_{1\mu}) + a_2(u_2, \mathcal{R}_{2\mu}) = (f_1, \mathcal{R}_1\mu) + (f_2, \mathcal{R}_2\mu), \quad \forall \mu \in W. \end{aligned} \tag{9.8}$$

Equation (9.7) reads in variational form as

$$\langle \mathcal{P}\lambda, \mu\rangle = \langle g, \mu\rangle, \quad \forall \mu \in W \tag{9.9}$$

which is, in turn, equivalent to Eq. $(9.8)_3$. Thus, any iterative procedure for the solution of the Steklov-Poincaré Eq. (9.7) results in such a procedure for Eq. (9.8), and vice versa. This is one of the basic aspects for iterative nonoverlapping DDMs. The most standard such decompositions are the Dirichlet-Neumann and the Neumann-Neumann methods. Let us display their variational forms based on the decomposition of Eq. (9.8). (See Reference 32.)

ALGORITHM 9.1

Given λ^k at iteration level k:

1. *Solve for $u_1^{k+1} \in V_1$:*

$$\begin{aligned} a_1\left(u_1^{k+1}, v_1\right) &= (f_1, v_1)\, \forall\, v_1 \in V_1^0 \\ u_1^{k+1} &= \lambda^k \quad \text{on } \Gamma \end{aligned}$$

2. *Solve for $u_2^{k+1} \in V_2$:*

$$\begin{aligned} a_2\left(u_2^{k+1}, v_2\right) &= (f_2, v_2)\, \forall\, v_2 \in V_2^0 \\ a_2\left(u_2^{k+1}, \mathcal{R}_2\mu\right) &= (f_2, \mathcal{R}_2\mu) + (f_1, \mathcal{R}_1\mu) \\ &\quad - a_1\left(u_1^{k+1}, \mathcal{R}_1\mu\right), \quad \forall\, \mu \in W \end{aligned}$$

3. *Update l^k: $\lambda^{k+1} = \theta\, r_{12}u_2^{k+1} + (1-\theta)\,\lambda^k,\ \theta > 0.$*

Algorithm 9.1 has an operator format as follows:

$$\lambda^{k+1} = \lambda^k + \theta \mathcal{P}_2^{-1}(-\mathcal{P}\lambda^k + g). \tag{9.10}$$

Moreover, it has a sequential character, as the subproblem on Ω_2 has to be solved before the step can be completed.

The so-called Neumann-Neumann method resolves this difficulty with respect to parallelism at the expense of introducing an additional Neumann solve with the residual of the transmission condition of Eq. $(9.5)_4$ on the right-hand side of the Neumann condition. As we will not use this method, we just display its operator form, which is

$$\lambda^{k+1} = \lambda^k + \theta\left(\delta_1 \mathcal{P}_1^{-1} + \delta_2 \mathcal{P}_2^{-1}\right)(g - \mathcal{P}\lambda^k). \tag{9.11}$$

Both iterations Eq. (9.10) and Eq. (9.11) can be interpreted as preconditioned damped Richardson iterations in the Hilbert space W. See Reference 32 for a detailed discussion of the methods and their discretizations and extensions.

9.2 Domain Decomposition and Optimization

9.2.1 Virtual Controls

The point we want to stress is that the transmission conditions of Eq. $(9.5)_3$ can be interpreted as exact controllability constraints wherein λ is viewed as a "control" function in the control space W or $L^2(\Gamma)$. As it is much more convenient to work in the control space $L^2(\Gamma)$, we take the second approach. We look for solutions λ with minimal norm; see Lions and Pironneau [24]. We may envision various ways to approach the equivalent controllability problem, such as the formulation

$$\inf \frac{1}{2} \sum_{i=1}^{2} |\lambda_i|^2_{L^2(\Gamma)} \tag{9.12}$$

such that

$$a_i(u_i, v_i) = (f_i, v_i) + b_i(\lambda_i, v_i), \quad \forall v_i \in V_i \tag{9.13}$$

$$r_{ji} u_i = r_{ij} u_j, \quad \forall i, j \tag{9.14}$$

$$\lambda_i + \lambda_j = 0, \quad \forall i, j. \tag{9.15}$$

The problem in Eq. (9.12) and Eq. (9.13) is an optimal control problem with equality state and control constraints of Eq. (9.14) and Eq. (9.15), respectively. The optimal controls that realize Equations (9.12) to (9.15) are artificial or *virtual* controls, the objective of the control problem being the continuity of traces along Γ.

In the general case of a multidomain splitting of Eq. (9.3), we define

$$\lambda_i = (\lambda_{ij})_{j \in \mathcal{I}_i} \tag{9.16}$$

and, recalling Eq. (9.4), we denote by

$$U_i = \bigcup_{j \in \mathcal{I}_i} U_{ij}, \qquad U = \bigcup_{i=1}^{I} U_i. \tag{9.17}$$

The requirement of Eq. (9.15) then reads as

$$\lambda_{ij} + \lambda_{ji} = 0, \quad \forall\, j \in \mathcal{I}_i, \tag{9.18}$$

$i = 1, \ldots, I$. Thus, in the case of multidomain splittings, the state and control constrained optimal control problem is given by

$$\begin{aligned}
&\inf \left\{ J(\lambda_i) := \sum_{i=1}^{I} \sum_{j\in\mathcal{I}_i} |\lambda_{ij}|^2_{U_{ij}} \right\} \\
&a_i(u_i, v_i) = (f_i, v_i) + b_i\,(\lambda_i, v_i), \quad \forall\, v_i \in V_i, \\
&r_{ji} u_i = r_{ij} u_j, \quad j \in \mathcal{I}_i, \\
&\lambda_{ij} + \lambda_{ji} = 0, \quad j \in \mathcal{I}_i, \quad i = 1, \ldots, I.
\end{aligned} \tag{9.19}$$

We add that $b_i(\lambda_i, v_i)$ is now replaced with $b_i(\lambda_i, v_i) = \sum_{j\in\mathcal{I}_i} (\lambda_{ij}, r_{ji} v_i)_{U_{ij}}$. Obviously, the problem of Eq. (9.19) is coupled in i via the adjacency structure of the decomposition. The main goal is to derive iterative procedures in order to decouple the constraints of Eq. $(9.19)_3$ and Eq. $(9.19)_4$.

We first consider a two-domain decomposition and eliminate the constraint $\lambda_i + \lambda_j$ by setting $\lambda_1 = -\lambda_2 = \lambda$. Then Eq. (9.19) reads

$$\begin{gathered}
\inf_{\lambda} \frac{1}{2} |\lambda|^2_{L^2(\Gamma)=:U} \\
a_i(u_i, v_i) = (f_i, v_i) + (-1)^{i+1} (\lambda_i,\, r_{ji}\, v_i)\ \forall\, v_i \in V_i, \quad i = 1, 2 \\
r_{21} u_1 = r_{12} u_2.
\end{gathered} \tag{9.20}$$

We introduce a Lagrangian relaxation of Eq. $(9.20)_3$ via a Lagrange multiplier $q \in L^2(\Gamma)$. Thus, we consider the Lagrangian equation

$$\mathcal{L}(\lambda, q) := \frac{1}{2} |\lambda|^2_U + (q, r_{21} u_1 - r_{12} u_2)_U \tag{9.21}$$

and then the saddle-point problem

$$\inf_{\lambda} \sup_{q} \mathcal{L}(\lambda, q) = -\inf_{q} \inf_{\lambda} \mathcal{L}(\lambda, q)$$

subject to Eq. $(9.20)_2$. We define the dual functional $\tilde{J}(q)$ as

$$\tilde{J}(q) := -\inf_{\lambda} \mathcal{L}(\lambda, q). \tag{9.22}$$

As the problem

$$\inf_{\lambda} \{\mathcal{L}(\lambda, q) =: J(q; \lambda)\} \quad \text{subject to Eq. } (9.20)_2 \tag{9.23}$$

admits a unique optimal solution λ^{opt}, the original problem reduces to the unconstrained minimization problem

$$\inf_{q} \{-J(q; \lambda^{opt})\}. \tag{9.24}$$

The gradient of $-J(q; \lambda^{opt})$ is $\nabla[-J(q; \lambda^{opt})] = -(r_{21}u_1 - r_{12}u_2)$. The gradient procedure for the Lagrange multiplier then is

$$q^{k+1} = q^k + p(r_{21}u_1 - r_{12}u_2) \tag{9.25}$$

where

$$a_i^*(p_i, \hat{p}_i) = (-1)^{i+1}\left(q^k, r_{ji}\hat{p}_i\right), \quad \forall\, \hat{p}_i \in V_i, \tag{9.26}$$

$$a_i(u_i, \hat{u}_i) = (-1)^i(r_{21}p_1 - r_{12}p_2, r_{ji}\hat{u}_i), \quad \forall\, \hat{u}_i \in V_i. \tag{9.27}$$

ALGORITHM 9.2

Given q^k,

1. *Solve Eq.* (9.26) *for p_i^k.*
2. *Solve Eq.* (9.27) *for u_i^k.*
3. *Update q^k in Eq.* (9.25).
4. *Go back to Step 1 and continue until finished.*

REMARK 9.1

1. Steps 1 and 2 are forwardly decoupled. Thus, the adjoint problems Eq. (9.26) can be solved in parallel first. Then the forward problems can be solved in parallel.
2. However, Eq. (9.26) and Eq. (9.27) do not correspond separately to an optimality system of an optimal control problem on Ω_i.

Thus, if we eliminate the constraint $\lambda_i + \lambda_j = 0$ in Eq. (9.19) we do obtain an iterative domain decomposition procedure that can be solved in parallel based on an optimization problem, in fact a virtual optimal control problem on Ω, but the local problems to solve on Ω_i do not correspond to local virtual optimal control problems on Ω_i.

It is desirable to directly apply existing optimization software developed for a broad variety of problems on standard domains. Hence, DDMs for optimal control problems that lead to local optimal control problems are preferred. It is obvious that certain extensions, variations, and alternatives may also be considered.

For instance, instead of using a Lagrangian relaxation as in Eq. (9.21), one may introduce an augmented Lagrangian equation

$$\mathcal{L}_r(\lambda, q) := \mathcal{L}(\lambda, q) + \frac{r}{2}\,|r_{21}u_1 - r_{12}u_2|_U^2. \tag{9.28}$$

See Glowinski and Le Tallec [7] for a general discussion of such methods. Again, we can derive a saddle-point type iteration analogous to Algorithm 9.2. Similar remarks as above apply to its parallel features (e.g., there is no decoupling into separate optimization problems).

Yet another variant is to consider just a penalization of the state constraints $r_{21}u_1 - r_{12}u_2$:

$$J_r(\lambda) = \frac{1}{2}\,|\lambda|_U^2 + \frac{r}{2}\,|r_{21}u_1 - r_{12}|^2. \tag{9.29}$$

Gradient procedures associated with Eq. (9.29) have been investigated by Gunzburger and Lee [10] and Gunzburger et al. [9].

Penalty-based iterative domain decomposition procedures are interesting in particular for nonlinear problems, where the existence and regularity of Lagrange multipliers is difficult to establish.

Consequently, we now consider the Lagrangian relaxation of Eq. $(9.19)_3$ *and* Eq. $(9.19)_4$ followed by a gradient procedure. This approach has been outlined in Reference 23. We obtain the following saddle-point problem:

$$\inf_{\lambda \in U} \sup_{\mu, q} \Bigg[\Bigg\{ \frac{1}{2} \sum_i \sum_{j \in \mathcal{I}_i} |\lambda_{ij}|^2_{U_{ij}} + \sum_i \sum_{j \in \mathcal{I}_i} (q_{ji}, r_{ji} u_i - r_{ij} u_j)_{U_{ij}}$$
$$+ \sum_i \sum_{j \in \mathcal{I}_i} (\mu_{ij}, \lambda_{ij} + \lambda_{ji})_{U_{ij}} \Bigg\} =: \mathcal{L}(\lambda, \mu, q) \Bigg] \quad \text{subject to}$$
$$a_i(u_i, v_i) = (f_i, v_i) + b_i(\lambda_i, v_i), \quad \forall\, v_i \in V_i, \tag{9.30}$$

where $\mathcal{L}(\lambda, \mu, q)$ is the Lagrangian equation for Eq. (9.19). Following standard derivations, we arrive at the following gradient procedure.

ALGORITHM 9.3
Given q^k, μ^k,

1. *Solve for p_i^k:*

$$a_i^*\left(p_i^k, \hat{p}_i\right) = \sum_{j \in \mathcal{I}_i} \left(q_{ji}^k - q_{ij}^k, r_{ji} \hat{p}_i\right)_{U_{ij}} \quad \forall\, \hat{p}_i \in V_i$$

2. *Solve for u_i^k:*

$$a_i\left(u_i^k, \hat{u}_i\right) = (f_i, \hat{u}_i) - \sum_{j \in \mathcal{I}_i} \left(r_{ji} p_i^k + \mu_{ij}^k + \mu_{ji}^k, r_{ji} \hat{u}_i\right)_{U_{ij}} \quad \forall\, \hat{u}_i \in V_i$$

3. *Update q^k, μ^k:*

$$\begin{aligned} q_{ji}^{k+1} &= q_{ji}^k + p\left(r_{ji} u_i^k - r_{ij} u_j^k\right) \\ \mu_{ij}^{k+1} &= \mu_{ij}^k + p\left(\lambda_{ij}^k + \lambda_{\mathbf{j}}^k\right), \\ \textit{with } \lambda_{ij}^k &= r_{ji}\, p_i^k + \mu_{ij}^k + \mu_{ji}^k. \end{aligned}$$

4. *Go back to Step 1 and continue until finished.*

This procedure has been proposed by Lions and Pironneau [23] for overlapping domain decompositions, in case no constraint on λ_{ij} applies. They remark that the method can be extended also to the nonoverlapping case. The arguments above provide the details.

In retrospective, going back to the original saddle-point formulation of Eq. (9.30) of Eq. (9.19), we realize that the cost, for a given q^k, μ^k can be written as

$$J(\mu^k, q^k; \lambda) := \sum_i J_i(\mu^k, q^k, \lambda_i). \tag{9.31}$$

The algorithm iterates the optimality conditions for the optimal control problem

$$\inf_{\lambda_i} J_i(\mu^k, q^k; \lambda_i) \quad \text{subject to } a_i(u_i, \hat{u}_i) = (f_i, \hat{u}_i) + \sum_{j \in \mathcal{I}_i} (\lambda_{ij}, r_{ji} \hat{u}_i), \quad \forall\, \hat{u}_i \in V_i. \tag{9.32}$$

REMARK 9.2 The Lagrangian relaxation of Eq. (9.30) followed by the saddle-point iteration given in Algorithm 9.3 results in a decomposition of the global optimality system into a local system that is itself an optimality system of the virtual optimal control problem of Eq. (9.32), defined on subdomain i. This is precisely the paradigm we wish to emphasize.

REMARK 9.3 It is important to note that if we eliminate the control constraints $\lambda_{ij} + \lambda_{ji} = 0$ first and then optimize with respect to the remaining virtual controls (just one function on $\Gamma_{ij} = \Gamma_{ji}\ \forall\ i, j$ such that Ω_i and Ω_j are adjacent), then the optimality conditions are coupled.

It appears to be very natural to stabilize and possibly accelerate Algorithm 9.3. The typical procedure is to consider an augmented Lagrangian equation instead of the original Lagrangian Eq. (9.30), where one introduces a penalty parameter $r > 0$ and considers

$$\mathcal{L}_r(\lambda, \mu, q) := \mathcal{L}(\lambda, \mu, q) + \frac{r}{2}\left\{ \sum_{i,j} |r_{ji}u_i - r_{ij}u_j|^2_{U_{ij}} + \sum_{i,j} |\lambda_{ij} + \lambda_{ji}|^2_{U_{ij}} \right\}. \tag{9.33}$$

It turns out, however, that the corresponding optimality systems would not decouple into a local penalized problem.

In order to derive an augmented Lagrangian method that leads to a decomposed set of optimization problems in each individual domain, we further relax the transmission condition $r_{ji}u_i = r_{ij}u_j$ by introducing yet another virtual control η

$$r_{ji}u_i = \eta_{ij}\ \forall\ i, j, i \in \mathcal{I},\ j \in \mathcal{I}_i, \tag{9.34}$$

$\eta_{ij} = \eta_{ji}$. We may then consider the Lagrangian relaxations of Eq. (9.34). Let us first look at the two-domain case with $\lambda_{ij} = -\lambda_{ji} = \lambda$ on Γ, $\eta_{ij} = \eta_{ji} = \eta$. We obtain the unconstrained problem

$$\inf_{(u_i),\eta}\ \sup_{q} \left\{ \mathcal{L}_\beta(u, \eta; q) =: \sum_{i=1}^{2} \frac{1}{2} a_i(u_i, u_i) - (f_i, u_i) + \sum_{i,j} (q_{ji}, r_{ji}u_i - \eta) + \frac{\beta}{2} \sum_{i,j} |r_{ji}u_i - \eta|^2 \right\}. \tag{9.35}$$

Now Eq. (9.35) is a saddle-point problem for which we may use various extensions of the classical Uzawa algorithm. See Glowinski and Le Tallec [7] for a very nice presentation of the general subject.

We use a fractional step method with respect to the gradient procedure for the dual problems as advocated in Reference 7 (ALG3), which results in the following iteration:

ALGORITHM 9.4

Given η^{k-1}, q^k,

1. *Solve for u_i^k, $i = 1, 2$*

$$a_i\left(u_i^k \hat{u}_i\right) + \beta\left(r_{ji}u_i^k, r_{ji}\hat{u}_i\right) = (f_i, \hat{u}_i) - \left(q_{ji}^k - \beta\eta^{k-1}, r_{ji}\hat{u}_i\right) \quad \forall\ \hat{u}_i \in V_i$$

2. *Update:*

$$q_{ji}^{k+\frac{1}{2}} = q_{ji}^k + \rho_k\left(r_{ji}u_i^k - \eta^{k-1}\right)$$

3. *Compute η^k as*

$$\eta^k = \frac{1}{2\beta}\left(q_{ij}^{k+\frac{1}{2}} + q_{ji}^{k+\frac{1}{2}}\right) + \frac{1}{2}\left(r_{ji}u_i^k + r_{ij}u_j^k\right)$$

4. *Update:*

$$q_{ji}^{k+1} = q_{ji}^{k+\frac{1}{2}} + \rho_k\big(r_{ji}u_i^k - \eta^k\big).$$

See Glowinski and Le Tallec [7] for a discussion of the convergence properties of this version of the saddle-point algorithm. For $\beta = \rho,\ \forall\, k$, we can completely eliminate the Lagrangian multipliers and obtain an iteration based on the boundary conditions

$$\frac{\partial u_i^{k+1}}{\partial \nu_{A_i}} + \beta r_{ji} u_i^{k+1} = -\frac{\partial u_j^k}{\partial \nu_{A_j}} + \beta r_{ij} u_j^k \tag{9.36}$$

for $i, j = 1, 2$. As in the previous, more general setting, consistency of this algorithm is easily assessed.

The decoupling of the original transmission conditions according to Eq. (9.36) is due to Lions [25].

9.2.2 The Basic Algorithm of P.-L. Lions

For easier reference, we now summarize the algorithm of Lions [25] described at the end of the last section and show convergence of the algorithm.

ALGORITHM 9.5
Given u_i^n, $\frac{\partial}{\partial \nu_i} u_i^n$ $i = 1, 2$ compute u_i^{n+1}, $\frac{\partial}{\partial \nu_i} u_i^{n+1}$, $i = 1, 2$ according to

$$-\Delta u_i^{n+1} = f_i \quad \text{in } \Omega_i \tag{9.37}$$

$$u_i^{n+1} = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \tag{9.38}$$

$$\frac{\partial}{\partial \nu_i} u_i^{n+1} + \beta u_i^{n+1} = -\frac{\partial}{\partial \nu_j} u_j^n + \beta u_j^n \quad \text{on } \Gamma,\ i, j = 1, 2. \tag{9.39}$$

We summarize in the following theorem.

THEOREM 9.1
Let $u \in H_0^1 \cap H^{\frac{3}{2}}(\Omega)$ and $u_i^n \in V_i$, $i = 1, 2$ be the solution of the global and the local problem, respectively, with

$$\lambda_{ij}^{n-1} = (1-\varepsilon)\left(-\frac{\partial}{\partial \nu_j} u_j^{n-1} + \beta u_j^{n-1}\right) + \varepsilon\left(\frac{\partial}{\partial \nu_i} u_i^{n-1} + \beta u_i^{n-1}\right) \tag{9.40}$$

and $\varepsilon \in [0, 1)$, $\lambda_{ij}^0 \in L^2(\Gamma)$ given. Then

$$\|u^n - u\|_{H^1} = \left(\sum_{i=1}^{2} \left\|u_i^n - u_i\right\|_{H^1(\Omega_i)}^2\right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty, \tag{9.41}$$

and the transmission conditions hold in the sense that, as $n \to \infty$,

$$\big(u_1^n - u_2^n\big)|_\Gamma \to 0 \quad \text{strongly in } L^2(\Gamma),$$

$$\frac{\partial u_1^n}{\partial \nu_1} + \frac{\partial u_2^n}{\partial \nu_2} \to 0 \quad \text{weakly in } L^2(\Gamma).$$

From the point of view of numerical simulations, the fact that the update of Eq. (9.39) involves a Neumann derivative of u_j^n on Γ is not very convenient. We can reformulate Eq. (9.39) as

$$\frac{\partial u_i^{n+1}}{\partial \nu_i} + \beta u_i^{n+1} = g_i^{n+1} \quad g_i^{n+1} = 2\beta u_j^n - g_j^n. \tag{9.42}$$

But

$$g_i^{n+1} = 2\beta u_j^n - g_j^n = -\frac{\partial u_j^n}{\partial \nu_j} - \beta u_j^n + 2\beta u_j^n = -\frac{\partial u_j^n}{\partial \nu_j} + \beta u_j^n.$$

Thus, Eq. (9.39) is equivalent to Eq. (9.42). This variant has been used by Deng [3], where a proof of convergence without a relaxation parameter can be found.

9.3 General Elliptic Problems and More General Splittings

9.3.1 The Problem Setting

In this section we consider elliptic problems more general than the Poisson problem studied in Section 9.2.2 in order to fix ideas. Let $\mathcal{A}$ be the differential operator of second order in $\Omega \subset \mathbb{R}^d$ given by

$$\mathcal{A}u = -\sum_{k,\ell} \frac{\partial}{\partial x_k}\left[a_{k\ell}(x)\,\frac{\partial u}{\partial x_\ell}\right] + c(x)\,u, \tag{9.43}$$

with Ω as above, $c(\cdot) \in L^\infty(\Omega)$, $c \geq 0$ a.e, $a_{ij} \in L^\infty(\Omega)$, and piecewise C^1, $a_{k\ell} = a_{\ell k}$ a.e. such that $A := (a_{ij})$ satisfies the standard coercivity assumptions.

We associate with $\mathcal{A}$ on Ω the bilinear form

$$a_\Omega(u, v) = \int_\Omega \left[\sum_{k,\ell} a_{k\ell}(x)\,\frac{\partial u}{\partial x_k}\,\frac{\partial u}{\partial x_\ell} + c(x)uv\right] dx \tag{9.44}$$

and denote the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)_\Omega$. The canonical weak formulation then has a unique solution, according to the Lax-Milgram lemma.

We decompose Ω into $I\,(\geq 2)$ arbitrary disjoint subdomains Ω_i, $i = 1, \ldots, I$, as in Eq. (9.3). For the sake of simplicity, we will assume that the subdomains are connected and the areas Γ_{ij} meet the boundary $\partial\Omega$ at right angles in order to have convex subdomains. This assumption can be relaxed, however. We also assume the regularity

$$u|_{\Omega_i} \in H^{3/2}(\Omega_i) \tag{9.45}$$

and set

$$\frac{\partial u}{\partial \nu_{A_i}} := \sum_{k,\ell}\left[\alpha_{k\ell}(x)\,\frac{\partial u}{\partial x_\ell}\right]\nu_{ik}, \quad i = 1, \ldots, I, \tag{9.46}$$

$\nu_i = (\nu_{i1}, \ldots, \nu_{id})$, and $A_i := A|_i$. We generalize Algorithm 9.5 to the following algorithm.

ALGORITHM 9.6
Given u_i^n, $\frac{\partial u_i^n}{\partial \nu_{A_i}}$ *on* Γ_{ij}, $1 \le i \ne j \le I$, *with* u_i^n, $\frac{\partial u_i^n}{\partial \nu_i} \in L^2(\Gamma_{ij})$,

1. *Compute* u_i^{n+1}, $i = 1, \ldots, I$ *according to*

$$\begin{aligned} \mathcal{A}_i\, u_i^{n+1} &= f_i \quad \text{in } \Omega_i, \\ u_i^{n+1} &= 0 \quad \text{on } \Gamma_i \\ \frac{\partial u_i^{n+1}}{\partial \nu_{A_i}} + \beta_{ij} u_i^{n+1} &= -\frac{\partial u_j^n}{\partial \nu_{Aj}} + \beta_{ij} u_j^n := \lambda_{ij}^n \quad \text{on } \Gamma_{ij} \end{aligned} \tag{9.47}$$

2. $n \to n+1$, *go to Step 1.*

We can prove the following result.

THEOREM 9.2
Let $u \in H_0^1 \cap H^{\frac{3}{2}}(\Omega)$ *with* $\frac{\partial u}{\partial \nu_{A_i}}|_{\Gamma_{ij}} \in L^2(\Gamma_{ij})$, $1 \le i \ne j \le I$, *be the solution to the global problem of Eq.* (9.2) *associated with Eq.* (9.44) *and* $u_i^n \in H^1_{\Gamma_i}(\Omega)$, $i = 1, \ldots, I$ *the solution to the local problems*

$$\begin{aligned} \mathcal{A}_i u_i^n &= f_i \quad \text{in } \Omega_i \\ u_i^n &= 0 \quad \text{on } \Gamma_i \\ \frac{\partial u_n^i}{\partial \nu_{A_i}} + \beta_{ij} u_i^n &= (1-\varepsilon)\left(-\frac{\partial u_j^{n-1}}{\partial \nu_{Aj}} + \beta_{ij} u_j^{n+1}\right) \\ &\quad + \varepsilon\left(\frac{\partial u_i^{n-1}}{\partial \nu_{A_i}} + \beta_{ij} u_i^{n-1}\right) \end{aligned} \tag{9.48}$$

with $\varepsilon \in [0, 1)$. *If*

$$\frac{\partial u_i^0}{\partial \nu_{A_i}} + \beta_{ij} u_i^0 \in L^2(\Gamma_{ij}), \quad 1 \le i \ne j \le m, \tag{9.49}$$

then

$$\|u^n - u\|_{H^1} = \left(\sum_{i=1}^{I} \left\|u_i^n - u_i\right\|^2_{H^1(\Omega_i)}\right)^{\frac{1}{2}} \to 0, \quad u \to \infty. \tag{9.50}$$

9.3.2 An *a Posteriori* Error Estimate

The convergence results established so far guarantee convergence. However, for a numerical implementation, and in particular for adaptivity with respect to the splitting, we need reasonable stopping criteria, first on the infinite dimensional level. Moreover, the question how the parameters of the algorithms have to be chosen in order to obtain fast convergence is not answered. Otto and Lube [31] and Lube et al. [26] have obtained *a posteriori* estimates for the algorithm of this section at least for the two-domain case. We adopt the notation of Section 4.

Consider the problem $a_\Omega(u, v) = L(v),\ \forall\, v \in H_0^1(\Omega) = V$. The DDM of the last section gives at iteration level $n+1$:

$$a_{\Omega_i}\big(u_i^{n+1}, v_i\big) + \int_\Gamma \beta_i u_i^{n+1} v_i \, d\Gamma = \int_\Gamma \lambda_j^n v_i \, d\Gamma + L_i(v_i), \tag{9.51}$$

$$\lambda_i^{n+1} = (\beta_i + \beta_j) u_j^{n+1} - \lambda_j^n \quad \text{on } \Gamma. \tag{9.52}$$

We consider the errors

$$e_i^n = u_i^n - u_i, \quad \eta_i^n = \lambda_i^n - \lambda_i, \tag{9.53}$$

and define

$$\begin{aligned} C(\beta_1, A, c) :&= \frac{2C(\|\beta_1\|_{L^\infty}, \|A\|_{L^\infty}, \|c\|_{L^\infty})}{\min\{m_1, m_2\}} \\ &= \min\{m_1, m_2\}^{-1} \ \max\left\{C_W \|tr_1\| \, \|\beta_1\|_{L^\infty}, \right. \\ &\qquad \left. \max\{\|A\|_{L^\infty}, \|c\|_{L^\infty}\} \cdot \left\|tr_2^{-1}\right\|\right\}. \end{aligned} \tag{9.54}$$

THEOREM 9.3

Let u be a solution of the global problem of Eq. (9.2) *associated with Eq.* (9.44) *and* u_i^{n+1} *be solutions of Eq.* (9.51) *and Eq.* (9.52) *for* $i = 1, 2$. *Let the errors* e_i^n, η_i^n, $i = 1, 2$, *be given as in Eq.* (9.53). *Then we have the a posteriori estimate*

$$\left\|e_1^{n+1}\right\|_{V_1} + \left\|e_2^n\right\|_{V_2} \le C(\beta_1, a, c)\left\|u_2^n - u_1^{n+1}\right\|_{H_{00}^{\frac{1}{2}}(\Gamma)}. \tag{9.55}$$

REMARK 9.4

1. Note that the error estimate is not symmetric with respect to the domains. But this asymmetry can easily be removed by starting at domain 2 and then combing the estimates.
2. The estimate of Eq. (9.55) allows one to optimize the error bound $C(\beta, a, c)$ with respect to the transmission coefficients β_i appearing in the Robin data. This is a major advantage because the convergence results do not give any hint in this direction.

9.3.3 Interpretation as a Damped Richardson Iteration

To fix ideas, we consider the simplest problem as in Eq. (9.38) and Eq. (9.39) We decompose u_i as follows. Let $G_i(f_i) =: z_i$ solve

$$\begin{aligned} -\Delta z_i &= f_i \quad \text{in } \Omega \\ z_i &= 0 \quad \text{on } \partial\Omega \cap \partial\Omega_i \\ \frac{\partial}{\partial \nu_i} z_i + \beta z_i &= 0 \quad \text{on } \Gamma, \quad i = 1, 2. \end{aligned} \tag{9.56}$$

Moreover, let $w_i := R_i \lambda_{ij}$ solve

$$\begin{aligned} -\Delta w_i &= 0 \quad \text{in } \Omega \\ w_i &= 0 \quad \text{on } \partial\Omega \cap \partial\Omega_i \\ \frac{\partial}{\partial \nu_i} w_i + \beta w_i &= \lambda_{ij} \quad \text{on } \Gamma, \ i = 1, 2, \ j \neq i. \end{aligned} \tag{9.57}$$

Finally, define $S_i(\lambda_{ij}) = w_i|_\Gamma$. The operator S_i is an example of a Steklov-Poincaré-type operator, and $u_i = w_i + z_i$ solves the original problem if $\lambda_{ij} = -\frac{\partial}{\partial \nu_j} u_j + \beta u_j, \ i \neq j$. We may then establish the following relation:

$$\lambda_{ij} + \lambda_{ji} - 2\beta S_j(\lambda_{ji}) = 2\beta G_j(f_j)_{\Gamma_{ij}} \tag{9.58}$$

$i, j = 1, 2, i \neq j$.

Equation (9.58) can be interpreted as a Steklov-Poincaré-type equation to be solved on the common interface. This interpretation of the nonoverlapping domain decomposition of Lions seems to not have been recognized in the literature. For easier reference, let us denote

$$\lambda := \lambda_{ij}, \quad \mu = \lambda_{ji}.$$

Then we may rewrite Eq. (9.58) as follows:

$$\mathcal{A} := \begin{bmatrix} I & -(2\beta\, S_2 - I) \\ -(2\beta\, S_1 - I) & I \end{bmatrix}$$

$$x := \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \quad b = \begin{pmatrix} 2\beta\, G_2\, f_2|_\Gamma \\ 2\beta\, G_1\, f_1|_\Gamma \end{pmatrix}.$$

Then Eq. (9.58) is the same as

$$\mathcal{A}x = b,$$

which is solved iteratively as follows for some $\theta \in \mathbb{R}$:

$$x^{n+1} = x^n - \theta(\mathcal{A}x^n - b). \tag{9.59}$$

It is seen that the damped Richardson iteration of Eq. (9.59) with damping parameter θ is precisely the underrelaxed Robin-type iteration considered in Section 4, where we set

$$\mathcal{T}(\lambda, \mu) = (2\beta u_2|_\Gamma - \mu, \quad 2\beta u_1|_\Gamma - \lambda) \tag{9.60}$$

and recall that $g_1 = \lambda$, $g_2 = \mu$.

9.3.4 A Schur Operator Formulation

The matrix representation of the Robin-type iteration in the two-domain case (discrete or continuous) will have the pattern

$$\begin{pmatrix} A_1 & 0 & B_{11} & 0 \\ 0 & A_2 & 0 & B_{22} \\ 0 & C_{12} & I & I - 2\beta S_2 \\ C_{21} & 0 & I - 2\beta S_1 & I \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \lambda_{12} \\ \lambda_{21} \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{9.61}$$

which, for more general splittings, extends to

$$\left(\begin{array}{ccc|ccc} A_1 & & 0 & B_{11} & \dots & B_{1q} \\ 0 & A_2 & 0 & B_{21} & \dots & B_{2q} \\ \vdots & & & \vdots & & \\ 0 & & A_p & B_{p1} & \dots & B_{pq} \\ \hline C_{11} & & C_p & \Sigma_1 & & 0 \\ C_{21} & & C_{2p} & 0 & \Sigma_2 & 0 \\ C_{q1} & & C_{qp} & 0 & & \Sigma_q \end{array}\right) \tag{9.62}$$

where the matrices (B_{ij}) and (C_{jk}) have a sparsity pattern according to the adjacency structure of the subdomains.

The structure of the operator in (9.62) remains after discretization. We note that the Robin-type decomposition discussed here differs from the more standard splittings in the structure of the Schur operator, which admits the typical format

$$\left(\begin{array}{ccc|c} A_1 & & & A_{1\Sigma} \\ & \ddots & & \vdots \\ & & A_p & A_{p\Sigma} \\ \hline A_{\Sigma,1} & \dots & A_{\Sigma,p} & A_{\Sigma\Sigma} \end{array}\right) \tag{9.63}$$

with the obvious definitions $(B_{i1} \dots B_{iq}) =: A_{i,\Sigma}, i = 1, \dots, p, (C_{1i}^T, \dots, C_{qi}^T)^T =: A_{\Sigma,i}, A_{\Sigma\Sigma} := \Sigma$. After discretization, the structure remains as in (9.63). The matrix representation (9.63) is the paradigm for most of the known domain decomposition techniques, whereas the special sparsity structure of the Robin method resulting in (9.62) does not seem to have been noticed in the literature.

It is important to note that equations with matrices according to (9.63) have been shown to possess the structure of the hierarchical matrices, or H-matrices, introduced by Dryja and Hackbusch [6]; see also Reference 11 for the particular situation of (9.63). Corresponding equations, after discretization, can be very effectively solved in order $O(n)$, where n is the order of the matrix, by direct solving.

9.4 A Serial One-Dimensional Problem

9.4.1 The Steklov-Poincaré Equation

To further explore the structure of the iteration and the Schur operator equation, we consider a serial one-dimensional situation. To this end, we decompose the interval $(0, \ell)$ into $N + 1$ subintervals

$$0 = \ell_0 < \ell_1 < \cdots < \ell_N < \ell_{N+1} = \ell,$$

and we set $L_i := (\ell_i, \ell_{i+1})$, $i = 0, \dots, N$. Let us assume that at $0 = \ell_0$, we have a Dirichlet condition, whereas at $\ell_{N+1} = \ell$ a Neumann condition is imposed. Let $u_i = u|_{L_i}$ be the restriction of u to L_i. Thus,

$$u_0(\ell_0) = 0, \quad u_N'(\ell_{N+1}) = 0. \tag{9.64}$$

The transmission conditions at the interfaces, represented by the breakpoints ℓ_i, $i = 1, \dots, N$, are

$$u_i(\ell_{i+1}) = u_{i+1}(\ell_{i+1}), \quad i = 0, \dots, N-1, \tag{9.65}$$

$$u_i'(\ell_{i+1}) = u_{i+1}'(\ell_{i+1}), \quad i = 0, \dots, N-1. \tag{9.66}$$

Obviously, the standard Robin–Robin-type domain decomposition procedure leads one to consider the relaxation of Eq. (9.65) and Eq. (9.66) as usual:

$$u_i'(\ell_{i+1}) + \beta u_i(\ell_{i+1}) = \lambda_{i,i+1} = u_{i+1}'(\ell_{i+1}) + \beta u_{i+1}(\ell_{i+1}), \quad i = 0, \dots, N \tag{9.67}$$

and

$$-u_i'(\ell_i) + \beta u_i(\ell_i) = \lambda_{i,i-1} = -u_{i-1}'(\ell_i) + \beta u_{i-1}(\ell_i), \quad i = 1, \ldots, N. \tag{9.68}$$

On the bounding intervals $i = 0, i = N$, we have to solve

$$\begin{aligned} -u_0'' + c_0 u_0 &= f_0 \quad \text{in } L_0 \\ u_0(0) &= 0 \\ u_0'(\ell_1) + \beta u_0(\ell_1) &= \lambda_{0,1} \end{aligned} \tag{9.69}$$

and

$$\begin{aligned} -u_N'' + c_N u_N &= f_N \quad \text{in } L_N \\ u_N'(\ell_{N+1}) &= 0 \\ -u_N'(\ell_N) + \beta u_N(\ell_N) &= \lambda_{N,N-1}, \end{aligned} \tag{9.70}$$

respectively, whereas in L_i $i = 2, \ldots, N-1$, we have

$$\begin{aligned} -u_i'' + c_i u_i &= f_i \quad \text{in } L_i \\ -u_i'(\ell_i) + \beta u_i(\ell_i) &= \lambda_{i,i-1} \\ u_i'(\ell_{i+1}) + \beta u_i(\ell_{i+1}) &= \lambda_{i,i+1}. \end{aligned} \tag{9.71}$$

Problems of Eq. (9.69), Eq. (9.70), and Eq. (9.71) are to be decomposed according to $u_i = z_i + w_i$ $i = 0, \ldots, N$ where z_i solves Eq. (9.69), Eq. (9.70), and Eq. (9.71) with $\lambda_{0,1} = \lambda_{i,i-1} = \lambda_{i,i+1} = \lambda_{N,N-1} = 0$, and w_i satisfies Eq. (9.69), Eq. (9.70), and Eq. (9.71) with $f_i = 0$. Define $z_i =: G_i f_i, i = 0, \ldots, N$

$$\lambda_i = \begin{pmatrix} \lambda_{i,i-1} \\ \lambda_{i,i+1} \end{pmatrix}, \quad i = 1, \ldots, N-1, \tag{9.72}$$

$$\mathcal{S}_i(\lambda_i) = \begin{bmatrix} w_i(\ell_i) \\ w_i(\ell_{i+1}) \end{bmatrix}, \quad i = 1, \ldots, N-1. \tag{9.73}$$

We introduce the elementary matrices $a^{ij}_{\ell k} = \delta_{i,\ell}\, \delta_{j,k}$, $E_{ij} = (a^{ij}_{\ell k})_{\ell,k}$ and define

$$\begin{aligned} h_i &:= 2\beta[E_{12} G_{i-1}(f_{i-1})(\ell_i) + E_{21} G_{i+1}(f_{i+1})(\ell_{i+1})] \\ \Gamma_{i,i+1} &:= 2\beta(E_{21} S_{i+1} - E_{11}) \\ \Gamma_{i,i-1} &:= 2\beta(E_{12} S_{i-1} - E_{22}). \end{aligned} \tag{9.74}$$

Then the recursion of λ can be written as

$$-\Gamma_{i,i-1}\, \lambda_{i-1} + \lambda_i - \Gamma_{i,i+1}\, \lambda_{i+1} = h_i, \tag{9.75}$$

$i = 2, \ldots, N-2$. Similar relations hold for final and initial values.

In summary, the Steklov-Poincaré equation associated with this serial problem (i.e., Eq. (9.75)), can be written as

$$\begin{pmatrix} I & -\Gamma_{12} & & & & \\ -\Gamma_{21} & I & -\Gamma_{22} & & & \\ & & \cdots & & & \\ & & & \Gamma_{N-1,N-2} & I & -\Gamma_{N-1,N} \\ & & & & -\Gamma_{N,N-1} & I \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{N-1} \\ \lambda_N \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N-1} \\ h_N \end{pmatrix}. \tag{9.76}$$

We may write Eq. (9.76) compactly as

$$[\mathcal{I} - \mathcal{G}]\, \Lambda = H. \tag{9.77}$$

We may use the methods of Ascher et al. [1] to solve this system. In particular, we may use a single shooting step on each interval L_i, $i = 0, \ldots, N$. Note that this can be done in parallel. Even though in the one-dimensional context of a decomposition in space this observation may not seem relevant, it turns out that in optimal control of hyperbolic problems the analogous time domain decomposition will benefit from parallelism. In any case, we will obtain as a numerical solution a discrete analog $\mathcal{G}_\Delta$, H_Δ of $\mathcal{G}$, and H given by Eq. (9.77) and Eq. (9.76), respectively, such that

$$(\mathcal{I} - \mathcal{G}_\Delta)\, \Lambda = H_\Delta. \tag{9.78}$$

The idea is to precondition the damped Richardson iteration of Eq. (9.77) by rewriting it as

$$\Lambda^{n+1} - \mathcal{G}_\Delta\, \Lambda^{n+1} = -\mathcal{G}_\Delta\, \Lambda^n + \Lambda^n - \Theta[(\mathcal{I} - \mathcal{G})\, \Lambda^n - H]$$

or as

$$\Lambda^{n+1} = \Lambda^n - \Theta(\mathcal{I} - \mathcal{G}_\Delta)^{-1}\, ((\mathcal{I} - \mathcal{G})\, \Lambda^n - H). \tag{9.79}$$

The interpretation of Eq. (9.79) is very close to the one given by Lions et al. [22] and Maday and Turinici [27]. Indeed, replacing the right-hand side of Eq. (9.79) by the residual

$$res^n = (\mathcal{I} - \mathcal{G})\, \Lambda^n - H, \tag{9.80}$$

we propagate the error on the coarse grid (using a one-step shooting or a two-point discretization), do an updating step as Eq. (9.79), and perform a local solution on a fine grid (or even an exact solution in this setting of a one-dimensional problem) to obtain the new residuals. By its nature, the process is completely parallel.

9.5 Distributed Control of Elliptic Problems

9.5.1 The Optimal Control Problem and its Corresponding Optimality System

We now concentrate on optimal control problems. To fix ideas, let us begin with the most standard elliptic optimal control problem:

$$\begin{cases} \min_{u \in U} \displaystyle\int_\Omega \left(|w(u) - w_d|^2 + \nu |u|^2\right) dx \\ \quad -\Delta w = f + u \quad \text{in}\, \Omega \\ \quad\quad w = 0 \quad \text{on}\, \partial\Omega, \end{cases} \tag{9.81}$$

where $U \subset L^2(\Omega)$ is closed and convex, $\Omega \subset \mathbf{R}^n$, $f \in L^2(\Omega)$. There is a unique solution to Eq. (9.81). The optimality system is given by:

$$\begin{cases} -\Delta w = f + u \quad \text{in } \Omega \\ -\Delta p = w - w_d \quad \text{in } \Omega \\ w = 0 = p \quad \text{on } \partial\Omega \\ \displaystyle\int_\Omega (p + \nu u)(v - u)dx \geq 0, \quad \forall v \in U. \end{cases} \tag{9.82}$$

In particular, for $U = L^2(\Omega)$ the inequality in Eq. (9.82) turns into an equality which then provides $u = -\frac{1}{\nu} p$.

9.5.2 Domain Decomposition: a Complex Reformulation

To obtain a reasonable decomposition procedure of Eq. (9.82), Benamou [2] and Després [4] developed the following embedding into a Helmholtz problem with complex coefficient $k \in \mathbb{C}$ as follows. Let z be the (complex valued) solution of

$$\begin{cases} -\Delta z + kz = g \quad \text{in } \Omega \\ z = 0 \quad \text{on } \partial\Omega \\ z \in H_0^1(\Omega; \mathbf{C}) \end{cases} \tag{9.83}$$

and observe that for

$$z = w + \frac{\mathbf{i}}{\sqrt{\nu}} p, \quad k = -\frac{\mathbf{i}}{\sqrt{\nu}}, \quad g = f - \frac{\mathbf{i}}{\sqrt{\nu}} w_d,$$

Eq. (9.83) is equivalent to Eq. (9.82) for $U = L^2(\Omega)$. Hence, a convergent domain decomposition of Eq. (9.83) will result in such an algorithm for Eq. (9.82). We are, therefore, entirely in the context of *virtual controls*. This observation leads one to suggest a Robin-type algorithm following the lines of the preceding sections. To this end, let us consider the complex model problem

$$\begin{cases} -\Delta z + kz = f \quad \text{in } \Omega \\ \dfrac{\partial z}{\partial \nu} + \beta z = h \quad \text{on } \Gamma_1 \\ z = 0 \quad \text{on } \Gamma_0 \end{cases} \tag{9.84}$$

with $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $k, \beta \in \mathbb{C}$, $f \in L^2(\Omega; \mathbb{C})$, $h \in L^2(\Gamma_{1j}, \mathbb{C})$. We are interested in solutions $z \in H^1(\Omega; \mathbb{C})$ of Eq. (9.84).

THEOREM 9.4 (Benamou [2])
Let $f \in L^2(\Omega; \mathbb{C})$, $h \in L^2(\Gamma_1; \mathbb{C})$ be given. Assume that $k, \beta \in \mathbb{C}$ satisfies

$$\mathrm{Re}(\beta) \geq 0, \quad \mathrm{Im}(\beta)\,\mathrm{Im}(k) \geq 0; \; \mathrm{Im}(k) + \mathrm{Im}(\beta) \neq 0.$$

Then the problem in Eq. (9.84) *admits a unique weak solution z in* $H^1(\Omega; \mathbb{C})$.

With this result at hand, we may now consider the following domain decomposition procedure, which, for the sake of simplicity, we write down for two subdomains in an unrelaxed form.

The general case may then be handled as in the preceding sections. Let $z_i^n \in H^1(\Omega; \mathbb{C})$ be the unique weak solution of

$$\begin{cases} -\Delta z_i^n + k z_i^n = f & \text{in } \Omega_i \\ \dfrac{\partial z_i^n}{\partial \nu_i} + \beta z_i^n = g_i^n & \text{on } \Gamma \\ z_i^n = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \end{cases} \tag{9.85}$$

with $f = L^2(\Omega; \mathbb{C})$ and $g_i^n \in L^2(\Gamma; \mathbb{C})$, $i = 1, 2$. Update the inhomogeneity g_i^n by

$$g_i^{n+1} = 2\beta z_j^n - g_j^n, \quad i = 1, 2, \quad j \neq i. \tag{9.86}$$

Together with Eq. (9.86), the transmission condition of Eq. (9.85) at $n + 1$ reads

$$\begin{aligned} \frac{\partial z_i^{n+1}}{\partial \nu_i} + \beta z_i^{n+1} &= g_i^{n+1} = 2\beta z_j^n - g_j^n \\ &= 2\beta z_j^n - \frac{\partial z_j^n}{\partial \nu_j} - \beta z_j^n \\ &= -\frac{\partial z_j^n}{\partial \nu_j} + \beta z_j^n, \quad i = 1, 2, \quad j \neq i, \end{aligned}$$

which, upon convergence, reduces to the correct transmission conditions for the global problem across the interface Γ, as in the real case. One shows the following (see Reference 18):

THEOREM 9.5
Let g_i^0 $i = 1, 2$ be given in $L^2(\Gamma; \mathbb{C})$, and let $\beta, k \in \mathbb{C}$ satisfy

$$\mathrm{Re}(\beta) \geq 0, \quad \mathrm{Re}(\bar{\beta} k) \geq 0, \quad \mathrm{Re}(\beta) + \mathrm{Re}(\bar{\beta} k) \geq 0.$$

Then as $n \to \infty$ the solutions z_i^n, $i = 1, 2$ of Eq. (9.85) and Eq. (9.86) converge strongly in $H^1(\Omega_i; \mathbb{C})$ to the restriction of z on Ω_i, where z is the solution of

$$\begin{cases} -\Delta z + kz = f & \text{in } \Omega \\ z = 0 & \text{in } \partial\Omega. \end{cases} \tag{9.87}$$

REMARK 9.5

1. The method can be extended to an arbitrary number of subdomains.
2. The method can be relaxed as in the preceding chapter.
3. The proof is an adaption of the one given by Benamou [2] combined with the updating procedure given by Deng [3].
4. The method also extends to more general second-order complex equations.

9.5.3 Methods for Elliptic Optimal Control Problems

We go back to the domain decomposition procedure of Eq. (9.85) and Eq. (9.86) and replace the complex quantities according to our setting above. We obtain

$$\begin{cases} -\Delta w_i^n + \dfrac{1}{\nu} p_i^n = f, \quad -\Delta p_i^n = w_i^n - w_d \quad \text{in } \Omega_i \\ \dfrac{\partial}{\partial \nu_j} w_i^n + \lambda w_i^n - \dfrac{\mu}{\sqrt{\nu}}\, p_i^n = g_i^n \quad \text{on } \Gamma \\ \dfrac{\partial}{\partial \nu_i} p_i^n + \lambda p_i^n + \mu \sqrt{\nu}\, w_i^n = h_i^n \quad \text{on } \Gamma \\ w_i = 0 = p_i \quad \text{on } \partial\Omega \cap \partial\Omega, \end{cases} \tag{9.88}$$

together with the updates

$$\begin{cases} g_i^{n+1} = 2\big(\lambda w_j^n - \frac{\mu}{\sqrt{\nu}} p_j^n\big) - g_j^n \\ h_i^{n+1} = 2\big(\lambda p_j^n + \mu\sqrt{\nu}\, w_j^n\big) - h_j^n. \end{cases} \tag{9.89}$$

The iteration of Eq. (9.88) and Eq. (9.89) contains, besides $\nu > 0$, the parameters λ, μ, which have to satisfy

$$\lambda \geq 0, \quad \mu \leq 0, \quad \lambda - \mu > 0. \tag{9.90}$$

The canonical choice in the context under consideration is $\lambda = 0$, $\mu = -\delta \in \mathbb{R}$, $\nu = 1$ and results in the iteration

$$\begin{cases} -\Delta w_i^n + p_i^n = f \quad \text{in } \Omega_i \\ -\Delta p_i^n = w_i^n - w_d \quad \text{in } \Omega_i \\ w_i = p_i = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \end{cases} \tag{9.91}$$

$$\begin{cases} \dfrac{\partial w_i^n}{\partial \nu_i} + \delta p_i^n = g_i^n \\ \dfrac{\partial p_i^n}{\partial \nu_i} - \delta w_i^n = h_i^n \end{cases} \tag{9.92}$$

$$\begin{cases} g_i^{n+1} = 2\delta p_j^n - g_j^n \\ h_i^{n+1} = -2\delta w_j - h_j^n. \end{cases} \tag{9.93}$$

Of course, Eq. (9.93) can equivalently be expressed as

$$\begin{cases} g_i^n = -\dfrac{\partial w_j^{n+1}}{\partial \nu_j} + \delta p_j^{n-1} \\ h_i^n = -\dfrac{\partial p_j^{n-1}}{\partial \nu_j} + \delta w_j^{n-1}. \end{cases} \tag{9.94}$$

9.5.4 An *a Posteriori* Error Estimate

We define $\mathbf{e}_n$ as

$$e^n := \|\tilde{w}_1^n\|_{V_1} + \frac{1}{\sqrt{\nu}}\|\tilde{p}_1^n\|_{V_1} + \|\tilde{w}_2^n\|_{V_2} + \frac{1}{\sqrt{\nu}}\|\tilde{p}_2^n\|_{V_2} \tag{9.95}$$

and

$$\mathcal{E}_{ij}^{n,n+1} = \left\| w_i^{n+1} - w_j^n \right\|_\Gamma^2 + \frac{1}{\nu} \left\| p_i^{n+1} - p_j^n \right\|_\Gamma^2, \tag{9.96}$$

where $\tilde{w}_i^n = w_i^n - w_i$, $\tilde{p}_i^n = p_i^n - p_i$. Here w_i, p_i are the global solutions, restricted to Ω_i, whereas w_i^n, p_i^n are obtained by the iterative domain decomposition method. Then we obtain the following theorem (see Reference 18).

THEOREM 9.6

Let the error measure be defined by Eq. (9.95) *and the mismatch of the consecutive iterates along* Γ *be given by Eq.* (9.96)*. Then, there is a constant C depending only on the geometry of* Ω_1, Ω_2 *and* $\beta > 0$ *along* Γ, *such that*

$$e^{n,n+1} \le C\left\{\mathcal{E}_{12}^{n,n+1} + \mathcal{E}_{21}^{n,n+1}\right\}^{1/2}. \tag{9.97}$$

9.6 Boundary Controls

9.6.1 Standard Setting

Let us consider a standard elliptic problem again but now with a control acting through the boundary on Ω:

$$\inf_{f \in L^2(\Gamma_1)} \frac{1}{2} \int_{\Gamma_1} f^2 \, d\gamma + \frac{\kappa}{2} \|w - w_d\|^2 \tag{9.98}$$

subject to

$$\begin{aligned} -\Delta\, w &= F \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \Gamma_0, &\quad \frac{\partial w}{\partial \nu} = f \quad \text{on } \Gamma_1, \end{aligned}$$

where $\partial\Omega = \Gamma_0 \dot{\cup} \Gamma_1$, $\mathrm{meas}(\Gamma_1) > 0$. The optimality condition for Eq. (9.98) is given by

$$f = -p \quad \text{on } \Gamma_1, \tag{9.99}$$

where p solves the adjoint problem

$$\begin{aligned} -\Delta\, p &= \kappa(w - w_d) \quad \text{in } \Omega, \\ p = 0 \quad \text{on } \Gamma_0, &\quad \frac{\partial p}{\partial \nu} = 0 \quad \text{on } \Gamma_1. \end{aligned} \tag{9.100}$$

The complete optimality system is expressed as a system of coupled problems:

$$\begin{aligned} -\Delta w &= F, \\ -\Delta p &= \kappa(w - w_d) \quad \text{in } \Omega, \\ w &= 0 = p \quad \text{on } \Gamma_0, \\ \frac{\partial w}{\partial \nu} + p = 0, \quad \frac{\partial p}{\partial \nu} &= 0 \quad \text{on } \Gamma_1. \end{aligned} \tag{9.101}$$

9.6.2 Domain Decomposition

As the coupling in Eq. (9.101) is both on the boundary for the adjoint variable and in the interior for the direct variable, a reformulation as a complex problem as in Eq. (9.84) is not possible. Still, the analysis in the last section leading to the decomposition method of Eq. (9.88) and Eq. (9.89) suggests the following procedure:

$$\begin{aligned}
-\Delta w_i^n = F_i, \quad -\Delta p_i = \kappa\big(w_i^n - w_{di}\big) \quad \text{in } \Omega_i \\
\frac{\partial}{\partial \nu_i} w_i^n + \lambda w_i^n + \mu p_i^n = g_i^n \quad \text{on } \Gamma \\
\frac{\partial}{\partial \nu_i} p_i^n + \lambda p_i^n - \mu w_i^n = h_i^n \quad \text{on } \Gamma \\
w_i^n = 0 = p_i^n \quad \text{on } \partial\Omega_i \cap \Gamma_0 =: \Gamma_{0,i}, \\
\frac{\partial w^n}{\partial \nu_i} = -p_i^n, \ \frac{\partial p_i^n}{\partial \nu_i} = 0 \quad \text{on } \partial\Omega_i \cap \Gamma_1 =: \Gamma_{1,i}
\end{aligned} \tag{9.102}$$

together with the updates

$$\begin{aligned}
g_i^{n+1} &= 2\big(\lambda w_j^n + \mu p_j^n\big) - g_j^n \\
h_i^{n+1} &= 2\big(\lambda p_j^n - \mu w_j^n\big) - h_j^n.
\end{aligned} \tag{9.103}$$

For simplicity we assume that $\text{meas}(\Gamma_{0,i}) > 0$ and $\text{meas}(\Gamma_{1,i}) > 0$. As always, Eq. (9.103) can be underrelaxed by the introduction of a parameter $\varepsilon \in [0, 1)$ such that

$$\begin{aligned}
g_i^{n+1} &= (1-\varepsilon)\big[2\big(\lambda w_j^n + \mu p_j^n\big) - g_j^n\big] + \varepsilon g_i^n \\
h_i^{n+1} &= (1-\varepsilon)\big[2\big(\lambda p_j^n - \mu w_j^n\big) - h_j^n\big] + \varepsilon h_j^n.
\end{aligned} \tag{9.104}$$

9.6.3 Convergence

THEOREM 9.7
For $\lambda = 0$ and $\mu > 0$, and $\lambda > 0$ and $\mu > 0$, where in the latter case μ/λ is sufficiently large, the errors of iterative scheme converge strongly to zero in the H^1-sense.

For a proof see Reference 18.

9.6.4 An *a Posteriori* Error Estimate

We consider the errors $\tilde{w}_i$, $\tilde{p}_i$ as in the preceding section. They satisfy the optimality system of Eq. (9.99) with $F_i = 0$, $w_{d_i} = 0$ and $\tilde{g}_i^n, \tilde{h}_i^n$ instead of g_i^n, h_i^n. For the sake of simplicity we consider the standard case $\lambda = 0$. Thus,

$$\begin{aligned}
-\Delta \tilde{w}_i^n = 0, \quad -\Delta \tilde{p}_i^n = \kappa \tilde{w}_i \quad \text{in } \Omega_i \\
\frac{\partial \tilde{w}_i^{n+1}}{\partial \nu_i} + \mu \tilde{p}_i^{n+1} = \tilde{g}_i^n \quad \text{on } \Gamma \\
\frac{\partial \tilde{p}_i^{n+1}}{\partial \nu_i} - \mu \tilde{w}_i^{n+1} = \tilde{h}_i^n \quad \text{on } \Gamma \\
\tilde{w}_i^n = 0 = \tilde{p}_i^n \quad \text{on } \Gamma_0 \cap \partial\Omega =: \Gamma_{0,i} \\
\frac{\partial \tilde{w}_i^n}{\partial \nu_i} = -\tilde{p}_i^n, \quad \frac{\partial \tilde{p}_i^n}{\partial \nu_i} = 0 \quad \text{on } \Gamma_1 \cap \partial\Omega_i =: \Gamma_{1,i}
\end{aligned} \tag{9.105}$$

together with

$$\tilde{g}_i^{n+1} = 2\mu \tilde{p}_j^n - \tilde{g}_j^n, \quad \tilde{h}_i^{n+1} = -2\mu \tilde{w}_j^n - h_j^n. \tag{9.106}$$

We may then introduce the total error

$$e^n := \sum_{i=1}^{2} \left[\left\| \tilde{w}_i^n \right\|_{V_i} + \left\| \tilde{p}_i^n \right\|_{V_i} \right] \tag{9.107}$$

and show the existence of a constant C such that

$$(e^n + e^{n+1})^2 \leq C \sum_{\substack{i,j=1 \\ i \neq j}} \left\{ \left\| w_i^{n+1} - w_j^n \right\|_\Gamma + \left\| p_i^{n+1} - p_j^n \right\|_\Gamma \right\} (e^n + e^{n+1}). \tag{9.108}$$

If we introduce the errors on the interface Γ by

$$\mathcal{E}_{ij}^{n,n+1} := \left\| w_i^{n+1} - w_j^n \right\|_\Gamma^2 + \left\| p_i^{n+1} - p_j^n \right\|_\Gamma^2, \tag{9.109}$$

then Eq. (9.108) becomes

$$e^n + e^{n+1} \leq C \left(\mathcal{E}_{12}^{n,n+1} + \mathcal{E}_{21}^{n,n+1} \right)^{1/2}. \tag{9.110}$$

Then, with

$$e^{n,n+1} := e^n + e^{n+1}, \tag{9.111}$$

we obtain the following *a posteriori* error estimate.

THEOREM 9.8
Let measure of error $e^{n,n+1}$ be defined by Eq. (9.111) *and Eq.* (9.107). *Let the mismatch of the iterates along the interface Γ be defined by Eq.* (9.110). *Then there is a constant C, depending solely on the geometry of Ω_1, Ω_2 and the parameter $\mu > 0$ along the interface, such that*

$$e^{n,n+1} \leq C \left(\mathcal{E}_{12}^{n,n+1} + \mathcal{E}_{21}^{n,n+1} \right)^{1/2}. \tag{9.112}$$

9.7 Elliptic Systems on Two-Dimensional Networks

This section is mainly concerned with domain decomposition in optimal elliptic systems on two-dimensional polygonal networks $\mathcal{P}$ in $\mathbb{R}^N$. The model description is taken from Reference 18.

A *two-dimensional polygonal network* $\mathcal{P}$ in $\mathbb{R}^N$ is a finite union of nonempty subsets $\mathcal{P}_i$, $i \in \mathcal{I}$, such that

1. Each $\mathcal{P}_i$ is a simply connected open polygonal subset of a plane Π_i in $\mathbb{R}^N$.
2. $\bigcup_{i \in \mathcal{I}} \overline{\mathcal{P}}_i$ is connected.
3. For all $i, j \in \mathcal{I}$, $\overline{\mathcal{P}}_i \bigcap \overline{\mathcal{P}}_j$ is either empty, a common vertex, or a whole common side.

The reader is referred to Nicaise [28], whose notation we adopt, for more details about such two-dimensional networks.

For each $i \in \mathcal{I}$ we fix once and for all a system of coordinates in Pi_i. Thus,

$$\prod_i = \{p_{i0} + x_1\eta_{i1} + x_2\eta_{i2}\}, \quad x_1,\ x_2 \in \mathbb{R},$$

where p_{i0} is the origin of coordinates in Π_i and $\eta_{i1},\ \eta_{i2}$ are orthonormal vectors that span Π_i. We assume that the boundary $\partial\mathcal{P}_i$ of $\mathcal{P}_i$ is the union of a finite number of linear segments $\overline{\Gamma}_{ij}$, $j = 1, \dots, N_i$. It is convenient to assume that Γ_{ij} is open in $\partial\mathcal{P}_i$. The collection of all Γ_{ij} are the *edges* of $\mathcal{P}$ and will be denoted by $\mathcal{E}$. An edge Γ_{ij} corresponding to an $e \in \mathcal{E}$ will be denoted by Γ_{ie}, and the *index set* $\mathcal{I}_e$ of e is $\mathcal{I}_e = \{i \,|\, e = \Gamma_{ie}\}$. The *degree* of an edge is the cardinality of $\mathcal{I}_e$ and is denoted by $d(e)$. For each $i \in \mathcal{I}_e$ we will denote by ν_{ie} the unit outer normal to $\mathcal{P}_i$ along Γ_{ie}. The coordinates of ν_{ie} in the given coordinate system of $\mathcal{P}_i$ are denoted by (ν_{ie}^1, ν_{ie}^2).

We partition the edges of $\mathcal{E}$ into two disjoint subsets $\mathcal{D}$ and $\mathcal{N}$, corresponding, respectively, to edges along which Dirichlet conditions hold and along which Neumann or transmission conditions hold. The Dirichlet edges are assumed to be *exterior edges*, that is, edges for which $d(e) = 1$. The Neumann edges consist of exterior edges $\mathcal{N}^{\text{ext}}$ and *interior edges* $\mathcal{N}^{\text{int}} := \mathcal{N}\backslash cN^{\text{ext}}$.

Let $m \geq 1$ be a given integer. For a function $W : \mathcal{P} \mapsto \mathbb{R}^m$, W_i will denote the restriction of W to $\mathcal{P}_i$, that is

$$W_i : \mathcal{P}_i \mapsto \mathbb{R}^m : x \mapsto W(x).$$

We introduce real $m \times m$ matrices

$$A_i^{\alpha\beta},\ B_i^{\beta},\ C_i, \quad i \in \mathcal{I},\ \alpha, \beta = 1, 2,$$

where

$$A_i^{\alpha\beta} = \left(A_i^{\beta\alpha}\right)^*, \quad C_i = C_i^*$$

and where the * superscript denotes transpose. For sufficiently regular $W,\ \Phi : \mathcal{P} \mapsto \mathbb{R}^m$ we define the symmetric bilinear form

$$a(W, \Phi) = \sum_{i\in\mathcal{I}} \int_{\mathcal{P}_i} \left[A_i^{\alpha\beta}\left(W_{i,\beta} + B_i^{\beta} W_i\right) \cdot \left(\Phi_{i,\alpha} + B_i^{\alpha}\Phi_i\right) + C_i W_i \cdot \Phi_i\right] dx, \tag{9.113}$$

where repeated lower case Greek indices are summed over 1,2. A subscript after a comma indicates differentiation with respect to the corresponding variable (e.g., $W_{i,\beta} = \partial W_i/\partial x_\beta$). The matrices $A_i^{\alpha\beta}$, B_i^{β}, C_i may depend on $(x_1, x_2) \in \mathcal{P}_i$, and $a(W, \Phi)$ is required to be $\mathcal{V}$-elliptic for an appropriate function space $\mathcal{V}$ specified below.

We shall consider the variational problem

$$a(W, \Phi) = \left\langle F, \Phi \right\rangle_V, \quad \forall\Phi \in \mathcal{V},\ 0 < t < T, \tag{9.114}$$

where $\mathcal{V}$ is a certain space of test functions and F is a given sufficiently regular function. The variational Eq. (9.114) obviously implies, in particular, that the W_i, $i \in \mathcal{I}$, formally satisfies the system of equations

$$\begin{aligned} -\frac{\partial}{\partial x_\alpha}\left[A_i^{\alpha\beta}\left(W_{i,\beta} + B_i^{\beta}W_i\right)\right] + \left(B_i^{\alpha}\right)^* A_i^{\alpha\beta}\left(W_{i,\beta} + B_i^{\beta}W_i\right) \\ + C_i W_i = F_i \quad \text{in } \mathcal{P}_i, i \in \mathcal{I}. \end{aligned} \tag{9.115}$$

To determine the space $\mathcal{V}$, we need to specify the conditions satisfied by W along the edges of $\mathcal{P}$. These conditions are of two types: *geometric edge conditions* and *mechanical edge conditions*. As usual, the space $\mathcal{V}$ is then defined in terms of the geometric edge conditions. At a Dirichlet edge we set

$$W_i = 0 \quad \text{on } e \text{ when } \Gamma_{ie} \in \mathcal{D}. \tag{9.116}$$

Furthermore, along each $e \in \mathcal{N}^{\text{int}}$ we impose the condition

$$Q_{ie} W_i = Q_{je} W_j \quad \text{on } e \text{ when } \Gamma_{ie} = \Gamma_{je},\, e \in \mathcal{N}^{\text{int}}, \tag{9.117}$$

where for each $i \in \mathcal{I}_e$, Q_{ie} is a real, nontrivial $p_e \times m$ matrix of rank $p_e \leq m$ with p_e independent of $i \in \mathcal{I}_e$. If $p_e < m$, additional conditions *may* be imposed, such as

$$\prod_{ie} W_i = 0 \quad \text{on } e, \forall\, i \in \mathcal{I}_e,\, e \in \mathcal{N}^{\text{int}}, \tag{9.118}$$

where Π_{ie} is the orthogonal projection onto the kernel of Q_{ie}. The geometric edge conditions are taken to be Eqs. (9.116) to (9.118), and the space $\mathcal{V}$ of test functions consists of sufficiently regular functions $\Phi : \mathcal{P} \mapsto \mathbb{R}^m$ that satisfy the geometric edge conditions.

By formal integration by parts in Eq. (9.114), it follows in particular that

$$\nu_{ie}^{\alpha} A_i^{\alpha\beta} \left(W_{i,\beta} + B_i^{\beta} W_i \right) = 0 \quad \text{on } e \text{ when } \Gamma_{ie} \in \mathcal{N}^{\text{ext}}. \tag{9.119}$$

For each $\Gamma_{ie} \in \mathcal{N}^{\text{int}}$, write $\Phi_i = \Pi_{ie}\Phi_i + \Pi_{ie}^{\perp}\Phi_i$, and let Q_{ie}^{+} denote the generalized inverse of Q_{ie}, that is, Q_{ie}^{+} is a $m \times p_e$ matrix such that

$$Q_{ie} Q_{ie}^{+} = I_{p_e}, \quad Q_{ie}^{+} Q_{ie} = \prod_{ie}^{\perp}.$$

Then we deduce

$$\sum_{i \in \mathcal{I}_e} \left(Q_{ie}^{+} \right)^{*} \nu_{ie}^{\alpha} A_i^{\alpha\beta} \left(W_{i,\beta} + B_i^{\beta} W_i \right) = 0 \quad \text{on } e \text{ if } e \in \mathcal{N}^{\text{int}}. \tag{9.120}$$

Conditions of Eq. (9.119) and Eq. (9.120) are called the *mechanical edge conditions*. We also refer to Eq. (9.117) and Eq. (9.120) as the geometric and mechanical *transmission conditions*, respectively.

To summarize, the edge conditions are composed of the geometric edge conditions of Eqs. (9.116) to (9.118), and the mechanical edge conditions of Eq. (9.119) and Eq. (9.120). The geometric transmission conditions are Eq. (9.117) and Eq. (9.118), whereas the mechanical transmission conditions are given by Eq. (9.120).

9.7.1 Examples

Example 9.1

Suppose that $m = 1$. In this case the matrices $A_i^{\alpha\beta}$, B_i^{α}, C_i, Q_{ie} reduce to scalars $a_i^{\alpha\beta}$, b_i^{α}, c_i, q_{ie}, where $a_i^{\alpha\beta} = a_i^{\beta\alpha}$. Set

$$A_i = \left(a_i^{\alpha\beta} \right), \quad b_i = \operatorname{col}\left(b_i^{\alpha} \right).$$

The system of Eq. (9.115) takes the form

$$-\nabla \cdot (A_i \nabla W_i) + [-\nabla \cdot (A_i b_i) + b_i^{*} A_i b_i + c_i] W_i = F_i. \tag{9.121}$$

Suppose that all $q_{ie} = 1$. The geometric edge conditions of Eq. (9.116) and Eq. (9.117) are then $W_i = 0$ on e when $e \in \mathcal{D}$ and $W_i = W_j$ on e when $e \in \mathcal{N}^{\text{int}}$, whereas the mechanical edge conditions are

$$\sum_{i \in \mathcal{I}_e} [\nu_{ie} \cdot (A_i \nabla W_i) + (\nu_{ie} \cdot A_i b_i) W_i] = 0 \quad \text{on } e \text{ when } e \in \mathcal{N}.$$

Example 9.2

(*Membrane networks in* $\mathbb{R}^3$.) In this case $m = N = 3$. For each $i \in \mathcal{I}$ set, $\eta_{i3} = \eta_{i1} \wedge \eta_{i2}$, where η_{i1}, η_{i2} are the unit coordinate vectors in Π_i. Suppose that $B_i = C_i = 0$, $Q_{ie} = I_3$, where I_3 denotes the identity matrix with respect to the $\{\eta_{ik}\}_{k=1}^{3}$ basis. With respect to this basis the matrices $A_i^{\alpha\beta}$ are given by

$$A_i^{11} = \begin{pmatrix} 2\mu_i + \lambda_i & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & \mu_i \end{pmatrix}, \quad A_i^{22} = \begin{pmatrix} \mu_i & 0 & 0 \\ 0 & 2\mu_i + \lambda_i & 0 \\ 0 & 0 & \mu_i \end{pmatrix},$$

$$A_i^{12} = \begin{pmatrix} 0 & \lambda_i & 0 \\ \mu_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_i^{21} = \begin{pmatrix} 0 & \mu_i & 0 \\ \lambda_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Write

$$\begin{aligned} W_i &= \sum_{k=1}^{3} W_{ik}\eta_{ik}, \quad w_i = W_{i\alpha}\eta_{i\alpha}, \\ \varepsilon_{\alpha\beta}(w_i) &= \frac{1}{2}(W_{i\alpha,\beta} + W_{i\beta,\alpha}), \\ \sigma_i^{\alpha\beta}(w_i) &= 2\mu_i\varepsilon_{\alpha\beta}(w_i) + \lambda_i\varepsilon_{\gamma\gamma}(w_i)\delta^{\alpha\beta}. \end{aligned}$$

The bilinear form of Eq. (9.113) may be written

$$a(W, \Phi) = \sum_{i\in\mathcal{I}} \int_{\mathcal{P}_i} \left[\sigma_i^{\alpha\beta}(w_i)\varepsilon_{\alpha\beta}(\phi_i) + \mu_i W_{i3,\alpha}\Phi_{i3,\alpha}\right] dx$$

where $\Phi_i = \sum_{k=1}^{3} \Phi_{ik}\eta_{ik} := \phi_i + \Phi_{i3}\eta_{i3}$. The corresponding system of Eq. (9.115) is

$$-\sigma_{i,\alpha}^{\alpha\beta}(w_i) = F_{i\beta}, \quad \beta = 1, 2, \quad -\mu_i W_{i3,\alpha\alpha} = F_{i3} \quad \text{in } \mathcal{P}_i,\ i \in \mathcal{I} \tag{9.122}$$

where $F_i = \sum_{k=1}^{3} F_{ik}\eta_{ik}$. The geometric edge conditions of Eq. (9.116) and Eq. (9.117) are

$$\begin{aligned} W_i &= 0 \quad \text{on } e \text{ when } \Gamma_{ie} \in \mathcal{D}, \\ W_i &= W_j \quad \text{on } e \text{ when } \Gamma_{ie} = \Gamma_{je}. \end{aligned} \tag{9.123}$$

The mechanical edge conditions of Eq. (9.120) are

$$\sum_{i\in\mathcal{I}_e} \left[\sigma_i^{\alpha\beta}(w_i)\eta_{i\beta} + \mu_i W_{i3,\alpha}\eta_{i3}\right]\nu_{ie}^{\alpha} = 0 \quad \text{on } e \text{ when } e \in \mathcal{N}. \tag{9.124}$$

The system of Eqs. (9.122) to (9.124) models the small, static deformation of a network of homogeneous isotropic membranes $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ in $\mathbb{R}^3$ of uniform density one, and Lamé parameters λ_i and μ_i under distributed loads F_i, $i \in \mathcal{I}$; $W_i(x_1, x_2)$ represent the displacement of the material particle situated at $(x_1, x_2) \in \mathcal{P}_i$ in the reference configuration; the transmission conditions have the interpretation of continuity of displacements and balance of forces at a junction. The reader is referred to Reference 17, where this model is introduced and analyzed. Figures 9.1 and 9.2 show the transmission of energy through the joint connecting three membranes in three dimensions. We have chosen to display the result of a dynamic process rather than a static elliptic one in order to highlight the situation before and after transmission has taken place. [The pictures were produced by Dr. W. Rathmann at the author's institute and InuTec.]

REMARK 9.6 We have also used this framework in order to include interfacial problems for Reissner-Mindlin plates and interfacial problems for coplanar thin plates. See Lagnese and Leugering [18] for details, where also the relation to the work by Le Dret [5] in the spirit of the Ciarlet-Destuynder method of asymptotic expansions is explained.

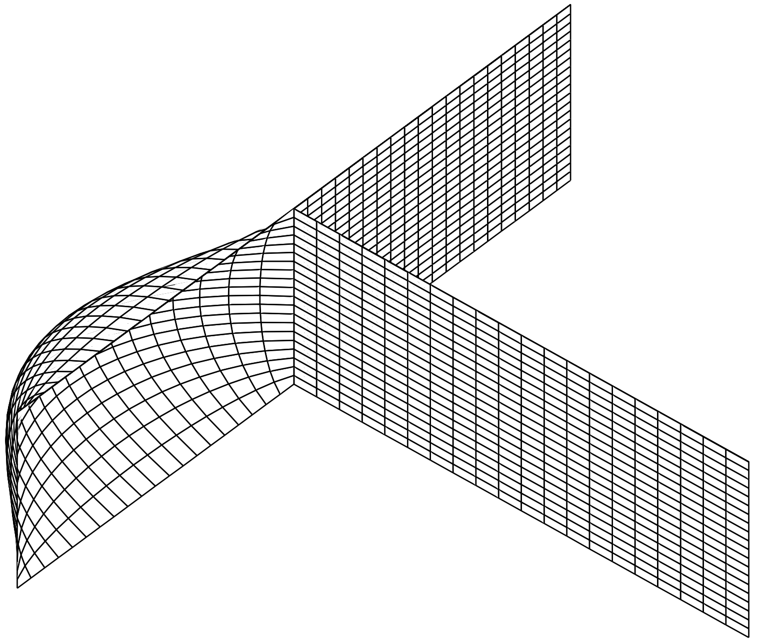

FIGURE 9.1: Interacting membranes: initial configuration.

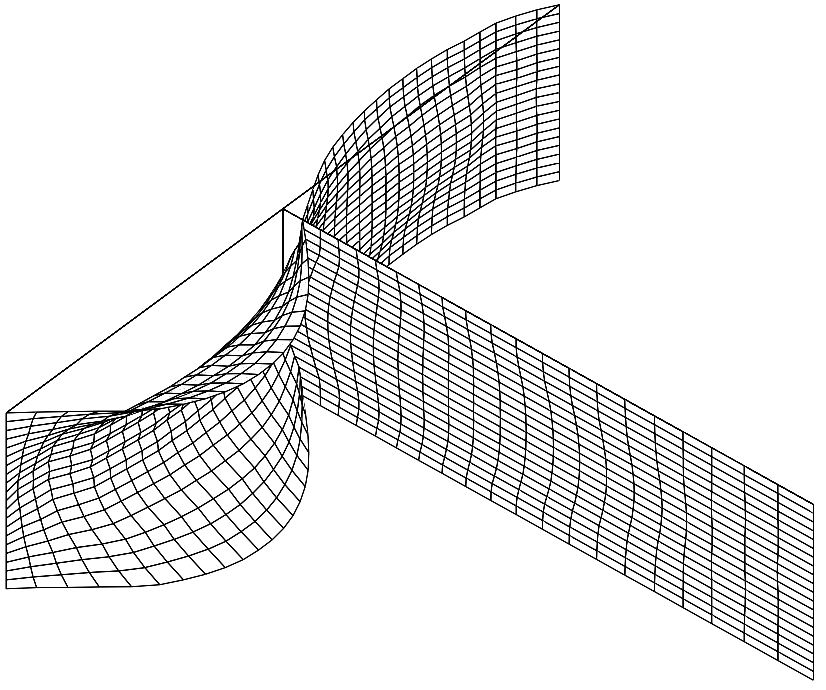

FIGURE 9.2: Interacting membranes: transmission effects.

9.7.2 Convergence of the Algorithm

To obtain a convergence result, we shall consider a relaxed version of the standard iteration step. Thus, we introduce a relaxation parameter $\epsilon \in [0, 1)$ and consider

$$\begin{aligned}&\left(Q_{ie}^{+}\right)^* D_{\nu_{ie}} W_i^{n+1} + k_e Q_{ie} W_i^{n+1} \\ &\quad = (1-\epsilon)\lambda_{ie}(W^n) + \epsilon\left[\left(Q_{ie}^{+}\right)^* D_{\nu_{ie}} W_i^n + k_e Q_{ie} W_i^n\right] := \lambda_{ie}^{\epsilon}(W^n). \end{aligned} \tag{9.125}$$

Thus, the $(n+1)$ local iterate is calculated as the solution of the variational equation

$$\begin{aligned}&a_i\left(W_i^{n+1}, \Phi_i\right) + \sum_{e \in \mathcal{N}_i^{\text{int}}} k_e \int_{\Gamma_{ie}} Q_{ie} W_i^{n+1} \cdot Q_{ie} \Phi_i d\Gamma \\ &\quad = \int_{\mathcal{P}_i} F_i \cdot \Phi_i dx + \sum_{e \in \mathcal{N}_i^{\text{int}}} \int_{\Gamma_{ie}} \lambda_{ie}^{\epsilon}(W^n) \cdot Q_{ie}\Phi_i d\Gamma, \quad \forall \Phi_i \in \mathcal{V}_i, \end{aligned} \tag{9.126}$$

where the starting data $\lambda_{ie}^{\epsilon}(W^0) \in L^2(\Gamma_{ie})$.

If $\lambda_{ie}^{\epsilon}(U^0) \in L^2(\Gamma_{ie})$, the problems of Eq. (9.126) for $n = 0, 1, \ldots$ have unique solutions with regularity

$$U_i^{n+1} \in \mathcal{V}_i, \quad \left(Q_{ie}^{+}\right)^* D_{\nu_{ie}} U_i^{n+1} \in L^2(\Gamma_{ie}) \quad \text{on } e \in \mathcal{N}_i^{\text{int}}.$$

Because the W_i^0 values are chosen so that $\lambda_{ie}^{\epsilon}(W^0) \in L^2(\Gamma_{ie})$, in order that $\lambda_{ie}^{\epsilon}(U^0) \in L^2(\Gamma_{ie})$ it is necessary that $\lambda_{ie}^{\epsilon}(W) \in L^2(\Gamma_{ie})$, which is equivalent to

$$\left(Q_{ie}^{+}\right)^* D_{\nu_{ie}} W_i \in L^2(\Gamma_{ie}) \quad \text{on } e \in \mathcal{N}_i^{\text{int}}, \quad i \in \mathcal{I}. \tag{9.127}$$

The regularity *assumption* of Eq. (9.127) is problematical in as much as precise knowledge of the regularity of solutions of Eq. (9.114) is generally unavailable. The regularity of W will depend on the F_i; the set of coefficients $A_i^{\alpha\beta}$, B_i^{α}, C_i; and the particular configuration and geometry of the two-dimensional network. For the case of interface problems for general elliptic equations on polygonal domains (i.e., when all $\mathcal{P}_i$ are coplanar), much is known and, in fact, Eq. (9.127) does not hold for all possible choices of the coefficients and the regions $\mathcal{P}_i$. The reader may consult the work of Nicaise and Sändig [29], [30]; Nicaise [28]; and the references therein. For a study of the Laplace and biharmonic operators on general two-dimensional networks, see Reference 28.

We have the following convergence result [18]:

THEOREM 9.9

Assume that the global and local bilinear forms are coercivity and that the global problem of Eq. (9.114) *has at least one solution having the regularity of Eq.* (9.127). *Let* $W = \{W_i\}_{i\in\mathcal{I}}$ *be such a solution. Suppose that* $k_e \equiv k$ *is independent of* $e \in \mathcal{N}^{\text{int}}$, *and let* $W_i^{n+1} \in \mathcal{V}_i$ *be the solution of Eq.* (9.126) *for* $n = 0, 1, \ldots$. *Then the following hold for the errors* $U_i^n = W_i^n - W_i$*:*

1. *For each* $\epsilon \in (0, 1)$,

$$\begin{aligned}&U_i^n \to U_i \;\; \textit{strongly in } H^1(\mathcal{P}_i), \quad \forall i \in \mathcal{I}, \\ &\textit{where } \; U = \{U_i\}_{i\in\mathcal{I}} \in \mathcal{V} \quad \textit{and} \quad a(U, \Phi) = 0, \forall \Phi \in \mathcal{V}\end{aligned}$$

2. *If* $\epsilon = 0$, *we have*

$$a_i\left(U_i^n, U_i^n\right) \to 0, \quad \forall i \in \mathcal{I}.$$

REMARK 9.7 The conclusion of part 1 of Theorem 9.9 is equivalent to saying that $U^n = \{U_i^n\}_{i\in\mathcal{I}}$ converges to zero in the quotient space $\mathcal{W}/cR$, where $\mathcal{W} = \Pi_{i\in\mathcal{I}}\mathcal{V}_i$ and $\mathcal{R}$ denotes the null space of the bilinear form $a(W, \Phi)$, that is,

$$\mathcal{R} = \{W \in \mathcal{V} : a(W, \Phi) = 0, \forall\Phi \in \mathcal{V}\}.$$

9.8 Extension and Remarks

The procedures outlined in this paper are completely general and apply to dynamic problems, first or second order in time, after certain canonical adoptions are made. One may include mechanical structures with different local dimension, such as beams coupled to plates, strings to membranes or shells, etc. Some modelling aspects along these lines are included in Lagnese, Leugering, and Schmidt [17]. See also the recent work of Horn and Leugering [33] on the coupling of nonlinear substructures. Nonlinear versions of nonoverlapping domain decomposition procedures are on their way. The methods also have a counterpart for overlapping domain decompositions. However, then, the controllability constraints that represent exact matching along interfaces are to be replaced by local exact controllability problems along "fattened interfaces." We refer to controls that realize the given optimal control problem on the global level as being "real controls," whereas again those controls that realize the exact (or relaxed matching) are referred to as "virtual controls." All that is said here can be extended to time-domain decomposition problems, which are the subject of current investigations. The concept for time-domain decompositions has been termed "pareal" by the late Lions (see Reference 22). For the extensions mentioned above, see the contributions by the author of this chapter and Lagnese and Leugering [13–16], Leugering [19–21], and the monograph of Reference 18.

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Chapter 10

Controllability of Parabolic and Hyperbolic Equations: Toward a Unified Theory

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Abstract This paper surveys the authors' and their collaborators' works on the attempt to unify the controllability theory of parabolic and hyperbolic equations. First of all, we show that the observability inequalities of the dual systems for those two types of equations of a different nature may be derived by means of a global Carleman-type estimate, which is based on a point-wise estimate for the corresponding principal operator. Next, we show that the null controllability of the heat equation may be obtained as the limit of the exact controllability of a family of singularly perturbed damped wave equations. Finally, we illustrate the complexity of the unification problem by analyzing a remarkable difference between the controllability of a class of hyperbolic-parabolic systems with control action entering the system through the wave component and the same problem but with control through the heat component.

10.1 Introduction

Let $T > 0$. Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2, consisting of the closure of two disjoint parts: Γ_0 and Γ_1, both relatively open in $\partial\Omega: \ \partial\Omega = \overline{\Gamma_0 \cup \Gamma_1},\ \Gamma_0 \cap \Gamma_1 = \emptyset$. Put $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$. Let ω be an open non-empty subset of Ω and denote by χ_ω the characteristic function of ω.

We begin with the following controlled heat equation with internal control

$$\begin{cases} y_t - \Delta y = \chi_\omega(x)u(t, x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{10.1}$$

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and the same problem but for the wave equation

$$\begin{cases} y_{tt} - \Delta y = \chi_\omega(x)u(t,x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } \Omega. \end{cases} \tag{10.2}$$

In Eq. (10.1), the *state* and *control spaces* are chosen to be $L^2(\Omega)$ and $L^2[(0,T) \times \omega]$, respectively; whereas in Eq. (10.2), $H_0^1(\Omega) \times L^2(\Omega)$ is the *state space*, and $L^2[(0,T) \times \omega]$ is the *control space*. Of course, the choice of these spaces is not unique.

The system in Eq. (10.1) is said to be null controllable (resp. approximately controllable) in $L^2(\Omega)$ if for any given $y_0 \in L^2(\Omega)$ (i.e., for any given $\varepsilon > 0$, $y_0, y_1 \in L^2(\Omega)$), one can find a control $u \in L^2[(0,T) \times \omega]$ such that the weak solution $y(\cdot) \in C([0,T]; L^2(\Omega)) \cap C([0,T]; H_0^1(\Omega))$ of Eq. (10.1) satisfies $y(T) = 0$ (resp. $|y(T) - y_1|_{L^2(\Omega)} \leq \varepsilon$). In the case of null controllability, the corresponding control u is called a null-control (with initial state y_0). Note, however, that, due to the smoothing effect of solutions to the heat equation, exact controllability for Eq. (10.1) is impossible (i.e., the above ε may not be taken to be zero).

Similarly, the system of Eq. (10.2) is said to be exactly controllable in $H_0^1(\Omega) \times L^2(\Omega)$ if for any (y_0, y_1), $(z_0, z_1) \in H_0^1(\Omega) \times \in L^2(\Omega)$, there is a control $u(\cdot) \in L^2[(0,T) \times \omega]$ such that the solution $y(\cdot) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of Eq. (10.2) satisfies that $y(T) = z_0$ and $y_t(T) = z_1$ in Ω.

There exists extensive literature on controllability problems. The controllability theory for finite dimensional linear systems was introduced by Kalman [9] at the beginning of the 1960s. Thereafter, many authors sought to develop it for more general systems, including infinite dimensional ones and its nonlinear and stochastic counterparts.

Early studies on controllability of partial differential equations (PDEs) can be found in References 3 to 7 (e.g., we refer to Russell's survey paper [24] for available results before 1978). In the past two decades, great progress has been made there. We mention only an incomplete list of related works, (References 1, 2, 8, 11, 20, 25, 33, 36) and the numerous references cited therein.

It is by now well known that the controllability theories for Eq. (10.1) and Eq. (10.2) are quite different. We will analyze the main differences in detail in Section 10.2. This, in turn, leads to a fundamental problem, that is, from the philosophical point of view it would be natural to expect to establish a unified theory, in some sense and to some extent, on the controllability of parabolic and hyperbolic equations. The main purpose of this paper is to survey the authors' and their collaborators' works in this respect. Our work includes three aspects. The first one is to develop a universal approach, based on the duality argument and the global Carleman estimate via point-wise estimate on the principal operator, to solve the controllability problem of parabolic and hyperbolic equations in a unified way. The second one is to analyze the relationship of the controllability theories between those two equations. The last one is to show the complexity of this unification problem by analyzing the controllability of a typical model of a coupled hyperbolic-parabolic system. The details on the above-mentioned three aspects will be explained in Sections 10.3 to 10.5.

10.2 Main Differences Between the Known Theories

First of all, let us recall the controllability of the system of Eq. (10.1):

THEOREM 10.1

([8], [18]) *For any $T > 0$ and any nonempty open subset ω of Ω, the system of Eq.* (10.1) *is null controllable and approximately controllable in $L^2(\Omega)$ on $[0,T]$ by means of control u in $L^2[(0,T) \times \omega]$.*

We emphasize again that, for any given $T > 0$, it is impossible to obtain exact controllability of the heat Eq. (10.1) by means of $L^2[(0,T) \times \omega]$-controls. On the other hand, because of the infinite speed of propagation of solutions to the heat equations for a finite heat pulse, the controllability time T and the controller ω in the system of Eq. (10.1) can be chosen as small as one likes. Especially, in this case, no geometric assumptions are needed to be imposed on the control set ω.

As for the controllability for the wave equation, we need to introduce the following notations. Fix $x_0 \in \mathbb{R}^n$, put

$$\Gamma_0 = \{x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0\}, \quad \Sigma_0 = (0, T) \times \Gamma_0, \tag{10.3}$$

where $\nu(x)$ denotes the unit outward normal vector of Ω at $x \in \Gamma$. For any set $S \in \mathbb{R}^n$ and $\varepsilon > 0$, put

$$\mathcal{O}_\epsilon(S) = \{y \in \mathbb{R}^n \mid |y - x| < \varepsilon \text{ for some } x \in S\}.$$

The following exact controllability result is well known.

THEOREM 10.2

([20]) *Let $T > 2 \max_{x \in \Omega} |x - x_0|$ and $\omega = \mathcal{O}_{\varepsilon_0}(\Gamma_0) \cap \Omega$ for some $\varepsilon_0 > 0$. Then the system of Eq.* (10.2) *is exactly controllable in $H_0^1(\Omega) \times L^2(\Omega)$ on* $[0, T]$ *by means of control u in $L^2[(0, T) \times \omega]$.*

It is notable that, due to the finite speed of propagation of solutions to the wave equation, one has to choose the controllability time T in Theorem 10.2 to be large enough, even in the context of approximate controllability. On the other hand, it is shown in Reference 2 that exact controllability of Eq. (10.2) is impossible without geometric conditions on the control set ω.

Various methods have been developed to address the controllability problem of PDEs. These methods have considerable "feedback" impact on some directions in PDE theory itself. The basic idea, however, to solve the controllability problem of PDE is based on the duality argument, which goes back to finite dimensional theory [9] (i.e., to reduce the problem to the obtainment of a suitable observability estimate for its dual system).

Replacing the time variable t by $T - t$, the dual system of Eq. (10.1) can be rewritten as follows:

$$\begin{cases} z_t - \Delta z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0 & \text{in } \Omega. \end{cases} \tag{10.4}$$

Similarly, one rewrites the dual system of Eq. (10.2) as

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega. \end{cases} \tag{10.5}$$

Thanks to the duality argument, it is easy to show that Theorems 10.1 and 10.2 are equivalent, respectively, to the following two observability results.

THEOREM 10.3

([8]), [18]) *For any $T > 0$ and any nonempty open subset ω of Ω, there is a constant $C > 0$ such that for all solutions of the system of Eq.* (10.4)*, it holds that:*

$$|z(T)|_{L^2(\Omega)} \le C|z|_{L^2[(0,T)\times\omega]}, \quad \forall\, z_0 \in L^2(\Omega). \tag{10.6}$$

THEOREM 10.4

([20]) *Let $T > 2 \max_{x \in \Omega} |x - x_0|$ and $\omega = \mathcal{O}_{\varepsilon_0}(\Gamma_0) \cap \Omega$ for some $\varepsilon_0 > 0$. Then there is a constant $C > 0$ such that for all solutions of the system of Eq.* (10.5)*, it holds that:*

$$|(z_0, z_1)|_{L^2(\Omega)\times H^{-1}(\Omega)} \le C|z|_{L^2[(0,T)\times\omega]}, \quad \forall\, (z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega). \tag{10.7}$$

To establish the observability estimate for PDEs, say Eq. (10.6) and Eq. (10.7), several powerful methods have been introduced. For example, for the heat equation, one uses the Carleman estimates, or the spectral method; for the wave equation, one uses the multiplier method, the microlocal analysis method, the Carleman estimates, or the spectral method. However, these methods are quite different and are very difficult to combine. Especially, these methods depend very strongly on the nature of the principal part of the related differential operator.

10.3 Global Carleman Estimate Via Point-Wise Estimate

It is well known that the Carleman estimate approach is one of the "common" methods applied to the controllability/observability problems of parabolic and hyperbolic equations. A Carleman estimate is simply a weighted energy method. However, at least formally, the Carleman estimate used to derive the observability inequality for parabolic equations is quite different from that for hyperbolic ones. Sometimes it is desired to develop a unified Carleman estimate method for different types of PDEs.

Stimulated by the works in References 17, 10, and 12, the second author (Xu Zhang) and his collaborators have developed a unified global Carleman estimate method, based on a fundamental point-wise estimate on the related principal operator, to derive boundary or internal observability inequalities for hyperbolic equations with various different lower-order terms or boundary conditions ([13], [21], [26], [28]–[30]). This method has the advantage, among others, to give an explicit estimate on the observability constant with respect to suitable Sobolev space norms of the coefficients in the equations, which is crucial for some control problems. This method can also be applied to other equations of the hyperbolic type, say the plate equations ([31]) and the Schrödinger equations ([14]–[16]). More importantly, recently the first author (Wei Li) successfully extended this method to parabolic equations ([19]). Therefore, in some sense, this method (i.e., the method of the global Carleman estimate via a point-wise estimate) can be regarded as a unified approach to treat the controllability problems of parabolic and hyperbolic equations. In the rest of this section, we will give a more detailed explanation on this method.

10.3.1 A Stimulating Example

The key idea of Carleman estimates is available in proving the stability of ordinary differential equations (ODEs). Indeed, consider the following ODE in $\mathbb{R}^n$:

$$\begin{cases} \dot{x}(t) = a(t)x(t), \quad t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{10.8}$$

The following simple result is well known:

THEOREM 10.5
Let $a \in L^\infty(0, T)$. Then there is a constant $C_T > 0$ such that for all solutions of Eq. (10.8)*, it holds that:*

$$\max_{t\in[0,T]} |x(t)| \le C_T|x_0|, \quad \forall\, x_0 \in \mathbb{R}^n. \tag{10.9}$$

A Carleman-type Proof of Theorem 10.5. Clearly, for any $\lambda \in \mathbb{R}$, from the first equation of Eq. (10.8), we have

$$\frac{d}{dt}(e^{-\lambda t}|x(t)|^2) = -\lambda e^{-\lambda t}|x(t)|^2 + 2e^{-\lambda t}\dot{x}(t)\cdot x(t) = [2a(t) - \lambda]e^{-\lambda t}|x(t)|^2. \tag{10.10}$$

Choosing λ large enough so that $2a(t) - \lambda \leq 0$ in Eq. (10.10), one finds

$$|x(t)| \leq e^{\lambda T/2}|x_0|, \quad t \in [0, T],$$

which proves Eq. (10.9).

REMARK 10.1 The first line of Eq. (10.10) can be rewritten as the following point-wise identity:

$$2e^{-\lambda t}\dot{x}(t) \cdot x(t) = \frac{d}{dt}[e^{-\lambda t}|x(t)|^2] + \lambda e^{-\lambda t}|x(t)|^2. \tag{10.11}$$

Note that $\dot{x}(t)$ is the principal operator of the first equation in Eq. (10.8). The main idea of Eq. (10.11) is to establish a point-wise identity (or estimate) on the principal operator $\dot{x}(t)$ in terms of the sum of a "divergence" term $\frac{d}{dt}[e^{-\lambda t}|x(t)|^2]$ and an "energy" term $\lambda e^{-\lambda t}|x(t)|^2$. As we see in the proof of Theorem 10.5, one chooses λ to be big enough to absorb the undesired terms, which is the key of all Carleman-type estimates. In the sequel, we use exactly the same method (i.e., as we mentioned before, the method of Carleman estimate via point-wise estimate), to derive the observability inequality of the wave and heat equations. The inspiration for this comes from the Russian literature, in particular Reference 17, Lemma 1, p. 124, which is used precisely to obtain unique continuation results (pp. 133–142).

10.3.2 The Wave Equations

We begin with the following fundamental point-wise estimate for the ultrahyperbolic operator $\sum_{i=1}^{n} \kappa_i \partial_{x_i x_i}$.

THEOREM 10.6
Let $u, \ell \in C^2(\mathbb{R}^m)$ ($m \in \mathbb{N}$). Let ψ and κ_i ($i = 1, 2, \ldots, m$) be real constants. Let $\theta = e^{\ell}$ and $v = \theta u$. Then it holds that:

$$\begin{aligned}
\theta^2 \left| \sum_{i=1}^{m} \kappa_i u_{x_i x_i} \right|^2 \geq{} & \sum_{i=1}^{m} \Bigg[-4 \sum_{j=1}^{m} \kappa_i \kappa_j \ell_{x_j} v_{x_i} v_{x_j} + 2 \sum_{j=1}^{m} \kappa_i \kappa_j \ell_{x_i} v_{x_j}^2 + 2\kappa_i \psi v_{x_i} v - \kappa_i \psi_{x_i} v^2 \\
& - 2 \sum_{j=1}^{m} \kappa_i \kappa_j \ell_{x_i} \left(\ell_{x_j}^2 - \ell_{x_j x_j}\right) v^2 \Bigg]_{x_i} \\
& + 4 \sum_{i,j=1}^{m} \kappa_i \kappa_j \ell_{x_i x_j} v_{x_i} v_{x_j} - 2 \sum_{i,j=1}^{m} \kappa_i (\psi + \kappa_j \ell_{x_j x_j}) v_{x_i}^2 \\
& + \Bigg\{ \sum_{i=1}^{m} \Bigg[2 \sum_{j=1}^{m} \kappa_i \kappa_j \left(\ell_{x_i x_i} \ell_{x_j}^2 - \ell_{x_i x_i} \ell_{x_j x_j} + 2\ell_{x_i} \ell_{x_j} \ell_{x_i x_j} - \ell_{x_i} \ell_{x_i x_j x_j} \right) + \kappa_i \psi_{x_i x_i} \\
& + 2\psi \kappa_i \left(\ell_{x_i}^2 - \ell_{x_i x_i} \right) \Bigg] - \psi^2 \Bigg\} v^2.
\end{aligned} \tag{10.12}$$

PROOF We borrow some idea from the proof of Reference 17, Lemma 1, p. 124. Recalling that $v = \theta u$ and using the equality

$$u_{x_i x_i} = \theta^{-1}\left[v_{x_i x_i} - 2\ell_{x_i} v_{x_i} + \left(\ell_{x_i}^2 - \ell_{x_i x_i}\right) v\right],$$

we obtain

$$\begin{aligned}
\theta^2\left|\sum_{i=1}^m \kappa_i u_{x_i x_i}\right|^2 &= \left|\sum_{i=1}^m \kappa_i [v_{x_i x_i} - 2\ell_{x_i} v_{x_i} + \left(\ell_{x_i}^2 - \ell_{x_i x_i}\right)v]\right|^2 \\
&= \left|\left\{\sum_{i=1}^m \kappa_i \left[v_{x_i x_i} + \left(\ell_{x_i}^2 - \ell_{x_i x_i}\right)v\right] - \psi v\right\} - 2\sum_{i=1}^m \kappa_i \ell_{x_i} v_{x_i} + \psi v\right|^2 \\
&\geq 2\left(\sum_{i=1}^m \kappa_i v_{x_i x_i}\right)\psi v - 4\left(\sum_{i=1}^m \kappa_i v_{x_i x_i}\right)\left(\sum_{i=1}^m \kappa_i \ell_{x_i} v_{x_i}\right) + 2\psi\sum_{i=1}^m \kappa_i\left(\ell_{x_i}^2 - \ell_{x_i x_i}\right)v^2 \\
&\quad - \psi^2 v^2 - 4\left(\sum_{i=1}^m \kappa_i \ell_{x_i} v_{x_i}\right)\left[\sum_{i=1}^m \kappa_i\left(\ell_{x_i}^2 - \ell_{x_i x_i}\right)v\right]. \qquad (10.13)
\end{aligned}$$

□

We transform each of the terms on the right-hand side of Eq. (10.13) so as to obtain expressions containing (up to a divergence) only $v_{x_i}^2$, $v_{x_i}v_{x_j}$, and v^2. We have

$$2\left(\sum_{i=1}^m \kappa_i v_{x_i x_i}\right)\psi v = -2\psi\sum_{i=1}^m \kappa_i v_{x_i}^2 + \sum_{i=1}^m \kappa_i \psi_{x_i x_i} v^2 + 2\sum_{i=1}^m \kappa_i(\psi v_{x_i} v)_{x_i} - \sum_{i=1}^m \kappa_i\left(\psi_{x_i} v^2\right)_{x_i}. \qquad (10.14)$$

Furthermore,

$$\begin{aligned}
-4\left(\sum_{i=1}^m \kappa_i v_{x_i x_i}\right)\left(\sum_{i=1}^m \kappa_i \ell_{x_i} v_{x_i}\right) &= -4\sum_{i,j=1}^m \kappa_i\kappa_j \ell_{x_j} v_{x_i x_i} v_{x_j} \\
= -2\sum_{i,j=1}^m \kappa_i\kappa_j \ell_{x_j x_j} v_{x_i}^2 &+ 4\sum_{i,j=1}^m \kappa_i\kappa_j \ell_{x_i x_j} v_{x_i} v_{x_j} + 2\sum_{i,j=1}^m \kappa_j\left(\kappa_i \ell_{x_j} v_{x_i}^2\right)_{x_j} \\
&- 4\sum_{i,j=1}^m \kappa_i(\kappa_j \ell_{x_j} v_{x_i} v_{x_j})_{x_i}. \qquad (10.15)
\end{aligned}$$

Furthermore still,

$$\begin{aligned}
-4\left(\sum_{i=1}^m \kappa_i \ell_{x_i} v_{x_i}\right)\left[\sum_{i=1}^m \kappa_i\left(\ell_{x_i}^2 - \ell_{x_i x_i}\right)v\right] &= -4\sum_{i,j=1}^m \kappa_i\kappa_j \ell_{x_i}\left(\ell_{x_j}^2 - \ell_{x_j x_j}\right)v_{x_i} v \\
= 2\sum_{i,j=1}^m \kappa_i\kappa_j\left(\ell_{x_i x_i}\ell_{x_j}^2 - \ell_{x_i x_i}\ell_{x_j x_j} + 2\ell_{x_i}\ell_{x_j}\ell_{x_i x_j} - \ell_{x_i}\ell_{x_i x_j x_j}\right)v^2 \\
- 2\sum_{i,j=1}^m \kappa_i\left[\kappa_j \ell_{x_i}\left(\ell_{x_j}^2 - \ell_{x_j x_j}\right)v^2\right]_{x_i}. \qquad (10.16)
\end{aligned}$$

Finally, combining Eqs. (10.13) to (10.16), we arrive at the desired estimate of Eq. (10.12). This completes the proof of Theorem 10.6.

REMARK 10.2 Theorem 10.6 is a sufficiently noteworthy variation of that in Reference 17, Lemma 1, p. 124. The main difference between them is that we do not return the function v in the right-hand side of Eq. (10.12) to u, unlike the Russian lemma, which has the variable u on both sides.

This not only greatly simplifies the computation, but also, as analyzed in Reference 13, plays a key role in establishing the observability estimate for the wave equations with Neumann boundary conditions.

REMARK 10.3 Of course, by choosing $m = 1 + n$, $t = x_{1+n}$, $\kappa_i = -1$ for $i = 1, 2, \ldots, n$, and $\kappa_{1+n} = 1$, one obtains from Theorem 10.6 a similar point-wise estimate but for the purely wave operator $\partial_{tt} - \Delta$.

From Theorem 10.6 and noting Remark 10.3, one may derive a boundary observability inequality for the following wave equations with lower-order terms:

$$\begin{cases} w_{tt} - \Delta w = q_1(t,x)w + q_2(t,x)w_t + \langle q_3(t,x), \nabla w\rangle & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0) = w_0, \quad w_t(0) = w_1 & \text{in } \Omega. \end{cases} \tag{10.17}$$

In Eq. (10.17), $q_i(\cdot)$ $(i = 1, 2, 3)$ are given functions allowed to be *time-variant* and *nonsmooth*.

More precisely, we have the following *a priori* estimate for solutions of Eq. (10.17).

THEOREM 10.7

([30]) *Let* $T > 2\max_{x\in\Omega} |x - x_0|$, $q_1 \in L^{n+1}(Q)$, $q_2 \in L^\infty(Q)$, *and* $q_3 \in L^\infty(Q; \mathbb{R}^n)$. *Then for any solution* $w \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ *of Eq.* (10.17), *it holds that*

$$|w_0|^2_{H_0^1(\Omega)} + |w_1|^2_{L^2(\Omega)} \le \mathcal{C}(r) \left|\frac{\partial w}{\partial \nu}\right|^2_{L^2(\Sigma_0)}, \quad \forall\, (w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega), \tag{10.18}$$

where Σ_0 *is given by Eq.* (10.3), *and* $\mathcal{C}(r)$ *is given by*

$$\mathcal{C}(r) = C\exp(Cr^2) \tag{10.19}$$

for some constant $C = C(T, \Omega) > 0$, *with* $r \triangleq |q_1|_{L^{n+1}(Q)} + |q_2|_{L^\infty(Q)} + |q_3|_{L^\infty(Q;\mathbb{R}^n)}$.

REMARK 10.4 We would like to point out that the estimate of Eq. (10.19) is not sharp. In fact, one may expect an estimate of the order of $e^{Cr^{1/2}}$, as indicated by Reference 36 for the case $n = 1$. How to derive a sharp estimate on Eq. (10.19) is an open problem.

By means of the duality argument, as a consequence of Theorem 10.7, one obtains immediately the exact boundary controllability of the following linear wave equation:

$$\begin{cases} y_{tt} - \Delta y = p_1(t,x)y + p_2(t,x)y_t + \langle p_3(t,x), \nabla y\rangle & \text{in } Q, \\ y = \chi_{\Gamma_0}(x)u(t,x) & \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } \Omega. \end{cases} \tag{10.20}$$

In Eq. (10.20), the "state space" and "control space" are chosen, respectively, to be $L^2(\Omega) \times H^{-1}(\Omega)$ and $L^2(\Gamma_0)$, where Γ_0 is given by Eq. (10.3).

We have the following result.

THEOREM 10.8

([30]) *Let* $T > 2\max_{x\in\Omega} |x - x_0|$, $p_1 \in L^{n+1}(Q)$, $p_2 \in W^{1,\infty}(Q)$, *and* $p_3 \in W^{1,\infty}(Q; \mathbb{R}^n)$. *Then for any given* (y_0, y_1) *and* $(z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *there is a control* $u \in L^2(\Sigma_0)$ *such that the weak solution* y *of Eq.* (10.20) *satisfies* $y(T) = z_0$ *and* $y_t(T) = z_1$ *in* Ω.

As mentioned before, thanks to Theorem 10.6 and noting Remark 10.3, one may also derive a boundary observability inequality for the following wave equations with purely homogenous Neumann boundary condition:

$$\begin{cases} w_{tt} - \Delta w = q_1(t,x)w + q_2(t,x)w_t + \langle q_3(t,x), \nabla w\rangle & \text{in } Q, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma, \\ w(0) = w_0, \quad w_t(0) = w_1 & \text{in } \Omega. \end{cases} \tag{10.21}$$

For this, we introduce the following assumption:
(H) *There exists a strictly convex function d: $\overline{\Omega} \to \mathbb{R}$, of class $C^3(\bar{\Omega})$, such that the following properties hold true:*

1. $$\left.\frac{\partial d}{\partial \nu}\right|_{\Gamma_1} = 0 \tag{10.22}$$

2. *The (symmetric) Hessian matrix $\mathcal{H}_d$ of $d(x)$ is a strictly positive definite on $\overline{\Omega}$*

3. $$\inf_{x\in\Omega} |\nabla d(x)| > 0. \tag{10.23}$$

We have the following observability result:

THEOREM 10.9
([13]) *Let assumption (H) hold true, and $\Gamma_0 = \Gamma/\overline{\Gamma_1}$. Let $q_1 \in L^{n+1}(Q)$, $q_2 \in L^\infty(Q)$, and $q_3 \in L^\infty(Q; \mathbb{R}^n)$. Then there is a constant $T_0 > 0$ so that for all $T > T_0$, solutions $w \in C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of Eq.* (10.21) *satisfy*

$$|w_0|^2_{H^1(\Omega)} + |w_1|^2_{L^2(\Omega)} \le C \int_0^T \int_{\Gamma_0} \left(w_t^2 + w^2\right) d\Gamma_0\, dt, \quad \forall\, (w_0, w_1) \in H^1(\Omega) \times L^2(\Omega). \tag{10.24}$$

REMARK 10.5 The above method can be extended to the general Riemann wave equations. We refer to Reference 26 for details.

10.3.3 The Heat Equations

Fix $a_{ij}(t,x) \in C^2(\overline{Q})$ to satisfy $a_{ij} = a_{ij}$ $(i, j = 1, 2, \dots, n)$. We begin with the following fundamental point-wise estimate for the ultraparabolic operator $-u_t - u_s + \sum_{i,j=1}^n (a_{ij}u_{x_i})_{x_j}$, which is the counterpart of Theorem 10.6.

THEOREM 10.10
([19]) *Let $u, \alpha \in C^2(\mathbb{R}^{2+n})$. Put $\theta = e^{\lambda\alpha}$ and $v = \theta u$. Then it holds that:*

$$\begin{aligned} &\theta^2 \left| -u_t - u_s + \sum_{i,j=1}^n (a_{ij}u_{x_i})_{x_j} \right|^2 \\ &\quad \ge \left\{ \sum_{i,j=1}^n \left[a_{ij}v_{x_i}v_{x_j} + \lambda(a_{ij})_{x_j}\alpha_{x_i}v^2 + \lambda a_{ij}\alpha_{x_ix_j}v^2 - \lambda^2 a_{ij}\alpha_{x_i}\alpha_{x_j}v^2\right] - \lambda(\alpha_t + \alpha_s)v^2 \right\}_t \\ &\quad + \left\{ \sum_{i,j=1}^n \left[a_{ij}v_{x_i}v_{x_j} + \lambda(a_{ij})_{x_j}\alpha_{x_i}v^2 + \lambda a_{ij}\alpha_{x_ix_j}v^2 - \lambda^2 a_{ij}\alpha_{x_i}\alpha_{x_j}v^2\right] - \lambda(\alpha_t + \alpha_s)v^2 \right\}_s \end{aligned}$$

$$+\sum_{j=1}^{n}\Bigg\{-2\sum_{i=1}^{n}\left(a_{ij}v_{x_i}v_t+a_{ij}v_{x_i}v_s+\lambda^2a_{ij}\alpha_{x_i}\alpha_t v^2+\lambda^2a_{ij}\alpha_{x_i}\alpha_s v^2\right)$$

$$+2\lambda\sum_{i,l,m=1}^{n}\big[a_{ij}a_{lm}\alpha_{x_i}v_{x_l}v_{x_m}-2a_{ij}a_{lm}\alpha_{x_l}v_{x_i}v_{x_m}-2a_{ij}a_{lm}\alpha_{x_lx_m}vv_{x_i}$$

$$+a_{ij}(a_{lm}\alpha_{x_lx_m})_{x_i}v^2+\lambda a_{ij}a_{lm}\alpha_{x_i}\alpha_{x_lx_m}v^2+\lambda a_{ij}(a_{lm})_{x_m}\alpha_{x_i}\alpha_{x_m}v^2\big]$$

$$-\lambda^2a_{ij}a_{lm}\alpha_{x_i}\alpha_{x_l}\alpha_{x_m}v^2\Bigg\}_{x_j}$$

$$+\Bigg\{\lambda(\alpha_t+\alpha_s)_t+\lambda(\alpha_t+\alpha_s)_s-\lambda\sum_{i,j=1}^{n}\big\{[(a_{ij})_{x_j}\alpha_{x_i}]_t+[(a_{ij})_{x_j}\alpha_{x_i}]_s+(a_{ij}\alpha_{x_ix_j})_t$$

$$+(a_{ij}\alpha_{x_ix_j})_s-\lambda(a_{ij}\alpha_{x_i}\alpha_{x_j})_t-\lambda(a_{ij}\alpha_{x_i}\alpha_{x_j})_s-2\lambda(a_{ij}\alpha_{x_i}\alpha_t)_{x_j}-2\lambda(a_{ij}\alpha_{x_i}\alpha_s)_{x_j}$$

$$+4\lambda\alpha_{ij}\alpha_{x_ix_j}\alpha_t+4\lambda a_{ij}\alpha_{x_ix_j}\alpha_s\big\}+2\lambda\sum_{i,j,l,m=1}^{n}\big\{\lambda^2(a_{ij}a_{lm}\alpha_{x_i}\alpha_{x_j}\alpha_{x_l})_{x_m}$$

$$-2\lambda^2a_{ij}a_{lm}\alpha_{x_i}\alpha_{x_j}\alpha_{x_lx_m}-\lambda[(a_{lm})_{x_m}a_{ij}\alpha_{x_i}\alpha_{x_j}]_{x_j}-\lambda(a_{ij}a_{lm}\alpha_{x_ix_j}\alpha_{x_l})_{x_m}$$

$$-2\lambda a_{ij}a_{lm}\alpha_{x_ix_j}\alpha_{x_lx_m}+2(a_{ij})_{x_j}a_{lm}\alpha_{x_i}\alpha_{x_lx_m}-[a_{ij}(a_{lm}\alpha_{x_lx_m})_{x_j}]_{x_i}\big\}\Bigg\}v^2$$

$$+2\lambda\sum_{i,j,l,m=1}^{n}[2a_{ij}(a_{lm}\alpha_{x_l})_{x_j}v_{x_i}v_{x_m}-(a_{ij}a_{lm})_{x_m}\alpha_{x_l}v_{x_i}v_{x_j}+a_{ij}a_{lm}\alpha_{x_lx_m}v_{x_i}v_{x_j}]. \qquad (10.25)$$

REMARK 10.6 Similar to Remark 10.3, by choosing u and α to be independent of s in Theorem 10.10, one obtains a similar point-wise estimate but for the heat operator $-u_t+\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j}$.

In the sequel, we suppose further that a_{ij} satisfies the following elliptic condition:

$$\sum_{i,j=1}^{n}a_{ij}\xi_i\xi_j\geq\lambda_0\sum_{i,j=1}^{n}\xi^2,\quad\forall\,(t,x,\xi_1,\ldots,\xi_n)\in\overline{Q}\times\mathbb{R}^n,$$

where $\lambda_0>0$ is a constant.

From Theorem 10.10 and noting Remark 10.6, similar to Theorem 10.7, one may derive a boundary observability inequality for the following heat equations with lower-order terms:

$$\begin{cases}w_t-\sum_{i,j=1}^{n}(a_{ij}w_{x_i})_{x_j}=q_1(t,x)w+\langle q_2(t,x),\nabla w\rangle & \text{in } Q,\\ w=0 & \text{on } \Sigma,\\ w(0)=w_0 & \text{in } \Omega.\end{cases}\qquad (10.26)$$

In Eq. (10.26), $q_1\in L^\infty(Q)$ and $q_2\in L^\infty(Q;\mathbb{R}^n)$.

More precisely, we have the following *a priori* estimate for solutions of Eq. (10.26).

THEOREM 10.11

([8]) *For any $T>0$ and any non-empty open subset γ_0 of $\partial\Omega$, there is a constant $C>0$ such that for all solutions of the system of Eq.* (10.26)*, it holds that:*

$$|w(T)|_{H_0^1(\Omega)}\leq C\left|\frac{\partial w}{\partial\nu}\right|_{L^2[(0,T)\times\gamma_0]},\quad\forall\,w_0\in L^2(\Omega). \qquad (10.27)$$

REMARK 10.7 We refer to Reference 19 for another application of Theorem 10.10. Also, we refer to Reference 27 for an interesting application of the global Carleman estimate for the heat equation to the optimal control problem.

As before, by means of the duality argument, one may obtain the null (boundary) controllability of the following general heat equations:

$$\begin{cases} y_t - \sum_{i,j=1}^{n} (a_{ij} y_{x_i})_{x_j} = p_1(t,x)y + \langle p_2(t,x), \nabla y \rangle & \text{in } Q, \\ y = \chi_{\gamma_0}(x)u(t,x) & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{10.28}$$

Indeed, one has the following result:

THEOREM 10.12

([8]) *For any $T > 0$ and any non-empty open subset γ_0 of $\partial\Omega$, assume $p_1 \in L^\infty(Q)$ and $p_2 \in W^{1,\infty}(Q; \mathbb{R}^n)$. Then for any given $y_0 \in L^2(\Omega)$, there is a control $u \in L^2[(0,T) \times \gamma_0]$ such that the weak solution y of Eq.* (10.28) *satisfies $y(T) = 0$ in Ω.*

REMARK 10.8 Using a similar method, one may obtain the internal observability and controllability of the general heat equations.

REMARK 10.9 It would be interesting to use the above method to derive a global Carleman estimate for the parabolic operator of higher order, say $\partial_t + (-\Delta)^m$, with $m \in \mathbb{N}$. However, this seems to be an open problem.

10.4 Controllability of the Heat Equation Via That of the Hyperbolic Equation

Now, a natural problem arises: what is the relationship between the controllability theories of parabolic and hyperbolic equations? Pioneer work in this respect was Russell's paper in 1973 [23]. In his paper, he showed that the exact controllability of wave Eq. (10.2) implies the null controllability of heat Eq. (10.1) with the same control set but in a short time.

Recently, López et al. [21] gave a "quantitative" version of Russell's result. More precisely, consider the following controlled dissipative wave equations:

$$\begin{cases} \varepsilon y_{\varepsilon,tt} - \Delta y_\varepsilon + y_{\varepsilon,t} = \chi_{\omega_\varepsilon} u_\varepsilon, & \text{in } Q, \\ y_\varepsilon = 0, & \text{on } \Sigma, \\ y_\varepsilon(0) = y_0, \quad y_{\varepsilon,t}(0) = y_1, & \text{in } \Omega. \end{cases} \tag{10.29}$$

In (10.29), $\varepsilon > 0$ is a small parameter (which is meant to tend to zero), $(y_\varepsilon, y_{\varepsilon,t}) \equiv [y_\varepsilon(t,x), y_{\varepsilon,t}(t,x)]$ is the state to be controlled, and $u_\varepsilon = u_\varepsilon(x,t)$ is the control. When $\omega_\varepsilon \equiv \omega$, the formal limit of Eq. (10.29) is the controlled heat Eq. (10.1). In Reference 21, it is shown that this formal observation can be understood rigorously as follows:

THEOREM 10.13

([21]) *Let $T > 0$, $\omega_\varepsilon \equiv \omega = \Omega \cap \mathcal{O}_\delta(\Gamma_0)$ with Γ_0 given by* (10.3) *and some $\delta > 0$. Then, there exists an $\varepsilon_0 = \varepsilon_0(T, \Omega, \omega) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the system of Eq.* (10.29) *is uniformly exactly* *controllable in time T. Furthermore, for any fixed* $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$*, the null-controls* u_ε *of Eq.* (10.29)*, which drive the system to rest at time* T*, may be chosen such that*

$$u_\varepsilon \to u \in L^2[(0, T) \times \omega] \text{ as } \varepsilon \to 0,$$

where u is a null-control for the limit of heat Eq. (10.1) *with initial datum* y_0.

To explain the main idea of the proof of this theorem, we need to recall the spectrum of the Laplacian

$$\begin{cases} -\Delta e_k = \mu_k e_k & \text{in } \Omega \\ e_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.30}$$

We know that Eq. (10.30) admits an increasing sequence of positive eigenvalues of finite multiplicity

$$0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_k \leq \cdots$$

tending to infinity. The eigenfunctions $\{e_k\}_{k\geq 1}$ may be chosen to constitute an orthonormal basis of $L^2(\Omega)$. Next, let us introduce the following subspaces of the space $H_0^1(\Omega) \times L^2(\Omega)$:

$$\begin{aligned} H_p^\varepsilon &= \left\{ U = (u, v) \in H_0^1(\Omega) \times L^2(\Omega) \;\middle|\; u, v \in \text{span}_{1\leq k\leq k(\varepsilon)}(e_k) \right\}, \\ H_h^\varepsilon &= \left\{ U = (u, v) \in H_0^1(\Omega) \times L^2(\Omega) \;\middle|\; u, v \in \text{span}_{k\geq k(\varepsilon)+1}(e_k) \right\}, \end{aligned} \tag{10.31}$$

where $k(\varepsilon)$ is such that $\frac{1}{4\mu_{k(\varepsilon)+1}} < \varepsilon \leq \frac{1}{4\mu_{k(\varepsilon)}}$. Finally, for any given $(u, v) \in H_0^1(\Omega) \times L^2(\Omega)$, we call its orthogonal projections over H_p^ε and H_h^ε, respectively, the parabolic and hyperbolic components.

Roughly speaking, the proof of Theorem 10.13 is as follows. We divide the time interval $[0, T]$ in three subintervals: $I_1 = [0, T/3]$, $I_2 = [T/3, 2T/3]$, and $I_3 = [2T/3, T]$. In the time interval I_1, we control to zero the parabolic component of the solution. Following the methods in Reference 18, this can be done uniformly with respect to $\varepsilon \to 0$. In the time interval I_2, we let the system in Eq. (10.29) evolve freely without control. In this way the parabolic components remain at rest and, due to the strong dissipativity of the system in Eq. (10.29) in its hyperbolic components, the size of the solution at time $t = 2T/3$ becomes exponentially small (i.e., of the order of $e^{-C/\varepsilon}$ as $\varepsilon \to 0$ for a suitable constant $C > 0$). Finally, in the interval I_3, we apply a control driving the whole solution to zero. When doing this we need the following observability estimate for the solutions of the dual system:

$$\begin{cases} \varepsilon z_{tt} - \Delta z - z_t = 0 & \text{in } Q \\ z = 0 & \text{on } \Sigma \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega, \end{cases} \tag{10.32}$$

that is,

THEOREM 10.14

([21]) *Let* ω *be given in Theorem 10.13. Then, for any* $T > 0$*, there exists* $\varepsilon(T) > 0$ *and positive constants* $C = C(T, \Omega, \omega) > 0$ *such that*

$$|z_0|^2_{L^2(\Omega)} + \varepsilon |z_1|^2_{H^{-1}(\Omega)} \leq C e^{C/\sqrt{\varepsilon}} \int_0^T \int_\omega z^2 dx dt \tag{10.33}$$

for all $0 < \varepsilon < \varepsilon(T)$ *and every solution of Eq.* (10.32).

According to this estimate, the control needed in the interval I_3 is exponentially large (of the order of $e^{C/\sqrt{\varepsilon}}$) with respect to the data of the solution at time $t = 2T/3$. However, because, in view of the analysis made in the interval I_2, these data are exponentially small, these two phenomena compensate, and the control turns out to be uniformly bounded and even exponentially small.

The proof of Theorem 10.14 is performed using again a global Carleman estimate, which is based on the following key point-wise estimate for the ultrahyperbolic operator $\partial_{tt} + \partial_{ss} - \Delta + k(\partial_t + \partial_s)$.

THEOREM 10.15

([21]) *Let* $u, \ell \in C^2(\mathbb{R}^{n+2})$. *Let* Ψ *and* k *be real constants. Let* $\theta = e^{\ell}$ *and* $v = \theta u$. *Then it holds that:*

$$
\begin{aligned}
&\theta^2|u_{tt} + u_{ss} - \Delta u + k(u_t + u_s)|^2 \\
&\geq \Bigg[(k - 2\ell_t)\left(|v_t|^2 - |v_s|^2 + \sum_{j=1}^{n}|v_{x_j}|^2\right) + 2(k - 2\ell_s)v_t v_s + 4\sum_{j=1}^{n}(\ell_{x_j} v_t v_{x_j}) \\
&\quad + 2\Psi v_t v + (k - 2\ell_t)(A + \Psi)|v|^2\Bigg]_t + \Bigg[(k - 2\ell_s)\left(|v_s|^2 - |v_t|^2 + \sum_{j=1}^{n}|v_{x_j}|^2\right) \\
&\quad + 2(k - 2\ell_t)v_t v_s + 4\sum_{j=1}^{n}(\ell_{x_j} v_s v_{x_j}) + 2\Psi v_s v + (k - 2\ell_s)(A + \Psi)|v|^2\Bigg]_s \\
&\quad - 2\sum_{j=1}^{n}\Bigg[2\sum_{i=1}^{n}(\ell_{x_i} v_{x_i} v_{x_j}) - \ell_{x_j}\sum_{i=1}^{n}|v_{x_i}|^2 + (k - 2\ell_t)v_t v_{x_j} + (k - 2\ell_s)v_s v_{x_j} \\
&\quad + \Psi v_{x_j} v + \ell_{x_j}\left(|v_t|^2 + |v_s|^2\right) - (A + \Psi)\ell_{x_j}|v|^2\Bigg]_{x_j} \\
&\quad + 2\left(-\Psi + \sum_{i=1}^{n}\ell_{x_i x_i} + \ell_{tt} - \ell_{ss}\right)|v_t|^2 + 2\left(-\Psi + \sum_{i=1}^{n}\ell_{x_i x_i} + \ell_{ss} - \ell_{tt}\right)|v_s|^2 \\
&\quad + 8\ell_{ts} v_t v_s - 8\sum_{j=1}^{n}(\ell_{t x_j} v_t v_{x_j} + \ell_{s x_j} v_s v_{x_j}) + 4\sum_{i,j=1}^{n}(\ell_{x_i x_j} v_{x_i} v_{x_j}) \\
&\quad + 2\left(\Psi - \sum_{i=1}^{n}\ell_{x_i x_i} + \ell_{tt} + \ell_{ss}\right)\sum_{j=1}^{n}|v_{x_j}|^2 + B|v|^2,
\end{aligned}
\tag{10.34}
$$

where

$$
A \triangleq \ell_t^2 - \ell_{tt} + \ell_s^2 - \ell_{ss} - \sum_{j=1}^{n}\left(\ell_{x_j}^2 - \ell_{x_j x_j}\right) - \Psi - k(\ell_t + \ell_s) \tag{10.35}
$$

and

$$
\begin{aligned}
B &= 2A\left(\Psi - \sum_{i=1}^{n}\ell_{x_i x_i}\right) + 2(\ell_{tt} + \ell_{ss})(A + \Psi) - (k - 2\ell_t)A_t - (k - 2\ell_s)A_s \\
&\quad - 2\sum_{j=1}^{n}(A_{x_j}\ell_{x_j}) + \Psi^2 - 2\Psi\sum_{i=1}^{n}\ell_{x_i x_i}.
\end{aligned}
\tag{10.36}
$$

REMARK 10.10 More recently, Theorem 10.13 was strengthened in Reference 22 under the Geometric Optics Control Condition introduced in Reference 2. It is notable that the proof of the corresponding observability estimate in Reference 22 is greatly simplified, compared with that of Theorem 10.14.

Note, however, that the system of Eq. (10.1) is null-controllable for any $T > 0$, and for any given nonempty subdomain ω of Ω, not necessarily the control set in Theorem 10.13. Therefore, it is of great interest to see whether one can reproduce this property from the controllability of Eq. (10.29). This was done in the recent work of Zhang [32] by allowing the control set ω_ε of Eq. (10.29) to change as ε tends to zero. More precisely, we have the following result:

THEOREM 10.16
([32]) *Let $T > 0$, ω be any given open nonempty subset of Ω and $\omega_\varepsilon = \omega \cup (\Omega \cap \mathcal{O}_\varepsilon(\partial\Omega))$. Then, for any given $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, one can choose the null-controls u_ε of Eq.* (10.29) *such that $u_\varepsilon \chi_{\omega_\varepsilon} \to u\chi_\omega$ in $L^2[(0,T)\times\Omega]$ as $\varepsilon \to 0$, where u is a null-control for Eq.* (10.1) *with initial datum y_0. Furthermore, it holds that:*

$$y_\varepsilon \xrightarrow{s} y \text{ in } L^2\left[0,T; H_0^1(\Omega)\right] \cap C([0,T]; L^2(\Omega)),$$
$$y_{\varepsilon,t} \xrightarrow{s} y_t \text{ in } L^2[(0,T)\times\Omega],$$

where y_ε and y are the corresponding solutions of Eq. (10.29) *and Eq.* (10.1)*, respectively.*

The proof of Theorem 10.16 is based on the following explicit observability estimate.

THEOREM 10.17
For any $T > 0$, there exist two positive constants $\varepsilon_2 = \varepsilon_2(T,\Omega)$ and $C = C(T,\Omega)$ such that

$$|z_0|^2_{L^2(\Omega)} + \varepsilon |z_1|^2_{H^{-1}(\Omega)} \le Ce^{C/\sqrt{\varepsilon}} \int_0^T \int_{\Omega\cap\mathcal{O}_\varepsilon(\partial\Omega)} z^2 dxdt \tag{10.37}$$

for all $0 < \varepsilon < \varepsilon_2$ and every solution of Eq. (10.32).

The proof of Theorem 10.17 is very close to that of Theorem 10.14 and is based on Theorem 10.15 again.

10.5 Spectral Analysis Method and the Complexity of the Unification Problem

Clearly, the spectral analysis method is the best common method to treat the controllability problems of parabolic and hyperbolic equations in one space dimension, because in this case the asymptotic behavior of eigenvalues and (generalized) eigenvectors for the underlying semigroup can be established. In this section, we will analyze the boundary controllability problems of a class of hyperbolic-parabolic systems. We will see that when the control action enters the system through different components (i.e., the wave and heat one), the corresponding controllability results are quite different. This phenomenon indicates that there would be a long way to go to establish a really unified controllability theory.

We begin with the following controlled $1-d$ linearized model for fluid-structure interaction:

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\infty)\times(0,1), \\ z_{tt} - z_{xx} = 0 & \text{in } (0,\infty)\times(-1,0), \\ y(t,1) = 0 & t \in (0,\infty), \\ z(t,-1) = u_1(t) & t \in (0,\infty), \\ y(t,0) = z(t,0), \quad y_x(t,0) = z_x(t,0) & t \in (0,\infty), \\ y(0) = y_0 & \text{in } (0,1), \\ z(0) = z_0, z_t(0) = z_1 & \text{in } (-1,0). \end{cases} \tag{10.38}$$

This system consists of a wave equation, arising on the interval $(-1, 0)$ with state (z, z_t), and a heat equation, that holds on the interval $(0, 1)$ with state y. The wave and heat components are coupled through an interface, the point $x = 0$, with transmission conditions imposing the continuity of (y, z) and (y_x, z_x). The control space of the system in Eq. (10.38) is $L^2(0, T)$. Apparently, the control $u_1(\cdot)$ enters the system through the wave endpoint $x = -1$. The state space of the system of Eq. (10.38) is the Hilbert space

$$\mathcal{H} \stackrel{\triangle}{=} \{(f, g, h) \mid (h, f) \in H^{-1}(-1, 1), \quad g \in L^2(-1, 0)\}$$

with the norm $|(f, g, h)|_{\mathcal{H}} = \sqrt{|(h, f)|^2_{H^{-1}(-1,1)} + |g|^2_{L^2(-1,0)}}$.

By combining a special energy estimate approach for the $1 - d$ wave equation and the known global Carleman estimate for the heat equation, the following result was proved in Reference 37:

THEOREM 10.18

([37]) *Let $T > 2$. Then for every $(y_0, z_0, z_1) \in \mathcal{H}$, there exists a control $u_1 \in L^2(0, T)$ such that the solution (y, z, z_t) of the system in Eq.* (10.38) *satisfies $y(T) = 0$ in $(0, 1)$ and $z(T) = z_t(T) = 0$ in $(-1, 0)$.*

One may expect that the same null-controllability result still holds for the following system:

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } (0, \infty) \times (0, 1), \\ z_{tt} - z_{xx} = 0 & \text{in } (0, \infty) \times (-1, 0), \\ y(t, 1) = u_2(t) & t \in (0, \infty), \\ z(t, -1) = 0 & t \in (0, \infty), \\ y(t, 0) = z(t, 0), \quad y_x(t, 0) = z_x(t, 0) & t \in (0, \infty), \\ y(0) = y_0 & \text{in } (0, 1), \\ z(0) = z_0, z_t(0) = z_1 & \text{in } (-1, 0), \end{cases} \tag{10.39}$$

where the control $u_2(\cdot)$ acts on the heat component instead of the wave one. However, this is not the case.

To see this, let us analyze the dual system of both Eq. (10.38) and Eq. (10.39), which can be written as follows:

$$\begin{cases} p_t - p_{xx} = 0 & \text{in } (0, T) \times (0, 1), \\ q_{tt} - q_{xx} = 0 & \text{in } (0, T) \times (-1, 0), \\ p(t, 1) = 0 & t \in (0, T), \\ q(t, -1) = 0 & t \in (0, T), \\ p(t, 0) = q(t, 0), \quad p_x(t, 0) = q_x(t, 0) & t \in (0, T), \\ p(0) = p_0 & \text{in } (0, 1), \\ q(0) = q_0, \quad q_t(0) = q_1 & \text{in } (-1, 0) \end{cases} \tag{10.40}$$

once the sense of time has been reversed. The system in Eq. (10.40) is well posed in the energy space

$$H \equiv H_0^1(-1, 1) \times L^2(-1, 0).$$

Indeed, the system in Eq. (10.40) can be written in an abstract form $Y_t = \mathcal{A}Y, Y(0) = Y_0$. Here $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is an unbounded operator defined as follows:

$$\mathcal{A}Y = (f_{xx}, h, g_{xx}), \quad \forall\, Y = (f, g, h) \in D(\mathcal{A}),$$

where $D(\mathcal{A}) \equiv \{(f, g, h) \in H \mid (g, f) \in H^2(-1, 1), h \in H^1(-1, 0), f \in H^3(0, 1), f_{xx}(1) = h(-1) = 0$, and $f_{xx}(0) = h(0)\}$. It is easy to show that operator $\mathcal{A}$ generates a contractive

C_0-semigroup in H with compact resolvent. Hence, $\mathcal{A}$ has a sequence of eigenvalues (in $\mathbb{C}$) tending to ∞.

The proof of Theorem 10.18 is based on the following observability estimate on Eq. (10.40).

LEMMA 10.1

([37]) *Let $T > 2$. Then there is a constant $C > 0$ such that every solution of Eq.* (10.40) *satisfies*

$$|[p(T), q(T), q_t(T)]|_H^2 \leq C|q_x(\cdot, -1)|^2_{L^2(0,T)}, \quad \forall\, (p_0, q_0, q_1) \in H. \tag{10.41}$$

However, the following negative result on the observability for the system of Eq. (10.40) in H implies the lack of boundary controllability of Eq. (10.39) in $\mathcal{H}$ from the heat component with a defect of infinite order.

THEOREM 10.19

([34], [35]) *Let $T > 0$ and $s \geq 0$. Then*

$$\sup_{(p_0,q_0,q_1)\in H/\{0\}} \frac{|[p(T), q(T), q_t(T)]|_H}{|p_x(\cdot, 1)|_{H^s(0,T)}} = +\infty,$$

where (p, q, q_t) is the solution of the system in Eq. (10.40) *with initial data (p_0, q_0, q_1).*

Theorem 10.19 is a consequence of the following spectral analysis result.

THEOREM 10.20

([35]) *The eigenvalues of $\mathcal{A}$ can be divided into two classes $\{\lambda_\ell^p\}_{\ell=\ell_1}^\infty$ and $\{\lambda_k^h\}_{|k|=k_1}^\infty$, where ℓ_1 and k_1 are suitable positive integers. Furthermore, the following asymptotic estimates hold as ℓ and k tend to ∞, respectively:*

$$\lambda_\ell^p = -\ell^2\pi^2 + 2 + O(\ell^{-1}), \quad \lambda_k^h = -\frac{1}{\sqrt{|1+2k|\pi}} + \left(\frac{1}{2} + k\right)\pi i + \frac{\operatorname{sgn}(k)}{\sqrt{|1+2k|\pi}} i + O(|k|^{-1}).$$

Moreover, there exist positive integers n_0, $\tilde{\ell}_1 \geq \ell_1$ and $\tilde{k}_1 \geq k_1$ such that

$$\{u_{j,0}, \ldots, u_{j,m_j-1}\}_{j=1}^{n_0} \bigcup \left\{u_\ell^p\right\}_{\ell=\tilde{\ell}_1}^\infty \bigcup \left\{u_k^h\right\}_{|k|=\tilde{k}_1}^\infty$$

form a Riesz basis of H, where $u_{j,0}$ is an eigenvector of $\mathcal{A}$ with respect to some eigenvalue μ_j with algebraic multiplicity m_j, $\{u_{j,1}, \ldots, u_{j,m_j-1}\}$ is the associated Jordan chain, and u_ℓ^p and u_k^h are eigenvectors of $\mathcal{A}$ with respect to eigenvalues λ_ℓ^p and λ_k^h, respectively.

The proof of Theorem 10.20 follows from a careful spectral analysis.

To analyze the boundary control and observation problem of the above simplified fluid-structure model from the heat component, we need to introduce the following two Hilbert spaces (recall

Theorem 10.20 for n_0, m_j, $u_{j,k}$, u_ℓ^p, u_k^h, $\tilde{\ell}_1$, and $\tilde{k}_1$):

$$V = \left\{ \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} a_{j,k} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell u_\ell^p + \sum_{|k|=\tilde{k}_1}^{\infty} b_k u_k^h \,\middle|\, a_{j,k}, a_\ell, b_k \in \mathbb{C}, \right.$$
$$\left. \sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} |k| e^{\sqrt{2|k|\pi}} |b_k|^2 < \infty \right\},$$
$$V' = \left\{ \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} a_{j,k} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell u_\ell^p + \sum_{|k|=\tilde{k}_1}^{\infty} b_k u_k^h \,\middle|\, a_{j,k}, a_\ell, b_k \in \mathbb{C}, \right.$$
$$\left. \sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} \frac{|b_k|^2}{|k| e^{\sqrt{2|k|\pi}}} < \infty \right\},$$

endowed with their canonical norms. Clearly, one is the dual of the other.

Thanks to Theorem 10.20 and Lemma 10.1, we have the following crucial boundary observability estimate for Eq. (10.40) through the heat component.

THEOREM 10.21

([35]) *For any $T > 2$, there is a constant $C > 0$ such that every solution of Eq.* (10.40) *satisfies*

$$|[p(T), q(T), q_t(T)]|_{V'}^2 \leq C |p_x(\cdot, 1)|_{L^2(0,T)}^2 \tag{10.42}$$

for all $(p_0, q_0, q_1) \in V'$.

Finally, denote by A the Laplacian $-\partial_{xx}$ on $(-1, 1)$ with homogeneous Dirichlet boundary conditions. We introduce the map $\mathcal{S}\colon \mathcal{H} \to H$ by $\mathcal{S}(f, g, h) = [A^{-1}(h, f), -g]$ for any $(f, g, h) \in \mathcal{H}$. It is easy to show that $\mathcal{S}$ is an isometric isomorphism from $\mathcal{H}$ onto H.

We have the following null-controllability result on the system in Eq. (10.39):

THEOREM 10.22

([35]) *Let $T > 2$. Then for every $(y_0, z_0, z_1) \in \mathcal{S}^{-1}V$, there exists a control $u_2 \in L^2(0, T)$ such that the solution (y, z, z_t) of the system in Eq.* (10.39) *satisfies $y(T) = 0$ in* $(0, 1)$ *and $z(T) = z_t(T) = 0$ in* $(-1, 0)$.

Obviously, the controllability result in Theorem 10.22 differs significantly from that in Theorem 10.18.

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Chapter 11

A Remark on Boundary Control on Manifolds

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Abstract We show that the method “SUR⇒ control” introduced in References 15 and 14 for boundary control in R^n can be modified to apply to compact manifolds with boundary.

11.1 Introduction

With the recent interest in differential geometric methods in boundary control theory (see References 4 and 5), it is natural to ask how easily existing treatments in R^n can be converted to manifolds. What are the main methods of boundary control in R^n? The method of multipliers and its extensions, Carleman estimates, associated with the names Lasiecka and Triggiani [8], [9], [10], [21], Lions and his school [11], Lagnese [7], Tataru [18], and others, in general, has the advantage of low differentiability requirements on the coefficients and its applicability to stabilization questions. The method of geometrical optics initiated in Reference 13 and essentially completed in Reference 1 needs C^∞ coefficients but yields optimal control times. Both the methods of multipliers and Reference 1 work with optimal spaces for Dirichlet controls. Moreover, both have to leave the finite energy space for Neumann controls. The method in Reference 13 has the advantage of simplicity (five pages). Taken together with Tataru's trace theorem [19], it gives optimal results for the case of Neumann controls, if controls are taken on the whole boundary. (The assumption of analytic coefficients in Reference 13 may be replaced by C^∞.)

Tataru's trace theorem [19] states that a finite energy solution of a second-order hyperbolic equation with C^∞ coefficients and L^2 right-hand side has an L^2 trace of the conormal derivative on any C^∞ noncharacteristic hypersurface, locally. This makes the “SUR” method described in this paper in many ways the preferred method for the Neumann problem, because the method does not necessitate leaving the finite energy space. It is expected that Tataru's trace theorem will ultimately be true with C^2 or C^3 coefficients. Anyhow, this smoothness assumption need be made only near the boundary, if we take into account M. Taylor's recent result on propagation of singularities [20], which assumes only C^2 coefficients for second-order hyperbolic equations.

11.2 The Case SUR⇒ Control in R^n

We here make use of a very versatile method of boundary control, motivated by Reference 17, introduced in References 15 and 14 in connection with the Schrödinger and wave equations, respectively which can be summarized by the principle

$$\text{(local) smoothing} + \text{uniqueness} + \text{reversibility} \Rightarrow \text{controllability}.$$

We first describe the method for control on the whole smooth boundary $\partial\Omega$ of the bounded domain Ω in R^n. We illustrate the method with linear second-order hyperbolic equations with C^∞ coefficients independent of t and with data in the finite energy space $H_1(\Omega) \times L^2(\Omega)$, but it will be clear that the method extends to higher-order equations as well, in a variety of spaces under appropriate conditions. We assume that the hyperbolic operator L is defined in all of $R^n \times [0, T]$: $Lu = u_{tt} - \Sigma a_{ij} u_{x_i x_j} - \Sigma a_i u_{x_i} - au$.

The idea of the proof is to extend the initial data to all of R^n, so that it still is in the finite energy space and so that its support is contained in a slightly larger set than $\bar{\Omega}$. *If this extension is carried out in a clever way*, it will turn out that the solution of the new initial value problem (on all of R^n) will vanish on $\bar{\Omega}$ for $t \geq T$, for some T. One then reads off the controls on $\partial\Omega \times [0, T]$. Let Ω_0 and Ω_1 be slight open enlargements of $\bar{\Omega}$ such that $\bar{\Omega}_0 \subset \Omega_1$. Let E be an extension operator that extends initial data of finite energy in Ω to data with support in Ω_0, so as to maintain finite energy. Let S be an operator that solves the pure initial value problem for the linear hyperbolic equation $Lu = 0$ in all of x-space and maps the extended Cauchy data (u, u_t) at time zero to the Cauchy data at time T. We assume that S is locally smoothing in the sense that the Cauchy data at time T will be smoother in Ω_1. This certainly will be the case if the coefficients of L are in C^∞ and we make the following assumption:

Bicharacteristic assumption 1: *Each bicharacteristic curve starting in* $\bar{\Omega}_0 \times \{0\}$ *eventually leaves (and stays out of) the cylinder* $\bar{\Omega}_1 \times (0, T)$ *through its lateral boundary.*

Second, we use the fact that the backward initial problem is solvable and call the operator mapping the Cauchy data at $t = T$ to that at $t = 0$ by S^{-1}. We note that S^{-1} preserves smoothness. Let φ be a smooth "cutoff" function with $\varphi \equiv 1$ near $\bar{\Omega}_0$ but zero near the complement of Ω_1. φ will also denote the operation of multiplying by φ.

Now let g be the given initial data. Consider the equation in Ω

$$f - S^{-1}\varphi SEf = g \tag{$*$}$$

for the unknown f. Denoting $S^{-1}\varphi SE$ by K, it is clear that K is smoothing, which makes the above equation a Fredholm equation. Now suppose a solution f exists. Then we define the extension $\tilde{g}$ of g to all of R^n by

$$\tilde{g} = Ef - S^{-1}\varphi SEf.$$

Now apply S to both sides

$$SEf - \varphi SEf = S\tilde{g} = 0 \text{ near } \Omega \text{ at } t = T.$$

Thus, if we solve the initial value problem in $R^n \times \{t \geq 0\}$, with initial data $\tilde{g}$, it will vanish near Ω at $t = T$. We could then continue the solution to be identically zero past $t = T$ in a slightly enlarged Ω. Thus, we could read off the lateral boundary data on $\partial\Omega \times [0, T]$ to obtain the right inhomogeneous boundary data that will steer the solution to zero in time T. (One, of course, hopes these will be in as good a space as possible.)

We would, of course, like to conclude that boundary controllability holds for all g with finite energy. This is where the "uniqueness" part comes in. Denoting the complement of Ω by Ω_c, we assume that the following uniqueness property holds: if $Lu = 0$ in $R^n \times (0, T)$ and $u \equiv 0$ in $\Omega_c \times (0, T)$, then $u \equiv 0$ in $R^n \times (0, T)$. This follows, for example, by Reference 12, provided u belongs to H_1. In the case of time-independent coefficients, it also follows from elliptic unique continuation. (See, for example, Reference 1, p. 1050.)

It is well known that one consequence of this uniqueness property is the approximate controllability of the system (see Section 11.4), and we will now make use of this property.

Suppose first that there is a one-dimensional null space to $I - K$. Then $I - K^*$ has a one-dimensional null space $\{\psi\}$. This follows from the Fredholm alternative (see Reference 3). In what follows, brackets represent the H_1 inner product. We will solve (*) for a slightly different right-hand side: $g - az$ (in the finite energy space) where a is a scalar. This will be possible if $< g - az, \psi >= 0$, which can be achieved by $< g, \psi > = a < z, \psi >$ or $a = < g, \psi > / < z, \psi >$. This works provided $< z, \psi > \neq 0$. Now, pick z sufficiently close to ψ so that $< z, \psi > \neq 0$ and in addition such that z can be steered to zero. (That we can do this follows from approximate controllability.) Then, because the initial data $(g - az)$ can also be steered to zero, the same follows for g.

Next, suppose the null space of $I - K$ is k dimensional. Suppose that $\psi_1, \ldots, \psi_k$ is a basis for the null space of $I - K^*$. We may assume this basis is orthonormal. Then try

$$\left\langle g - \sum_{i=1}^{k} a_i z_i, \psi_j \right\rangle = 0$$

$$< g, \psi_j > = \sum_{i=1}^{k} a_i < z_i, \psi_j > .$$

This can be solved for the a_i, provided $\det\{< z_i, \psi_j >\} \neq 0$, which is the case if the z_i are chosen sufficiently close to the ψ_i. The z_i should also be chosen so that they can be steered to zero at $t = T$. We know this is possible by the approximate controllability. Then proceed as in the case $k = 1$.

Now, in what space are the controls? We notice that there are three components to the solution obtained: (a) the solution to the pure initial value problem with data Ef, where f is a solution to the Fredholm equation; (b) the solution to the pure initial value problem with data Kf; (c) the part due to approximate controllability.

Components (b) and (c) may be taken to be as smooth as desired if the coefficients are C^∞. Hence, the smoothness of the controllers is determined by (a) (i.e., the traces of the boundary operators acting on the solution of a pure initial value problems). It is here that Tataru's trace theorem comes in for the case of Neumann controls! *From the procedure it is clear that if the initial data have finite energy, the system will maintain finite energy as the controls are implemented.*

What if the controls are not on the whole boundary? Suppose the boundary consists of two parts, an active, or outer, part ∂_A on which the controls are imposed and a reflective, or inner, part ∂_R, which we may think of as due to an obstacle reflecting rays. We assume that $\overline{\partial_A} \cap \overline{\partial_R} = \emptyset$. Then we replace the solution of the pure initial value problem in $R^n \times (0, T)$ by the solution of the exterior mixed problem with obstacle boundary ∂_R. Using the propagation of singularities for such problems as governed by generalized broken (reflected) bicharacteristics (see References 6 and 16), we can still obtain a smoothing operator S as before. The uniqueness theorem now follows from Reference 12. Otherwise, everything works as before, provided we make two changes:

1. When making the "slight extensions" Ω_j of Ω, we make these across only ∂_A, leaving ∂_R fixed, and define $\partial_A \Omega_j = \partial \Omega_j \setminus \partial_R$.

2. We replace bicharacteristic assumption 1 by the following assumption:
   Bicharacteristic assumption 2: *Every generalized broken (reflected) bicharacteristic starting in $\overline{\Omega_0} \times \{0\}$ eventually leaves (and stays out of) $\overline{\Omega_1} \times (0, T)$ through the "active" or "outer" part of its lateral boundary $\partial_A \Omega_1 \times (0, T)$.*

The value of T must, of course, be increased.

11.3 The Case of a Manifold with Boundary

Again, let us first consider the case of control on the whole boundary. Let $\bar{\Omega}$ be a smooth compact Riemannian manifold with boundary. (We assume connectedness.) Then $\bar{\Omega}$ can be extended to a larger open manifold M. (See Section 11.5.) Again we introduce "slight" extensions of $\bar{\Omega}$ as follows:

$$\bar{\Omega} \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2 \subset M.$$

We will illustrate the method for $Lu = u_{tt} - \Delta u = 0$. Here Δ denotes the Laplace-Beltrami operator. *We assume that every bicharacteristic starting in* $\bar{\Omega}_0 \times \{0\}$ *eventually leaves (and stays out of)* $\bar{\Omega}_1 \times (0, T)$ *through its lateral boundary and hits* $\partial\Omega_2 \times (0, \infty)$. Let $\varphi \equiv 1$ near $\bar{\Omega}_0$ but equal to zero near $M\backslash\Omega_1$. Furthermore, let $\rho(x)$ be a smooth function in M such that $\rho \equiv 1$ in $\bar{\Omega}_1$, $0 < \rho < 1$ in $\Omega_2\backslash\bar{\Omega}_1$ and $\rho \equiv 0$ in $M\backslash\Omega_2$. In particular, $\rho = 0$ in $\partial\Omega_2$.

The dirty trick: consider the modified operator $L_\rho u = u_{tt} - \rho\Delta u$. Let us notice that the bicharacteristic curves of L_ρ agree with those of L over $\bar{\Omega}_1$ but over $\Omega_2\backslash\bar{\Omega}_1$ they are asymptotic to vertical lines projecting on $\partial\Omega_2$. (See Figure 11.1.)

The procedure for the proof is as follows:

We pick Ω_0, Ω_1, Ω_2 and T so that the assumptions (in italics) hold. Consequently, the "modified bicharacteristics" of L_ρ will hit the plane $t = T$ over the set $\bar{\Omega}_2\backslash\bar{\Omega}_1$. Then the procedure described for R^n will work here, too. Reflective boundaries can again be introduced but with a larger T.

Imitating the procedure for R^n introduced in Section 11.2, Ω_2 takes the place of R^n, roughly speaking. We now need the solution operator S for a hyperbolic problem in $\Omega_2 \times [0, \infty)$ with initial data having support in Ω_0. The unique solution to this is guaranteed by Theorems 23.2.4 and 24.1.1 of Reference 6. Indeed, let us introduce the auxiliary mixed problem for $\Omega_{2-\varepsilon} \times [0, \infty)$, for initial data having support in Ω_0 and zero Dirichlet data, where $\overline{\Omega_0} \subset \Omega_{2-\varepsilon} \subset \overline{\Omega_{2-\varepsilon}} \subset \Omega_2$ and where $\partial\Omega_{2-\varepsilon}$ is "sufficiently close" to Ω_2. If that is the case, the support of the solution will never reach $\partial\Omega_{2-\varepsilon}$ during the time interval $[0, T]$ because the speed of propagation will be too slow for that to happen. The solution can then be extended by zero to the rest of $\Omega_2 \times [0, T]$, and the proof proceeds as before.

In addition to the possibility of adding a reflective boundary ∂_R as in Section 11.3, more complicated manifolds (with boundaries) can also be handled by the simple expedient of embedding them in a

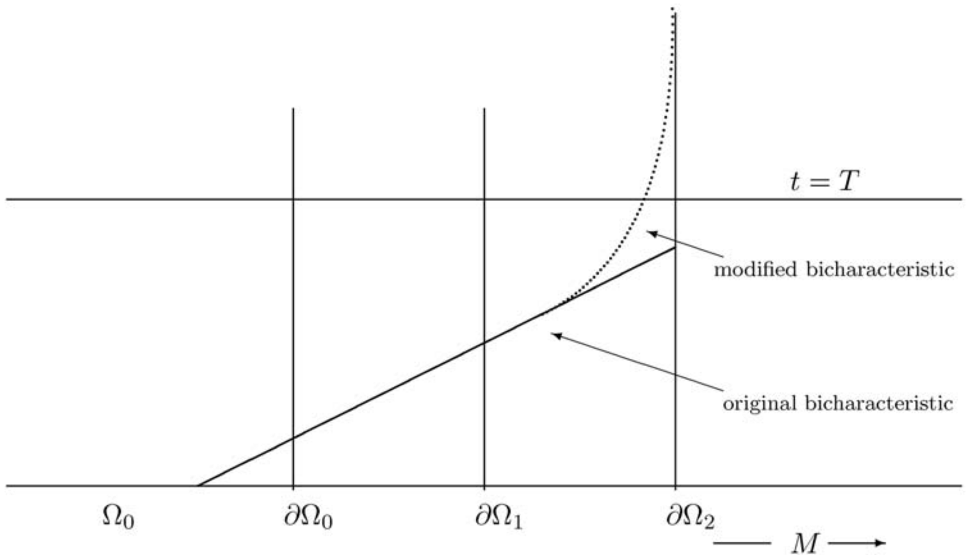

FIGURE 11.1: The dirty trick.

larger $\bar{\Omega}$. In particular, piecewise smooth boundaries (i.e., "deformed polyhedra") can be handled (using Tataru's trace theorem) as long as the reflective boundaries are smooth. Of course, if we take that approach, the control time may not be optimal.

REMARK

1. For the case of manifolds with a "reflective" boundary, the "mixed problem" step should be where zero Dirichlet data are prescribed on $\partial_A \Omega_{2-\varepsilon}$ and zero Dirichlet or Neumann data on ∂_R.

2. The conditions involving generalized broken bicharacteristics are usually not easy to check. However, if one assumes that all bicharacteristics tangent to $\partial_R \times (0, T)$ are tangent to finite order, one may simply omit the word "generalized" from the assumptions, thus dealing with only "ordinary" reflected bicharacteristics. This situation holds, for example, if ∂_{Ω_R} has negative second fundamental form. (In the references, these curves are simply referred to as "generalized bicharacteristics.")

11.4 Approximate Controllability

We first assume that $\Omega \subset R^n$. We would like to prove that the smooth functions $y(x, t)$ in the cylinder

$$\bar{\Omega} \times [0, T] = Q$$

vanishing near the top and bottom of Q are such that Ly is dense in $L^2(Q)$. To see that this implies approximate controllability, let us first suppose that the functions Ly fill out all of $L^2(Q)$. Then solve the initial value problem for the given initial data (extended to a smooth function with compact support) for all space and time. Call the solution $v(x, t)$. Let $\varphi(t)$ be a cutoff function such that $\varphi \equiv 1$ near $t = 0$ and $\varphi \equiv 0$ near $t = T$, and take $L[\varphi(t)v] = F(x, t)$. Then let w solve $Lw = F(x, t)$ with Cauchy data zero at $t = 0$ and at $t = T$, and set $u(x, t) = \varphi v - w$. Now if Lf is only dense, u will not be an exact solution to the equation, but the error can be converted into an error of small energy norm in the terminal data by solving a nonhomogeneous equation with zero initial data. Or, by solving the equation backward, the error could be added to the initial data.

Suppose that Ly is not dense in $L^2(Q)$. Then there exists an element g in $L^2(Q)$ orthogonal (in L^2) to the closure of the linear manifold Ly. With (,) denoting the L^2 inner product (or its extension), we have $(Ly, g) = 0$. Now extending g by zero to the rest of $R^n \times [0, T]$, and y smoothly to a function in $R^n \times [0, T]$, we see that $(y, L^*g) = 0$ for all y; hence, $L^*g = 0$ in $R^n \times [0, T]$, g vanishing in $(R^n \times [0, T]) \setminus Q$. By the propagation of singularities theorem g must be smooth. By the uniqueness theorem in Reference 12, $g = 0$, giving rise to a contradiction.

This proof can easily be modified to apply to a general Ω (not in R^n) by use of the techniques of Section 11.3.

11.5 Extending Manifolds

Assertion from Section 11.3: let $\bar{\Omega}$ be a smooth compact Riemannian manifold with boundary. (We assume connectedness.) Then $\bar{\Omega}$ can be extended to a larger open manifold M.

PROOF We assume the boundary $\partial\Omega$ has a finite number of smooth components. We are going to construct the extension one component at a time. Hence, we may as well assume that $\partial\Omega$ has one component, B, which is itself a manifold with an induced metric tensor g. Then near B we can introduce Fermi (or "normal") coordinates (see Reference 2 for example). From each point on B draw the unique geodesic leaving B normal and pointing inward. For all points within a certain distance ϵ from B, we can thus introduce "semiglobal" coordinates $(d, b) \in [0, \epsilon] \times B$, d representing the distance to B along the geodesic. The extension of $\bar{\Omega}$ is accomplished by replacing the original boundary strip by its "double" $(d, b) \in [-\epsilon, \epsilon] \times B$. ∎

We hope to return to this subject.

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Chapter 12

Model Structure and Boundary Stabilization of an Axially Moving Elastic Tape

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Abstract Linear and nonlinear equations of motion for a thin elastic tape moving axially between two sets of rollers with speed $c > 0$, and subject to a positive tension, were derived in an earlier article, and well-posedness was demonstrated in the linear case. Instability of the nominal equilibrium state for c sufficiently large was also demonstrated for the linear model, and it was shown that such instability can be overcome with active boundary control, exercised through movable rollers, synthesized via feedback on boundary data. In the present article we revisit this system, examining first of all some structural aspects for the fixed roller system, including the form of the adjoint system. In addition, we characterize a set of stabilizing boundary feedback controls for the movable roller configuration in a somewhat different manner than before, and we provide, for the corresponding closed loop systems, a simplified proof of well-posedness and a proof of uniform exponential stability.

12.1 Setting; Equations of Motion

We consider a thin tape composed of an elastic material moving axially between two sets of rollers located at $x = 0$ and $x = L > 0$; the tape moves to the left with speed $c > 0$. We treat the tape as an Euler-Bernoulli beam with density per unit length $\rho > 0$ and bending moment $A > 0$. We suppose in addition that the strip is subject to a tension $\kappa > 0$. The equilibrium position of the tape is the x-axis, $w = 0$, and displacements from that equilibrium are represented by $w = w(x, t)$. In deriving the equations of motion we will assume any needed smoothness properties of $w(x, t)$. Questions of well-posedness and regularity have been considered in Reference 12 and will also be treated in a later section here.

Focusing attention on a material particle moving with the tape, we see that its transverse velocity is $\frac{\partial w}{\partial t} - c\,\frac{\partial w}{\partial x}$. Using this in the standard kinetic energy expression the total energy may be taken to be

$$\mathcal{E}[w(\cdot,t)] = \frac{1}{2}\int_0^L E(x,t)\,dx,$$

$$E(x,t) = \rho\left[\left(\frac{\partial w}{\partial t} - c\,\frac{\partial w}{\partial x}\right)(x,t)\right]^2 + A\left[\frac{\partial^2 w}{\partial x^2}('')\right]^2 + \kappa\left[\frac{\partial w}{\partial x}('')\right]^2, \tag{12.1}$$

the symbol $''$ being used here and elsewhere to indicate repetition of previously indicated functional arguments. In some situations it is desirable to assume reinforced restoring forces to ensure bounded amplitude solutions. For this purpose the quadratic energy $\mathcal{E}$ is augmented by addition of a quartic term to the quadratic integrand to obtain

$$\tilde{E}(x,t) = \rho\left[\left(\frac{\partial w}{\partial t} - c\,\frac{\partial w}{\partial x}\right)(x,t)\right]^2 + A\left[\frac{\partial^2 w}{\partial x^2}('')\right]^2 + \kappa\left[\frac{\partial w}{\partial x}('')\right]^2 + B\left[\frac{\partial w}{\partial x}('')\right]^4,$$

where B is a positive constant. Standard variational techniques then give, as the equation of motion, the nonlinear partial differential equation

$$\rho\left(\frac{\partial^2 w}{\partial t^2} - 2c\frac{\partial^2 w}{\partial t\partial x} + c^2\frac{\partial^2 w}{\partial x^2}\right) + A\frac{\partial^4 w}{\partial x^4} - \left[\kappa + 6B\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial x^2} = 0. \tag{12.2}$$

Equations of this type have been studied extensively, but in a somewhat different context, in References 13 to 15. In the present paper we confine attention to the linear case corresponding to $B = 0$ and the quadratic energy in Eq. (12.1). To simplify the subsequent analysis, we note that if we set $\tilde{c} = c\sqrt{\rho}, \tilde{t} = t/\sqrt{\rho}$ then ρ disappears from Eq. (12.2). We make this change of variable and then rename $\tilde{t}$ and $\tilde{c}$ as t and c again. Then we have the simpler linear partial differential equation

$$\left(\frac{\partial^2 w}{\partial t^2} - 2c\frac{\partial^2 w}{\partial t\partial x}\right) + A\frac{\partial^4 w}{\partial x^4} + (c^2 - \kappa)\frac{\partial^2 w}{\partial x^2} = 0. \tag{12.3}$$

For fixed rollers the boundary conditions take the form

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = w(L,t) = \frac{\partial w}{\partial x}(L,t) = 0. \tag{12.4}$$

We do not specifically assume that the tape has a periodic structure as it would (e.g., in the case of a band saw blade), nor do we assume the tape to be wrapped around the rollers at the ends, for which case it is known (see, e.g., Reference 1) that the radius of curvature of the rollers has a definite influence on system dynamics.

With the boundary conditions in Eq. (12.4) $\frac{\partial^4 w}{\partial x^4}$ and $-\frac{\partial^2 w}{\partial x^2}$ are unbounded, positive self-adjoint operators, and we are led to expect that for any given tension $\kappa > 0$, instabilities may arise for a sufficiently large speed $c > 0$. Indeed, this has been known for some time (see, e.g., References 8, 16, and 17 and further references in Reference 12). In Reference 12, with the aid of the symbolic manipulation capabilities of MATLAB®, this was confirmed for a "prototype system," of which the above can be considered a perturbation. There, we also proposed a family of closed loop boundary control mechanisms, realized mathematically in the form of modified boundary conditions corresponding, physically, to controlled movable rollers, to overcome such instabilities. We showed that the open and closed loop systems correspond to strongly continuous semigroups on appropriate state spaces, but we were not able to demonstrate uniform exponential stability.

In the present paper we further examine the properties of the fixed roller system; in particular we develop the equations for the adjoint system, which are seen to have a somewhat unusual form. We also revisit the boundary stabilization problem, obtaining somewhat modified stabilization criteria as compared with those in Reference 12, and we establish uniform exponential decay, with respect to a norm related to system energy, for the corresponding closed loop semigroups.

12.2 System Structure

An analysis of the structure of the system just described allows us, in Theorem 12.1 below, to identify some rather unusual features. Introducing a new variable

$$z \equiv \frac{\partial w}{\partial t} - c\,\frac{\partial w}{\partial x}, \tag{12.5}$$

solutions of the partial differential Eq. (12.3) are embedded in those of the first-order system

$$\begin{aligned} \frac{\partial z}{\partial t} &= c\,\frac{\partial z}{\partial x} + \kappa\,\frac{\partial^2 w}{\partial x^2} - A\,\frac{\partial^4 w}{\partial x^4}, \\ \frac{\partial w}{\partial t} &= z + c\,\frac{\partial w}{\partial x}. \end{aligned} \tag{12.6}$$

The "finite energy" state space $\mathcal{H}_\mathcal{E}$ consists of pairs (z, w) such that $z \in L^2(0, L),\ \ w \in H_0^2(0, L)$, the latter being defined as the closed subspace of $H^2(0, L)$ obtained by requiring $w = 0$ and $\frac{dw}{dx} = 0$ at $x = 0, L$. In this space an "energy inner product" may be introduced in the form of an integral (in which, to avoid an excessively complicated notation, we assume all variables with "~" to be conjugated where complex)

$$\left\langle \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} \tilde z \\ \tilde w \end{pmatrix} \right\rangle_\mathcal{E} = \int_0^L \left(z\tilde z + \kappa\,\frac{dw}{dx}\frac{d\tilde w}{dx} + A\,\frac{d^2 w}{dx^2}\frac{d^2 \tilde w}{dx^2} \right) dx. \tag{12.7}$$

In Reference 12, it is shown that this system corresponds to a strongly continuous semigroup which is, in fact, a strongly continuous group, in a space equivalent to $\mathcal{H}_\mathcal{E}$. In the present context the semigroup generator associated with Eq. (12.6) is

$$T\,(z, w) \equiv \begin{pmatrix} c\,\frac{dz}{dx} + \kappa\,\frac{d^2 w}{dx^2} - A\,\frac{d^4 w}{dx^4} \\ z + c\,\frac{dw}{dx} \end{pmatrix}. \tag{12.8}$$

The domain, $\mathcal{D}(T)$, of T consists of $(z, w) \in \mathcal{H}_\mathcal{E}$ for which $T(z, w)$ again lies in $\mathcal{H}_\mathcal{E}$. Because the w component of $T(z, w)$ must lie in $H_0^2(0, L)$, $z + c\,\frac{dw}{dx} \in H_0^2(0, L)$ and both $z + c\,\frac{dw}{dx}$ and $\frac{dz}{dx} + c\,\frac{d^2 w}{dx^2}$ must vanish at $x = 0, L$. Because $\frac{dw}{dx}$ also vanishes there, z vanishes there. Summarizing,

$$\mathcal{D}(T) = \left\{ (z, w) \,\middle|\, w \in H^4(0, L) \cap H_0^2(0, L),\ \ z + c\,\frac{dw}{dx} \in H_0^2(0, L) \right\}. \tag{12.9}$$

It will be observed that $\mathcal{D}(T)$ involves four boundary conditions at each of $x = 0$ and $x = L$. The physical constraints on the system correspond to $w = 0$, $\frac{dw}{dx} = 0$; the remaining conditions, $z + c\,\frac{dw}{dx} = 0$, $\frac{dz}{dx} + c\,\frac{d^2 w}{dx^2} = 0$, $x = 0, L$ represent *consistency conditions* that must be satisfied in order for $\mathcal{H}_\mathcal{E}$ to remain invariant under the action of the semigroup generated by T.

THEOREM 12.1

The adjoint operator T^* corresponds to the system and domain

$$T^* \begin{pmatrix} \tilde z \\ \tilde w \end{pmatrix} = -\begin{pmatrix} c\,\frac{d\tilde z}{dx} + \kappa\,\frac{d^2 \tilde w}{dx^2} - A\,\frac{d^4 \tilde w}{dx^4} \\ \tilde z + c\,\frac{d\tilde w}{dx} - q(\cdot, \tilde w) \end{pmatrix}, \tag{12.10}$$

$$\mathcal{D}(T^*) = \left\{ (\tilde z, \tilde w) \,\middle|\, \tilde w \in H^4(0, L) \cap H_0^2(0, L),\ \tilde z + c\,\frac{d\tilde w}{dx} - q(\cdot, \tilde w) \in H_0^2(0, L) \right\}, \tag{12.11}$$

where $q(x, \tilde{w})$ is the solution of the boundary value problem

$$A\,\frac{d^4 q}{dx^4} - \kappa\,\frac{d^2 q}{dx^2} = 0, \tag{12.12}$$

$$q(0) = q(L) = 0, \quad \frac{dq}{dx}(0) = c\,\frac{d^2\tilde{w}}{dx^2}(0), \quad \frac{dq}{dx}(L) = c\,\frac{d^2\tilde{w}}{dx^2}(L). \tag{12.13}$$

The operator T^* generates a strongly continuous group in the state space $\mathcal{H}_{\mathcal{E}}$.

REMARK 12.1 The specific conditions characterizing the domain in Eq. (12.11) are developed below.

PROOF We compute

$$\begin{aligned}
\left\langle T\,(z,w)\,, \begin{pmatrix}\tilde{z}\\ \tilde{w}\end{pmatrix}\right\rangle_{\mathcal{E}} &= \int_0^L \left[\left(c\,\frac{dz}{dx} + \kappa\,\frac{d^2w}{dx^2} - A\,\frac{d^4w}{dx^4}\right)\tilde{z} + \kappa\left(\frac{dz}{dx} + c\,\frac{d^2w}{dx^2}\right)\frac{d\tilde{w}}{dx}\right.\\
&\quad \left. + A\left(\frac{d^2z}{dx^2} + c\,\frac{d^3w}{dx^3}\right)\frac{d^2\tilde{w}}{dx^2}\right] dx\\
&= \int_0^L z\left(-c\,\frac{d\tilde{z}}{dx} - \kappa\,\frac{d^2\tilde{w}}{dx^2} + A\,\frac{d^4\tilde{w}}{dx^4}\right) - \kappa\frac{dw}{dx}\left(\frac{d\tilde{z}}{dx} + c\,\frac{d^2\tilde{w}}{dx^2}\right)\\
&\quad - A\frac{d^2w}{dx^2}\left(\frac{d^2\tilde{z}}{dx^2} + c\,\frac{d^3\tilde{w}}{dx^3}\right) dx + \left(c\,z\tilde{z} + \kappa\,z\frac{d\tilde{w}}{dx} + A\,\frac{dz}{dx}\frac{d^2\tilde{w}}{dx^2}\right.\\
&\quad - A\,z\frac{d^3\tilde{w}}{dx^3} + \kappa\,\frac{dw}{dx}\tilde{z} + \kappa c\,\frac{dw}{dx}\frac{d\tilde{w}}{dx} - A\,\frac{d^3w}{dx^3}\tilde{z} + A\,\frac{d^2w}{dx^2}\frac{d\tilde{z}}{dx}\\
&\quad \left. + Ac\,\frac{d^2w}{dx^2}\frac{d^2\tilde{w}}{dx^2}\right)\Bigg|_0^L .
\end{aligned} \tag{12.14}$$

Integrating by parts twice with use of Eq. (12.12) and Eq. (12.13), one sees that the second integral in Eq. (12.14) may be rewritten in the form

$$\begin{aligned}
&-\int_0^L \left[z\left(c\,\frac{d\tilde{z}}{dx} + \kappa\,\frac{d^2\tilde{w}}{dx^2} - A\,\frac{d^4\tilde{w}}{dx^4}\right) + \kappa\,\frac{dw}{dx}\left(\frac{d\tilde{z}}{dx} + c\,\frac{d^2\tilde{w}}{dx^2} - \frac{dq}{dx}\right)\right.\\
&\quad \left. + A\,\frac{d^2w}{dx^2}\left(\frac{d^2\tilde{z}}{dx^2} + c\,\frac{d^3\tilde{w}}{dx^3} - \frac{d^2q}{dx^2}\right)\right] dx.
\end{aligned} \tag{12.15}$$

The boundary terms at the end of Eq. (12.14) can be put in the vector-matrix form

$$\begin{pmatrix} z & \frac{dz}{dx} & \frac{dw}{dx} & \frac{d^2w}{dx^2} & \frac{d^3w}{dx^3}\end{pmatrix}
\begin{pmatrix} c & 0 & \kappa & 0 & -A\\ 0 & 0 & 0 & A & 0\\ \kappa & 0 & \kappa c & 0 & 0\\ 0 & A & 0 & Ac & 0\\ -A & 0 & 0 & 0 & 0\end{pmatrix}
\begin{pmatrix} \tilde{z}\\ \frac{d\tilde{z}}{dx}\\ \frac{d\tilde{w}}{dx}\\ \frac{d^2\tilde{w}}{dx^2}\\ \frac{d^3\tilde{w}}{dx^3}\end{pmatrix}. \tag{12.16}$$

∎

For the relationship $\langle T\binom{z}{w},\ \binom{\tilde{z}}{\tilde{w}}\rangle_{\mathcal{E}} = \langle\binom{z}{w},\ T^*\binom{\tilde{z}}{\tilde{w}}\rangle_{\mathcal{E}}$ to hold, Eq. (12.16) should be zero. Because pairs $(z, w) \in \mathcal{D}(T)$ satisfy $z(0) = z(L) = 0,\ \frac{dw}{dx}(0) = \frac{dw}{dx}(L) = 0$, for such (z, w) that boundary form reduces to

$$\left(\frac{dz}{dx}\ \frac{d^2w}{dx^2}\ \frac{d^3w}{dx^3}\right)\begin{pmatrix} 0 & 0 & A \\ 0 & A & Ac \\ -A & 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{z} \\ \frac{d\tilde{z}}{dx} \\ \frac{d^2\tilde{w}}{dx^2} \end{pmatrix}. \tag{12.17}$$

Because $\frac{d^3w}{dx^3}$ is arbitrary at $x = 0, L$ we must include

$$\tilde{z}(0) = \tilde{z}(L) = 0 \tag{12.18}$$

among the boundary conditions characterizing $\mathcal{D}(T^*)$. Then Eq. (12.17) reduces to

$$\left(\frac{dz}{dx}\ \frac{d^2w}{dx^2}\right)\begin{pmatrix} 0 & A \\ A & Ac \end{pmatrix}\begin{pmatrix} \frac{d\tilde{z}}{dx} \\ \frac{d^2\tilde{w}}{dx^2} \end{pmatrix}. \tag{12.19}$$

The requirement that the second component of $T(z, w)$ (i.e., $z + c\,\frac{dw}{dx}$), should lie in $H_0^2(0, L)$ implies

$$\frac{dz}{dx} + c\,\frac{d^2w}{dx^2} = 0, \quad x = 0, L.$$

Thus, Eq. (12.19) becomes

$$\left(\frac{d^2w}{dx^2}\right)(-c\ 1)\begin{pmatrix} 0 & A \\ A & Ac \end{pmatrix}\begin{pmatrix} \frac{d\tilde{z}}{dx} \\ \frac{d^2\tilde{w}}{dx^2} \end{pmatrix} = A\,\frac{d^2w}{dx^2}\,\frac{d\tilde{z}}{dx}.$$

Because $\frac{d^2w}{dx^2}$ is arbitrary at $x = 0, L$ and $A > 0$, we obtain an additional pair of adjoint boundary conditions

$$\frac{d\tilde{z}}{dx}(0) = \frac{d\tilde{z}}{dx}(L) = 0. \tag{12.20}$$

For $(\tilde{z}, \tilde{w}) \in \mathcal{D}(T^*)$ the second component of $T^*(\tilde{z}, \tilde{w})$ must lie in $H_0^2(0, L)$. These consistency conditions correspond to the additional boundary conditions

$$\tilde{z} + c\,\frac{d\tilde{w}}{dx} - q = 0, \quad \frac{d\tilde{z}}{dx} + c\,\frac{d^2\tilde{w}}{dx^2} - \frac{dq}{dx} = 0, \quad x = 0, L. \tag{12.21}$$

The first condition in Eq. (12.21), given Eq. (12.11) and Eq. (12.18), is equivalent to the first part of Eq. (12.13), while the second, given Eq. (12.11) and Eq. (12.20), is equivalent to the second part of Eq. (12.13).

Because $\mathcal{H}_{\mathcal{E}}$ is a Hilbert space, and hence reflexive, the semigroup generator property of T^* follows from Corollary 10.6, Section 1.10, in Reference 9. The fact that Eq. (12.3) is invariant under replacement of t by $-t$ and x by $L - x$ implies that the strongly continuous semigroup e^{Tt} generated by the operator T is, in fact, a strongly continuous group of bounded operators. Then this is true for T^* as well. The adjoint semigroup, in the sense of that term ordinarily used in differential equations, is given by $e^{-T^*t} = (e^{-Tt})^*$. The proof is complete.

REMARK 12.2 Using elementary techniques one sees that the solution of Eq. (12.12) and Eq. (12.13) is given by

$$\begin{aligned} q(x, \tilde{w}) = {} & c\,\frac{d^2\tilde{w}}{dx^2}(0)\,x + \alpha\left\{\frac{A}{\kappa}\left[e^{-\left(\frac{\kappa}{A}\right)^{1/2}x} - 1\right] + \left(\frac{A}{\kappa}\right)^{1/2} x\right\} \\ & + \beta\left\{\frac{A}{\kappa}\left[e^{\left(\frac{\kappa}{A}\right)^{1/2}x} - 1\right] - \left(\frac{A}{\kappa}\right)^{1/2} x\right\}, \end{aligned}$$

where α and β are such that

$$c\,\frac{d^2\tilde{w}}{dx^2}(0)\,L + \alpha\left\{\frac{A}{\kappa}\left[e^{-(\frac{\kappa}{A})^{1/2}L} - 1\right] + \left(\frac{A}{\kappa}\right)^{1/2} L\right\} + \beta\left\{\frac{A}{\kappa}\left[e^{(\frac{\kappa}{A})^{1/2}L} - 1\right] - \left(\frac{A}{\kappa}\right)^{1/2} L\right\} = 0, \tag{12.22}$$

$$c\,\frac{d^2\tilde{w}}{dx^2}(0) + \alpha\left[-\left(\frac{A}{\kappa}\right)^{1/2} e^{-(\frac{\kappa}{A})^{1/2}L} + \left(\frac{A}{\kappa}\right)^{1/2}\right] + \beta\left[\left(\frac{A}{\kappa}\right)^{1/2} e^{(\frac{\kappa}{A})^{1/2}L} - \left(\frac{A}{\kappa}\right)^{1/2}\right] = c\,\frac{d^2\tilde{w}}{dx^2}(L). \tag{12.23}$$

Thus, $q(x,\tilde{w})$ in Eq. (12.10) and Eq. (12.11) has the form (reducing to 0 when $c = 0$)

$$q(x,\tilde{w}) = c\left[\frac{d^2\tilde{w}}{dx^2}(0)\,q_0(x) + \frac{d^2\tilde{w}}{dx^2}(L)\,q_L(x)\right], \tag{12.24}$$

where $q_0(x)$ and $q_L(x)$, respectively, correspond to the solution of Eq. (12.22) and Eq. (12.23) with c, $\frac{d^2\tilde{w}}{dx^2}(0)$, $\frac{d^2\tilde{w}}{dx^2}(L)$ replaced by $1,\ 1,\ 0$ and $1,\ 0,\ 1$, respectively; symmetry considerations imply $q_L(x) = -q_0(L-x)$. Thus, in the formula in Eq. (12.10) for T^*, the boundary values of $\frac{d^2\tilde{w}}{dx^2}$ are "fed back" into the functional expression of Eq. (12.10) of the operator T^* via Eq. (12.24).

12.3 The Closed-Loop Semigroup

We know from work in Reference 12 that the system of Eq. (12.3) and Eq. (12.4) becomes unstable for certain values of the axial velocity parameter c. This leads to the desirability of showing that stability can be regained with the use of active boundary controls synthesized in terms of collocated boundary data. Our emphasis on use of boundary data alone for feedback purposes is, of course, prompted by the realization that data from the moving belt at points other than boundary points is ordinarily unavailable without the introduction of advanced measurement techniques unlikely to be economically viable in most application contexts.

In the present work we define a *controlled endpoint* of the moving tape system to be either $x = 0$ or $x = L$ with boundary conditions of the form

$$\frac{\partial^2 w}{\partial x^2} = u, \quad \frac{\partial^3 w}{\partial x^3} = v \tag{12.25}$$

replacing the boundary conditions of Eq. (12.4) at the endpoint in question. In Reference 12 we showed that the system cannot be stabilized by taking $x = L$ to be the only controlled endpoint. Consequently, as in Reference 12, we take $x = L$ to be a fixed endpoint and $x = 0$ to be a controlled endpoint. Furthermore, we suppose the controls u and v to be synthesized via *linear collocated boundary feedback relations* of a particular form so that Eq. (12.25) become closed loop boundary conditions

$$\frac{\partial^2 w}{\partial x^2} = u = \alpha\,\frac{\partial^2 w}{\partial x\,\partial t}(0,t), \quad \frac{\partial^3 w}{\partial x^3}(0,t) = a\,\frac{\partial w}{\partial x}(0,t) + b\,\frac{\partial w}{\partial t}(0,t). \tag{12.26}$$

The time derivatives present in Eq. (12.26) (with coefficients α, b, which will presently be taken to be positive) correspond to an assumption that the rollers at the controlled end, though free to rotate and to move in the vertical direction, in doing so are subjected to opposing "frictional" forces. These may arise naturally in the physical configuration or may be introduced as part of an active control implementation.

We find it convenient now to use a slightly different, but equivalent, state space for our system as compared with the space $\mathcal{H}_{\mathcal{E}}$ used in the previous section. We define $\mathcal{H}$ to consist of function pairs $\mathbf{z} = \left(\begin{smallmatrix} w \\ \zeta \end{smallmatrix}\right)$ in $H_r^2(0, L) \times L^2(0, L)$, where

$$H_r^2(0, L) = \left\{ w \in H^2(0, L) \middle| w(L) = \frac{dw}{dx}(L) = 0 \right\}. \tag{12.27}$$

The space $\mathcal{H}$ is equipped with the "energy" inner product (in which we take $\rho = 1$ as discussed before Eq. [12.3])

$$\left\langle \begin{pmatrix} w \\ \zeta \end{pmatrix}, \begin{pmatrix} \hat{w} \\ \hat{\zeta} \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^L \left[A \frac{d^2 w}{dx^2} \frac{d^2 \hat{w}}{dx^2} + \kappa \frac{dw}{dx} \frac{d\hat{w}}{dx} + \left(\zeta - c\frac{dw}{dx} \right) \overline{\left(\hat{\zeta} - c\frac{d\hat{w}}{dx} \right)} \right] dx \tag{12.28}$$

and the related norm

$$\|\mathbf{z}\|_{\mathcal{H}} = (\langle \mathbf{z}, \mathbf{z} \rangle_{\mathcal{H}})^{\frac{1}{2}} = \left\{ \left\langle \begin{pmatrix} w \\ \zeta \end{pmatrix}, \begin{pmatrix} \hat{w} \\ \hat{\zeta} \end{pmatrix} \right\rangle_{\mathcal{H}} \right\}^{\frac{1}{2}}. \tag{12.29}$$

The closed loop system may then be written in first-order form as

$$\frac{d\mathbf{z}}{dt} = \mathcal{A}\mathbf{z}, \quad \mathcal{A}\mathbf{z} \equiv \begin{bmatrix} \zeta \\ -A\frac{d^4 w}{dx^4} - (c^2 - \kappa)\frac{d^2 w}{dx^2} + 2c\,\frac{d\zeta}{dx} \end{bmatrix}, \tag{12.30}$$

the domain of $\mathcal{A}$ being

$$\mathcal{D}(\mathcal{A}) = \{\mathbf{z} \in \mathcal{H} w \in H^4(0, L), \quad \zeta \in H_r^2(0, L), \quad \text{Eq. (12.26) satisfied.}\}. \tag{12.31}$$

THEOREM 12.2
For $c^2 - \kappa > 0$ the constants α, a, b in Eq. (12.26) *can be selected so that the operator $\mathcal{A}$ with domain of Eq.* (12.31) *is dissipative and generates a C_0 semigroup $S(t)$ of contractions on $\mathcal{H}$.*

REMARK 12.3 A comparable result was proved in Russell [12], with the boundary conditions at $x = 0$ characterized in a different way.

PROOF We compute, with extensive algebraic manipulation,

$$
\begin{aligned}
\operatorname{Re}\langle \mathcal{A}\mathbf{z},\ \mathbf{z}\rangle_{\mathcal{H}} &= \operatorname{Re}\left[A\left\langle \frac{d^2\zeta}{dx^2},\ \frac{d^2w}{dx^2}\right\rangle + \kappa\left\langle \frac{d\zeta}{dx},\ \frac{dw}{dx}\right\rangle\right. \\
&\quad \left. + \left\langle -A\frac{d^4w}{dx^4} - (c^2-\kappa)\frac{d^2w}{dx^2} + c\,\frac{d\zeta}{dx},\ \zeta - c\,\frac{dw}{dx}\right\rangle\right] \\
&= \left[A\left(\frac{d\zeta}{dx}\frac{d^2\overline{w}}{dx^2} - \zeta\frac{d^3\overline{w}}{dx^3}\right) + Ac\left(\frac{d^3w}{dx^3}\frac{d\overline{w}}{dx} - \frac{1}{2}\left|\frac{d^2w}{dx^2}\right|^2\right)\right. \\
&\quad \left. - (c^2-\kappa)\left(\frac{dw}{dx}\overline{\zeta} - \frac{c}{2}\left|\frac{dw}{dx}\right|^2\right) + \frac{c}{2}|\zeta|^2\right]_0^L \\
&= -\frac{Ac}{2}\left|\frac{d^2w}{dx^2}(L)\right|^2 - A\alpha\left(1 - \frac{c\alpha}{2}\right)\left|\frac{d\zeta}{dx}(0)\right|^2 - \left(\frac{c}{2} - Ab\right)|\zeta(0)|^2 \\
&\quad - \frac{c}{2}\left(Aa + c^2 - \kappa\right)\left|\frac{dw}{dx}(0)\right|^2 + \operatorname{Re}\left[(Aa - Acb)\zeta(0)\frac{d\overline{w}}{dx}(0)\right. \\
&\quad \left. + (c^2-\kappa)\overline{\zeta}(0)\frac{dw}{dx}(0)\right].
\end{aligned}
\tag{12.32}
$$

From this it is clear that if the coefficients α, a, b are chosen so that the above quadratic form is strictly negative definite, the operator $\mathcal{A}$ becomes dissipative. In particular, we can choose α, a, b to satisfy

$$
0 \le \alpha \le \frac{2}{c}, \quad 0 < \frac{c^2-\kappa}{2Ac} \le b \le \frac{c}{2A}, \quad a = c\,b - \frac{1}{A}(c^2-\kappa). \tag{12.33}
$$

Then, Eq. (12.32) becomes

$$
\operatorname{Re}\langle \mathcal{A}\mathbf{z},\ \mathbf{z}\rangle_{\mathcal{H}} = -c_1\left|\frac{d^2w}{dx^2}(L)\right|^2 - c_2\left|\frac{d\zeta}{dx}(0)\right|^2 - c_3\,|\zeta(0)|^2 - c_4\left|\frac{dw}{dx}(0)\right|^2, \tag{12.34}
$$

wherein

$$
c_1 = \frac{Ac}{2}, \quad c_2 = A\alpha\left(1 - \frac{c\alpha}{2}\right), \quad c_3 = \frac{c}{2} - Ab, \quad c_4 = \frac{c}{2}(Aa + c^2 - \kappa)
$$

are all nonnegative. ∎

Applying a modification of the standard Lumer-Phillips theory developed in References 6 and 7, we see that $\mathcal{A}$ generates a strongly continuous semigroup of contractions in $\mathcal{H}$ if $0 \notin \sigma(\mathcal{A})$. For this it is sufficient to show that, given any $F = (\phi, \psi)^* \in \mathcal{H}$ we can find $Y = (w, \zeta)^*$ in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}Y = F$, that is,

$$
\zeta - (0)\,w = \phi, \quad -A\frac{d^4w}{dx^4} - (c^2-\kappa)\frac{d^2w}{dx^2} + 2c\,\frac{d\zeta}{dx} - (0)\,\zeta = \psi. \tag{12.35}
$$

Substituting the first equation into the second we have

$$
\frac{d^4w}{dx^4} + \frac{1}{A}(c^2-\kappa)\frac{d^2w}{dx^2} = \frac{2c}{A}\left(\frac{d\phi}{dx} - \psi\right) \equiv f, \tag{12.36}
$$

and we have the desired result if Eq. (12.36) is solvable for arbitrary $f \in L^2(0, L)$ which, in turn, is true just in case the homogeneous version of Eq. (12.36) with the boundary conditions has only the zero solution in $\mathcal{D}(\mathcal{A})$ as given by Eq. (12.31). Because $c^2 - \kappa > 0$, we may take γ to be the positive square root of $\frac{1}{A}(c^2 - \kappa)$; then solutions of Eq. (12.36) with $f = 0$ satisfying Eq. (12.4) take the form

$$w(x) = a_1 [1 - \cos\gamma(L - x)] - a_2 [\gamma(L - x) - \sin\gamma(L - x)] . \tag{12.37}$$

Because the first equation in (12.35) gives $\zeta(x) \equiv 0$ when $\phi(x) \equiv 0$, the boundary conditions at $x = 0$ become

$$\frac{d^2w}{dx^2}(0) = 0, \quad \frac{d^3w}{dx^3}(0) = a\,\frac{dw}{dx}(0). \tag{12.38}$$

Substituting Eq. (12.37) into Eq. (12.38) we obtain the system

$$\begin{bmatrix} \cos(\gamma L) & \sin(\gamma L) \\ \gamma^2 \sin(\gamma L) - a\ \sin(\gamma L) & a - \gamma^2 \cos(\gamma L) - a\ \cos(\gamma L) \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for which the coefficient matrix has the determinant

$$-\gamma^2 - a[2\cos^2(\gamma L) - \cos(\gamma L) - 1]. \tag{12.39}$$

If $\mathbf{z} = (w,\ \zeta)^* = (w,\ 0)^*$, with $w(x)$ satisfying the homogeneous counterpart of Eq. (12.36), we have $\mathcal{A}\mathbf{z} = 0$. Because $c_1 = \frac{Ac}{2} > 0$ in Eq. (12.34), we conclude $\frac{d^2w}{dx^2}(L) = 0$; applying this in Eq. (12.37) gives $a_1 = 0$. The conditions in Eq. (12.33) together with $c^2 - \kappa > 0$ also imply $c_4 > 0$ in Eq. (12.34), from which we conclude that $\frac{dw}{dx}(0) = 0$. Using this and $a_1 = 0$ with Eq. (12.37) we have

$$a_2 \frac{d}{dx}[\gamma(L - x) - \sin\gamma(L - x)]|_{x=0} = a_2\gamma[\cos(\gamma L) - 1] = 0.$$

Then either we have $a_2 = 0$ directly or we have $\cos(\gamma L) = 1$, implying the determinant in Eq. (12.39) is $-\gamma^2 \neq 0$, from which we also conclude $a_1 = a_2 = 0$. In any case the homogeneous equation has only the zero solution, and we conclude, via Eq. (12.36), that Eq. (12.35) has a solution for $F = (\phi, \psi)^* \in \mathcal{H}$ and thus that the theorem holds.

12.4 Uniform Exponential Stability of the Closed Loop System

In this section we will establish the uniform exponential stability of the closed loop semigroup $S(t)$ whose existence was established in Theorem 12.2. For initial states $\mathbf{z}(0) = \mathbf{z}_0 \in \mathcal{H}$ the mild solution of the system (12.30) is

$$\mathbf{z}(t) = S(t)\mathbf{z}_0, \quad t \geq 0.$$

Because the energy $E(t)$ in Eq. (12.1) is also given by $E(t) = \frac{1}{2}\,\|\mathbf{z}(t)\|^2_{\mathcal{H}}$, exponential decay of $\|S(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}$ is equivalent to uniform exponential decay of $E(t)$. In this direction we have the following result.

THEOREM 12.3

The C_0-semigroup $S(t)$ on the Hilbert space $\mathcal{H}$ is exponentially stable, that is, there exist positive constants M and γ such that

$$\|S(t)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq M\,e^{-\gamma t}, \quad t \geq 0. \tag{12.40}$$

PROOF From Reference 3 we know that the strongly continuous semigroup $S(t)$ has a decay rate of the form in Eq. (12.40) if and only if (**R** indicates the real axis; i **R** indicates the imaginary axis; $\mathcal{I}$ is the identity operator on $\mathcal{H}$)

$$i\,\mathbf{R} \cap \sigma(\mathcal{A}) = \phi, \quad \sup_{\beta\in\mathbf{R}} \|(i\beta\,\mathcal{I} - \mathcal{A})^{-1}\|\infty. \tag{12.41}$$

We will establish these by contradiction. If we suppose that the second condition of Eq. (12.41) fails, then the Principle of Uniform Boundedness (see, e.g., Reference 2) implies that there exists a sequence $\{\beta_n\} \in \mathbf{R}$ with $\lim_{n\to\infty} \beta_n = \infty$ and a corresponding sequence $\{\mathbf{z}_n\} \in \mathcal{D}(\mathcal{A})$ with $\|\mathbf{z}_n\|_{\mathcal{H}} = 1$ such that

$$\lim_{n\to\infty} \|(i\beta\,\mathcal{I} - \mathcal{A})\,\mathbf{z}_n\|_{\mathcal{H}} = 0,$$

that is,

$$i\beta_n w_n - \zeta_n \equiv f_n \to 0 \;\text{ in }\; H_r^2(0, L), \tag{12.42}$$

$$i\beta_n\zeta_n + A\frac{d^4w_n}{dx^4} + (c^2-\kappa)\frac{d^2w_n}{dx^2} - 2c\,\frac{d\zeta_n}{dx} \equiv g_n \to 0 \;\text{ in }\; L^2(0, L). \tag{12.43}$$

From Eq. (12.32), Eq. (12.34), and Eq. (12.42) it follows that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \operatorname{Re}\langle(i\beta_n\mathcal{I} - \mathcal{A})\,\mathbf{z}_n,\ \mathbf{z}_n\rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty}\left[c_1\left|\frac{d^2w_n}{dx^2}(L)\right|^2 + c_2\left|\frac{d\zeta_n}{dx}(0)\right|^2 + c_3\,|\zeta_n(0)|^2 + c_4\left|\frac{dw_n}{dx}(0)\right|^2\right], \end{aligned}$$

from which it follows that

$$\left|\frac{d^2w_n}{dx^2}(L)\right|,\quad \left|\frac{d\zeta_n}{dx}(0)\right|,\quad |\zeta_n(0)|,\quad \left|\frac{dw_n}{dx}(0)\right| \to 0, \quad \text{as } n\to\infty. \tag{12.44}$$

Eliminating ζ_n from the identities in Eq. (12.42) and Eq. (12.43), we obtain the identity in w_n:

$$-\beta_n^2 w_n + A\frac{d^4w_n}{dx^4} + (c^2-\kappa)\frac{d^2w_n}{dx^2} - 2ic\beta_n\frac{dw_n}{dx} \equiv g_n + i\beta_n f_n + 2c\frac{df_n}{dx}. \tag{12.45}$$

Using the multiplier method (see References 5 and 4), we take the inner product of Eq. (12.45) with $2x\frac{dw_n}{dx}$ in $L^2(0, L)$, integrate by parts, use Eq. (12.44) with the boundary conditions of Eq. (12.26), and use the convergence indicated in Eq. (12.42) and (12.43) to obtain

$$\begin{aligned} &\|\beta_n w_n\|^2 - (c^2-\kappa)\left\|\frac{dw_n}{dx}\right\|^2 + 3A\left\|\frac{d^2w_n}{dx^2}\right\|^2 - 4i\,\beta_n c\int_0^L x\left|\frac{d\zeta_n}{dx}\right|^2 dx \\ &+ 2A\,\frac{d^2w_n}{dx^2}(0)\frac{d\overline{w_n}}{dx}(0) - AL\left|\frac{d^2w_n}{dx^2}\right|^2 \to 0. \end{aligned}$$

Restricting to the real part of this expression we have

$$\|\beta_n w_n\|^2 - (c^2-\kappa)\left\|\frac{dw_n}{dx}\right\|^2 + 3A\left\|\frac{d^2w_n}{dx^2}\right\|^2 \to 0 \quad \text{as } n\to\infty. \tag{12.46}$$

Next we take the inner product of Eq. (12.45) with w_n in $L^2(0, L)$ and use the boundary conditions at $x = L$ (see Eq. [12.27) to get

$$\begin{aligned} &-\|\beta_n w_n\|^2 - (c^2-\kappa)\left\|\frac{dw_n}{dx}\right\|^2 + A\left\|\frac{d^2w_n}{dx^2}\right\|^2 \\ &-A\frac{d^3w_n}{dx^3}(0)\overline{w}_n(0) + A\frac{d^2w_n}{dx^2}(0)\frac{d\overline{w}_n}{dx}(0) - (c^2-\kappa)\frac{dw_n}{dx}(0)\overline{w}_n(0) \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Again, using Eq. (12.44) with Eq. (12.26), the above reduces to

$$-\|\beta_n w_n\|^2 - (c^2 - \kappa) \left\| \frac{dw_n}{dx} \right\|^2 + A \left\| \frac{d^2 w_n}{dx^2} \right\|^2 \to 0 \quad \text{as } n \to \infty. \tag{12.47}$$

Combining Eq. (12.46) with Eq. (12.47) we conclude that

$$\|\beta_n w_n\|, \quad \left\| \frac{dw_n}{dx} \right\|, \quad \left\| \frac{d^2 w_n}{dx^2} \right\| \to 0 \quad \text{as } n \to 0,$$

which further implies, using Eq. (12.42), that

$$\|\zeta_n\| \to 0 \quad \text{as } n \to \infty.$$

But these two conclusions contradict the assumption $\|\mathbf{z}_n\|_{\mathcal{H}} = 1$ for all n. Therefore, the second condition of Eq. (12.41) must hold. ∎

To prove the first condition of Eq. (12.41) we again use a contradiction argument. Assuming $i\beta \in \sigma(\mathcal{A})$, there exists a sequence $\{\mathbf{z}_n\} \in \mathcal{D}(\mathcal{A})$ with $\|\mathbf{z}_n\|_{\mathcal{H}} = 1$ for all n such that

$$\lim_{n\to\infty} \|(i\beta\mathcal{I} - \mathcal{A})\,\mathbf{z}_n\|_{\mathcal{H}} = 0.$$

Then a repetition of the above argument with β_n replaced by β leads to the same contradiction and the proof is complete.

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Chapter 13

Nonlinear Perturbations of Partially Controllable Systems

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Abstract We study the question of controllability of nonlinear systems in situations where the linearized system is only partially controllable. We discuss examples where such partial controllability is imposed by the physics of the problem. We show by means of examples that characterizing the set of reachable states for the nonlinear system is, in general, a difficult problem. Finally, we discuss the application of the center manifold theorem and results on controllability of "slow" modes.

13.1 Introduction

We consider a differential system of the form

$$\dot{x} = Ax + B(x) + f(t), \tag{13.1}$$

where A is a linear operator in a Banach space X, B vanishes at quadratic order as $x \to 0$, and f is a forcing term in a given class of functions from $[0, T]$ to a subspace of X. The question of controllability is whether it is possible to steer the system from a given state $x(0) = x_0$ to another given state $x(T) = x_f$.

If x_0 and x_f are "small" in a suitable sense, we can expect to be able to use perturbation arguments. Indeed, if the system is fully controllable, and if suitable technical hypotheses hold, the implicit function theorem can be used to show that the nonlinear system is locally controllable. The situation is more complex, however, if the linearized system is only partially controllable. The absence of full controllability may be imposed by physical considerations. For instance, a part of the system may describe material or microstructural properties that cannot be modified by external control inputs. The question of reachable states in the nonlinear system then becomes quite difficult, in general.

In Section 13.2 of this chapter, we discuss some examples from polymer rheology where the physics of the problem restricts controllability. Section 13.3 gives a discussion of several examples that illustrate the variety of behaviors that may occur when partially controllable systems are perturbed by nonlinearities. Finally, in Section 13.4, we turn to cases where the linear system can be decomposed into a "slow" controllable part and a rapidly decaying part that is not controllable.

We discuss the existence of perturbed center manifolds for the controlled problem and results regarding controllability of slow modes.

13.2 Examples

We consider the motion of a nonlinear fluid of Maxwell type. The equation of motion is

$$\rho[\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}] = \operatorname{div} \mathbf{T} - \nabla p + \mathbf{f}, \tag{13.2}$$

where $\mathbf{v}$ is the velocity, p the pressure, $\mathbf{T}$ the extra stress tensor, ρ the density, and $\mathbf{f}$ a given body force. In addition, we have the incompressibility condition

$$\operatorname{div} \mathbf{v} = 0, \tag{13.3}$$

and a constitutive equation

$$\lambda[\mathbf{T}_t + (\mathbf{v} \cdot \nabla)\mathbf{T}] + \mathbf{T} + \mathbf{q}(\nabla\mathbf{v}, \mathbf{T}) = \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]. \tag{13.4}$$

Here μ is the viscosity, λ is the relaxation time, and $\mathbf{q}$ is a nonlinear term that depends on the specific constitutive model. We assume that

$$|\mathbf{q}(\nabla\mathbf{v}, \mathbf{T})| \leq C(|\nabla\mathbf{v}|^2 + |\mathbf{T}|^2), \tag{13.5}$$

if $|\nabla\mathbf{v}|$ and $|\mathbf{T}|$ are sufficiently small.

Control inputs for a viscoelastic flow are by means of the body force $\mathbf{f}$ or by means of terms in boundary conditions. Several papers in the literature establish controllability of the velocity in linear viscoelastic media [4–8], but this leaves open the question whether the stresses can be controlled. Contrary to a claim in Reference 3, it turns out that the inability to change the constitutive behavior limits controllability [11].

Consider the linearized form of the constitutive equation,

$$\lambda\mathbf{T}_t + \mathbf{T} = \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T). \tag{13.6}$$

It is immediately clear that if the initial condition for Eq. (13.6) has the form

$$\mathbf{T}(0) = \nabla\mathbf{w}_0 + (\nabla\mathbf{w}_0)^T, \tag{13.7}$$

and if $\mathbf{w}$ satisfies

$$\lambda\mathbf{w}_t + \mathbf{w} = \mu\mathbf{v}, \tag{13.8}$$

then

$$\mathbf{T} = \nabla\mathbf{w} + (\nabla\mathbf{w})^T \tag{13.9}$$

for all time. The restricted form in Eq. (13.9) forces $\mathbf{T}$ to be traceless at each point, and it imposes differential equations relating the components of $\mathbf{T}$. Hence, for the linearized system, $\mathbf{T}$ is (at best) controllable only within a restricted subspace. Unfortunately, however, this subspace does not remain invariant when nonlinearities are included in the constitutive law.

Our second example concerns molecular theories for viscoelastic fluids. The simplest of such models are known as dumbbell models. They represent a polymer molecule by two beads connected by a spring, and the crucial quantity is the probability distribution function $\psi(\mathbf{R}, \mathbf{x}, t)$, which describes the probability that a polymer molecule at position $\mathbf{x}$ and time t has end-to-end vector $\mathbf{R}$. ψ is a symmetric function of $\mathbf{R}$ and satisfies

$$\int_{\mathbb{R}^3} \psi(\mathbf{R}, \mathbf{x}, t)\, d\mathbf{R} = 1. \tag{13.10}$$

The evolution of ψ is governed by a diffusion equation of the form

$$\psi_t + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\psi = \alpha \Delta_{\mathbf{R}}\psi + \operatorname{div}_{\mathbf{R}}\,[-(\nabla_x \mathbf{v}) \cdot \mathbf{R}\psi + \beta \mathbf{F}(\mathbf{R})\psi], \tag{13.11}$$

where α and β are constants, and $\mathbf{F}(\mathbf{R})$ is the force in the spring connecting the beads. The stress tensor in the equation of motion is given by

$$\mathbf{T}(\mathbf{x}, t) = \gamma \int_{\mathbb{R}^3} \mathbf{R} \otimes \mathbf{F}(\mathbf{R})\psi(\mathbf{R}, \mathbf{x}, t)\, d\mathbf{R}, \tag{13.12}$$

where γ is a constant, and $\otimes$ denotes the dyadic product.

In the absence of fluid motion, Eq. (13.11) has the equilibrium solution

$$\psi_0(\mathbf{R}) = \frac{e^{-U(\mathbf{R})}}{\int_{\mathbb{R}^3} e^{-U(\mathbf{R}')}\, d\mathbf{R}'}, \tag{13.13}$$

where

$$\nabla_R U(\mathbf{R}) = \frac{\beta}{\alpha}\mathbf{F}(\mathbf{R}). \tag{13.14}$$

The linearization of Eq. (13.11) at this solution yields

$$\psi_t = \alpha \Delta_{\mathbf{R}}\psi + \operatorname{div}_{\mathbf{R}}\,[\beta \mathbf{F}(\mathbf{R})\psi] - \operatorname{div}_{\mathbf{R}}\,[(\nabla_x \mathbf{v}) \cdot \mathbf{R}\psi_0]. \tag{13.15}$$

We can expand a solution of Eq. (13.15) in spherical harmonics:

$$\psi(\mathbf{R}, \mathbf{x}, t) = \sum_{L,m} \psi_{Lm}(|\mathbf{R}|, \mathbf{x}, t) Y_L^m(\theta, \phi). \tag{13.16}$$

Because ψ_0 is radially symmetric, it follows that the inhomogeneous term in Eq. (13.15) contributes only terms with $L = 2$. All contributions to ψ corresponding to spherical harmonics of higher order are consequently independent of the evolution of the velocity and not accessible to control inputs. The inclusion of nonlinear terms, however, destroys the decoupling of spherical harmonics.

13.3 Examples of Reachable Sets

The purpose of this section is to present a number of elementary examples of nonlinear control problems where the linearization is only partially controlled. As we shall see, the qualitative nature of the reachable set in the nonlinear case can vary considerably, making it appear unlikely that general theorems can be formulated.

We start with the problem

$$\dot{x} = f(t), \quad \dot{y} + \lambda y = x^2, \tag{13.17}$$

with given initial conditions $x(0) = x_0$, $y(0) = y_0$. Obviously, we can choose $f(t)$ to make $x(t)$ any function we want, whereas in the linearized problem the control has no effect on y. In the nonlinear case, we have

$$y(t) = e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)} x(s)^2 \, ds. \tag{13.18}$$

The set which can be reached at time T is

$$\left\{(x, y) \mid y > e^{-\lambda T} y_0\right\} \tag{13.19}$$

(unless $x_0 = 0$ in which case the point (0, 0) is also included). If, on the other hand, we replace x^2 in Eq. (13.17) by x^3, then the reachable set becomes the entire plane.

Now, let us expand the system to

$$\dot{x} = f(t), \quad \dot{y} + \lambda y = x^2, \quad \dot{z} + \mu z = x^2, \tag{13.20}$$

where $\lambda > \mu > 0$. Again, we can choose f to make $x(t)$ any function we want it to be, and we then have

$$\begin{aligned} y(t) &= e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)} x(s)^2 \, ds, \\ z(t) &= e^{-\mu t} z_0 + \int_0^t e^{-\mu(t-s)} x(s)^2 \, ds. \end{aligned} \tag{13.21}$$

It is not difficult to verify that now the reachable set at time T is (for $x_0 \neq 0$) defined by the inequalities

$$\begin{aligned} y(T) &> e^{-\lambda T} y_0, \\ z(T) &> e^{-\mu T} z_0, \\ y(T) - e^{\lambda T} y_0 < z(T) &- e^{\mu T} z_0 < e^{(\lambda-\mu)T}\left[y(T) - e^{-\lambda T} y_0\right]. \end{aligned} \tag{13.22}$$

Now consider the system

$$\dot{x} = u, \quad \dot{u} = f(t), \quad \dot{y} + \lambda y = u^2, \tag{13.23}$$

with initial conditions $x(0) = x_0$, $u(0) = u_0$, $y(0) = y_0$. Again, we can reach any final value of $x(T)$ and $u(T)$ that we would like to reach, but we have the relationship

$$I := x(T) - x_0 = \int_0^T u(s) \, ds. \tag{13.24}$$

For a given value of I, we find that

$$P := \int_0^T e^{-\lambda(T-s)} u(s)^2 \, ds \tag{13.25}$$

is minimized when

$$u(s) = \frac{I \lambda e^{-\lambda s}}{1 - e^{-\lambda T}}, \tag{13.26}$$

and the corresponding value of P is

$$P_0 = \frac{I^2 \lambda}{e^{\lambda T} - 1}. \tag{13.27}$$

Consequently the reachable set is (except for exceptional values when equality is allowed) described by the inequality

$$y(T) - e^{-\lambda T} y_0 > \frac{[x(T) - x(0)]^2 \lambda}{e^{\lambda T} - 1}. \tag{13.28}$$

Let us now add an additional equation

$$\dot{x} = u, \quad \dot{u} = f(t), \quad \dot{y} + \lambda y = u^2, \quad \dot{z} + \mu z = u^2, \tag{13.29}$$

with initial conditions $x(0) = x_0$, $u(0) = u_0$, $y(0) = y_0$, $z(0) = z_0$. Now the issue in determining the reachable set is how the value of

$$I = \int_0^T u(s)\, ds \tag{13.30}$$

restricts the values of

$$P = \int_0^T e^{-\lambda(T-s)} u(s)^2\, ds \tag{13.31}$$

and

$$Q = \int_0^T e^{-\mu(T-s)} u(s)^2\, ds. \tag{13.32}$$

To characterize the set of possible values P and Q, we determine its boundary. Points on the boundary correspond to functions u for which the linearized map

$$v \mapsto \left[2 \int_0^T e^{-\lambda(T-s)} u(s) v(s)\, ds, 2 \int_0^T e^{-\mu(T-s)} u(s) v(s)\, ds\right] \tag{13.33}$$

is not surjective from

$$\left\{ v \in L^2(0,T) \middle| \int_0^T v(s)\, ds = 0 \right\} \tag{13.34}$$

to $\mathbb{R}^2$. This is the case if a linear combination

$$\alpha e^{\lambda s} u(s) + \beta e^{\mu s} u(s) \tag{13.35}$$

is a constant. If $\lambda > \mu$, then the condition that u be square integrable imposes the restriction

$$\frac{\alpha}{\beta} > -e^{(\mu-\lambda)T} \quad \text{or} \quad \frac{\alpha}{\beta} < -1. \tag{13.36}$$

The corresponding u can be written as

$$u(s) = \frac{I\mu}{k\left(\frac{\alpha}{\beta} e^{\lambda s} + e^{\mu s}\right)}, \tag{13.37}$$

where

$$k = {}_2F_1\left(\frac{\mu}{\mu-\lambda}, 1, \frac{\lambda - 2\mu}{\lambda - \mu}, -\frac{\alpha}{\beta}\right) - e^{-\mu T} {}_2F_1\left[\frac{\mu}{\mu-\lambda}, 1, \frac{\lambda-2\mu}{\lambda-\mu}, -\frac{\alpha}{\beta} e^{(\lambda-\mu)T}\right]. \tag{13.38}$$

(${}_2F_1$ denotes the hypergeometric function.)

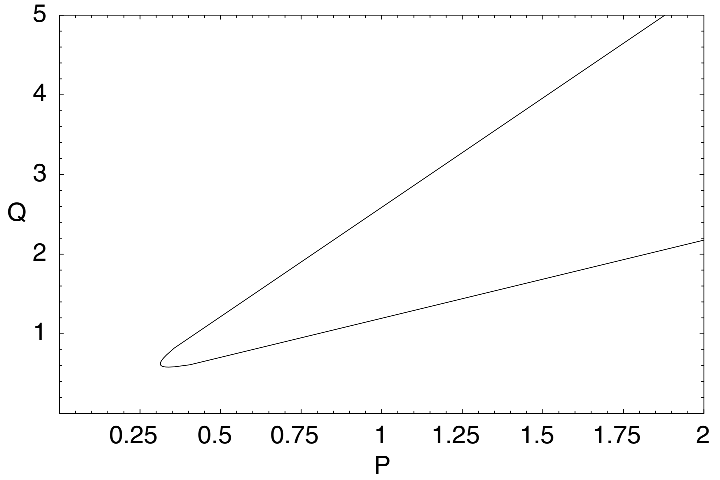

FIGURE 13.1: Plot of the accessible region (above and to the right of the curve shown) in the (P, Q) plane for $\lambda = 2$, $\mu = 1$, $T = 1$, $I = 1$.

Figure 13.1 shows the accessible set in the (P, Q) plane for the case $\lambda = 2$, $\mu = 1$, $T = 1$, and $I = 1$.

In all the examples above, the reachable set turned out to have non-empty interior. This does not need to be so. We consider a special case of the molecular dumbbell model introduced in the previous section. We consider the evolution of the distribution function associated with one specific fluid particle, and we assume that the spring force is linear. Then Eq. (13.11) is specialized to

$$\psi_t = \alpha \Delta_{\mathbf{R}} \psi + \mathrm{div}_{\mathbf{R}} [-\mathbf{L}(t) \cdot \mathbf{R}\psi + \beta \mathbf{R}\psi], \tag{13.39}$$

where $\mathbf{L}(t)$ is the velocity gradient at time t and at the position that the given fluid particle occupies at this time. It is readily shown that if the initial condition for ψ has the form

$$\psi(\mathbf{R}, 0) = \gamma_0 e^{-\mathbf{R}\cdot\mathbf{A}_0\mathbf{R}}, \tag{13.40}$$

then a solution can be found in the form

$$\psi(\mathbf{R}, t) = \gamma(t) e^{-\mathbf{R}\cdot\mathbf{A}(t)\mathbf{R}}, \tag{13.41}$$

see Reference 9. So, regardless of any control input $\mathbf{L}(t)$, we cannot drive ψ off this invariant manifold if the initial condition lies on it.

Our final example concerns a special case of Eq. (13.4). We consider the upper convected Maxwell model

$$\lambda\left[\mathbf{T}_t + (\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T\right] + \mathbf{T} = \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]. \tag{13.42}$$

We shall ignore the question of constraints relating $\mathbf{T}$ at different points in space and merely ask the more limited question of point-wise constraints on the values of $\mathbf{T}$. In the linearized case, the incompressibility condition forces $\mathbf{T}$ to have zero trace. In the nonlinear case, we integrate Eq. (13.42) to determine constraints on $\mathbf{T}$. Let $\mathbf{F}(\mathbf{x}, t, s)$ be such that

$$\mathbf{F}(\mathbf{x}, s, s) = \mathbf{I}, \quad \mathbf{F}_t(\mathbf{x}, t, s) + (\mathbf{v}\cdot\nabla)\mathbf{F}(\mathbf{x}, t, s) = (\nabla\mathbf{v})\mathbf{F}(\mathbf{x}, t, s). \tag{13.43}$$

Then the solution of Eq. (13.42) is given by

$$\mathbf{T}(\mathbf{x}, t) = \frac{\mu}{\lambda^2}\int_{-\infty}^{t} e^{-(t-s)/\lambda}[\mathbf{F}(\mathbf{x}, t, s)\mathbf{F}(\mathbf{x}, t, s)^T - \mathbf{I}]\, ds. \tag{13.44}$$

The restriction on $\mathbf{F}(\mathbf{x}, t, s)\mathbf{F}(\mathbf{x}, t, s)^T$ is that it is symmetric, positive definite, and of determinant one (the latter follows from incompressibility). We can conclude that

$$\frac{\lambda}{\mu}\mathbf{T} + \mathbf{I} \tag{13.45}$$

is a convex linear combination of positive definite symmetric matrices with determinant one. As we show in the appendix, this imposes the restriction

$$\det\left(\frac{\lambda}{\mu}\mathbf{T} + \mathbf{I}\right) \geq 1. \tag{13.46}$$

This condition is more restrictive than the Rutkevich stability condition of Reference 12, which merely requires $\frac{\lambda}{\mu}\mathbf{T} + \mathbf{I}$ to be positive definite. The inequality of Eq. (13.46) must hold regardless of any initial condition on $\mathbf{T}$. The imposition of an initial condition restricts the stresses further.

13.4 Center Manifolds

We now consider a system of the form

$$\begin{aligned} \dot{x} &= Ax + p(x, y) + f(t), \\ \dot{y} &= By + q(x, y), \end{aligned} \tag{13.47}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. We make the following assumptions:

1. All eigenvalues λ of A satisfy $\operatorname{Re}\lambda \geq M_1$, all eigenvalues of B satisfy $\operatorname{Re}\lambda \leq M_2 < 0$, and there exists an integer $k \geq 1$ such that $M_2 < kM_1$.

2. p and q are smooth functions and for small $|x|$ and $|y|$ we have $|p(x, y)| + |q(x, y)| \leq C(|x|^2 + |y|^2)$.

3. The control f is of the form

$$f(t) = \sum_{i=1}^{n} \gamma_i f_i(t), \tag{13.48}$$

where the $f_i(t)$ are given functions. We assume that the f_i are smooth on $(-\infty, \infty)$ and have compact support in $[0, T]$. Moreover, the mapping

$$\Gamma := (\gamma_1, \ldots, \gamma_n) \mapsto \int_0^T e^{A(t-s)} f(s)\, ds \tag{13.49}$$

is invertible.

If we linearize the system at $x = 0$, $y = 0$, then clearly only the x-part is controllable. On the other hand, the y-part of the system is stable, and control of y may be practically irrelevant if T is large enough to allow decay of y. In the nonlinear case, the uncontrolled system will have a center-unstable manifold of the form $y = g(x)$. We can expect that if we start with initial data on or close to this manifold, and the control is small enough, then we will remain close to this manifold. In this fashion, we should be able to control x, and we may have no need to control y.

In the presence of the control, we can use the usual trick of augmenting our system by the equation $\dot{\Gamma} = 0$. A version of the center manifold theorem suitable for our purposes is given in Reference 2.

The result proved there yields the existence of a locally invariant manifold of the form $y = g(x, \Gamma, t)$, where g is smooth at least of class $C^{k-1,1}$, and $|g(x, \Gamma, t)| = o(|x| + |\Gamma|)$ as $|x| + |\Gamma| \to 0$. Moreover, the construction of the invariant manifold implies that $g(x, \Gamma, t)$ is independent of t for $t < 0$, so that before onset of the control, g simply defines a center unstable manifold for the uncontrolled system. Finally, a standard perturbation argument shows that the reduced system

$$\dot{x} = Ax + p[x, g(x, \Gamma, t)] + \sum_{i=1}^{n} \gamma_i f_i(t) \tag{13.50}$$

is controllable in a neighborhood of the origin (i.e., if $|x_0|$ and $|x_f|$ are sufficiently small, there exists a Γ such that the solution with initial condition $x(0) = x_0$ satisfies $x(T) = x_f$).

We now consider the existence of spectral separation in the physical examples introduced earlier. Consider the motion of a viscoelastic Maxwell fluid. The linearized equations are

$$\begin{aligned} \rho \mathbf{v}_t &= \operatorname{div} \mathbf{T} - \nabla p + \mathbf{f}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \lambda \mathbf{T}_t + \mathbf{T} &= \mu[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]. \end{aligned} \tag{13.51}$$

We consider these equations on a bounded domain $\Omega \subset \mathbb{R}^3$ with the Dirichlet boundary condition $\mathbf{v} = 0$. By combining the equation of motion with the constitutive law, we obtain an equation for the velocity alone,

$$\begin{aligned} \rho(\lambda \mathbf{v}_{tt} + \mathbf{v}_t) &= \mu \Delta \mathbf{v} - \nabla(\lambda p_t + p) + \mathbf{f}_t + \lambda \mathbf{f}, \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned} \tag{13.52}$$

The eigenspectrum for this equation (the controllable part of the system) is given by

$$\rho(\lambda r^2 + r) = \mu q_n, \tag{13.53}$$

where $q_n < 0$ are the eigenvalues of the Stokes operator. If $\rho < -4\lambda\mu q_1$, then all the eigenvalues r have real part equal to $-1/(2\lambda)$. On the other hand, the spectrum of the uncontrolled part of the system consists of the point $-1/\lambda$. Consequently, a spectral separation condition as required by the center manifold theorem holds.

In contrast to problems involving ordinary differential equations (ODEs), serious technical issues need to be resolved to apply center manifold methods to hyperbolic partial differential equations (PDEs) such as those governing viscoelastic flows. See Reference 10 for some efforts in this direction.

Let us now consider the spectral problem associated with the diffusion equation (13.11). The linearization at the rest state is

$$\psi_t = \alpha \Delta_{\mathbf{R}} \psi + \operatorname{div}_{\mathbf{R}}(\beta \mathbf{R} \psi), \tag{13.54}$$

if we assume a linear spring. The associated eigenvalue problem is

$$\lambda \psi = \alpha \Delta_{\mathbf{R}} \psi + \operatorname{div}_{\mathbf{R}}(\beta \mathbf{R} \psi). \tag{13.55}$$

As usual, we can separate variables, and we end up with the radial problem

$$\lambda \psi = \alpha \left[\psi_{RR} + \frac{2}{R} \psi_R - \frac{L(L+1)}{R^2} \psi \right] + \beta(R \psi_R + 3\psi), \tag{13.56}$$

where $R = |\mathbf{R}|$. The solution of this equation, which is regular at the origin, is given by

$$\psi = R^L \, {}_1F_1\left(\frac{3+L}{2} - \frac{\lambda}{2\beta}, \frac{3}{2} + L, -\frac{\beta R^2}{2\alpha} \right), \tag{13.57}$$

where ${}_1F_1$ denotes the confluent hypergeometric function. If we require ψ to vanish faster than any power of R at infinity, then $L/2 + \lambda/(2\beta)$ must be a nonpositive integer $-N$ (see formula [13.1.5] in Reference 1). Consequently, the eigenvalues are

$$\lambda_{N,L} = -\beta(2N + L). \tag{13.58}$$

The corresponding eigenfunction is

$$\begin{aligned}
\psi &= R^L\, {}_1F_1\left(N + L + \frac{3}{2}, L + \frac{3}{2}, -\frac{\beta R^2}{2\alpha}\right) \\
&= R^L \exp\left(-\frac{\beta R^2}{2\alpha}\right) {}_1F_1\left(-N, L + \frac{3}{2}, \frac{\beta R^2}{2\alpha}\right) \\
&= \binom{N + L + 1/2}{N} R^L \exp\left(-\frac{\beta R^2}{2\alpha}\right) L_N^{L+1/2}\left(\frac{\beta R^2}{2\alpha}\right).
\end{aligned} \tag{13.59}$$

$L_N^{L+1/2}$ denotes the generalized Laguerre polynomial (see formulae [13.1.27] and [22.5.54] in Reference 1).

Physically, only even values of L are relevant, and the amplitude of the mode with $N = L = 0$ is prescribed due to the interpretation of ψ as a probability density, that is, the constraint of Eq. (13.10). Consequently, the least stable eigenvalue corresponds to $N = 1$, $L = 0$ and $N = 0$, $L = 2$. The perturbation in Eq. (13.15) resulting from the presence of a velocity gradient lies in the (five-dimensional) eigenspace corresponding to $N = 0$, $L = 2$.

Appendix: A Result from Linear Algebra

We shall prove the following result:

THEOREM 13.1

A symmetric matrix $\mathbf{A}$ *is a convex linear combination of positive definite matrices with determinant 1 if and only if* $\mathbf{A}$ *is positive definite and* $\det \mathbf{A} \geq 1$.

We first prove that the condition is sufficient. Without loss of generality, we can choose our basis such that $\mathbf{A}$ is diagonal. Now pick a 2×2 block of $\mathbf{A}$, which is, say, of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{13.60}$$

and let $\mathbf{A}(\mu)$ be the matrix obtained by replacing this block in $\mathbf{A}$ with

$$\begin{pmatrix} \lambda_1 & \mu \\ \mu & \lambda_2 \end{pmatrix}. \tag{13.61}$$

If $\det \mathbf{A} > 1$, then $\det \mathbf{A}(0) > 1$. On the other hand, $\det \mathbf{A}(\mu)$ is negative if $|\mu|$ is large. Consequently, there exist $\mu_1 > 0$, $\mu_2 > 0$ such that

$$\det \mathbf{A}(\mu_1) = \det \mathbf{A}(-\mu_2) = 1. \tag{13.62}$$

It follows that

$$\mathbf{A} = \frac{\mu_2}{\mu_1 + \mu_2}\mathbf{A}(\mu_1) + \frac{\mu_1}{\mu_1 + \mu_2}\mathbf{A}(-\mu_2). \tag{13.63}$$

To prove necessity, we start with the following lemma.

LEMMA 13.1
Let $\mathbf{A}$, $\mathbf{B}$ *be positive definite symmetric matrices with* $\det \mathbf{B} \geq \det \mathbf{A}$, $\mathbf{B} \neq \mathbf{A}$. *Then*

$$\frac{d}{d\lambda} \det [(1-\lambda)\mathbf{A} + \lambda \mathbf{B}] \Big|_{\lambda=0} > 0. \tag{13.64}$$

The derivative in question is equal to $\det \mathbf{A} \operatorname{tr} [\mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})]$, so it suffices to prove that

$$\operatorname{tr} (\mathbf{A}^{-1}\mathbf{B} - \mathbf{I}) > 0. \tag{13.65}$$

We note that $\operatorname{tr} (\mathbf{A}^{-1}\mathbf{B}) = \operatorname{tr} (\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2})$. The latter matrix is symmetric and positive definite with determinant greater or equal to 1. Let λ_i be its eigenvalues, and let m be the dimension of the matrix. Then $\lambda_i > 0$,

$$\prod_{i=1}^{m} \lambda_i \geq 1, \tag{13.66}$$

and we need to prove that

$$\sum_{i=1}^{m} \lambda_i > m. \tag{13.67}$$

To show this, we simply need to consider the problem of maximizing $\prod_{i=1}^{m} \lambda_i$ for a given $\sum_{i=1}^{m} \lambda_i$. The usual method of Lagrange multipliers yields the result that the maximum is achieved when the λ_i values are equal.

An immediate consequence of the lemma is that for two distinct positive matrices $\mathbf{A}$ and $\mathbf{B}$, we have

$$\det [\lambda \mathbf{A} + (1-\lambda)\mathbf{B}] > \min(\det \mathbf{A},\ \det \mathbf{B}) \tag{13.68}$$

for $0 < \lambda < 1$. The necessity of the condition in the theorem now follows by recursive application of this result.

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Chapter 14

On Junctions in a Network of Canals

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Abstract In this paper we derive a new model for the junctions in a network of canals individually modeled by Saint-Venant equations.

14.1 Introduction

Jack Lagnese, Guenter Leugering, and the author have been collaborating for over a decade on the modeling and analysis of a variety of linked structures in which distinct components are joined together. In particular, this collaboration, spurred on by the focused energy of Jack Lagnese, led to a monograph [4]. More recently in Reference 3, stimulated by the earlier paper of Coron et al. [1], Leugering and Schmidt considered networks of canals. Our approach, partially in the framework of Reference 4, drew heavily on results from the literature on conservation laws to obtain results on stabilization of flows in a simple star configuration of canals. The model used derived the Saint-Venant equations, first introduced in Reference 5, from a variational principle that also yielded conditions governing the dynamics of the flow through the junctions at which canals meet. Capturing the character of the junctions at which various components of a linked structure meet has repeatedly presented a challenge. In the case of the canal networks, the conditions at junctions that were used in Reference 3 did not take into account the angles at which the canals meet nor any other features of the geometry of the junction.

In this paper we present a new approach to the modeling of the canal junctions better reflecting the geometry. The model presents challenging new, and as yet unresolved, analytic problems. In particular, this author does not know of any results covering the basic existence theory of the kinds of boundary value problems to which the model given below leads that involve partial differential equations along each canal and ordinary differential equations (ODEs) at the junctions.

14.2 Derivation of the Model

We consider m slender canals corresponding to rectangular domains D_i (with $i = 1, \ldots, m$) flowing into a small junction region occupying a domain D_0. Let $\Sigma_i = \partial D_i \cap \partial D_0$, assumed to be a line

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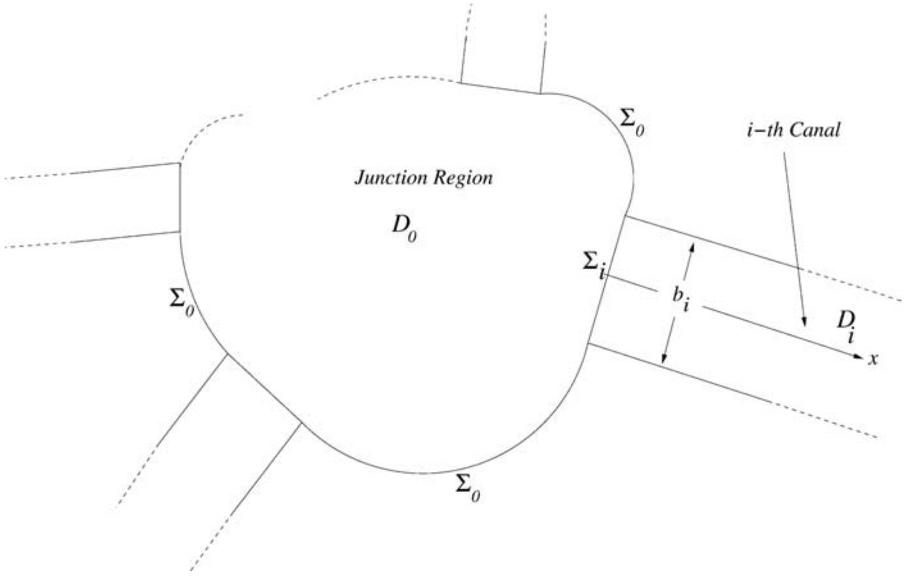

FIGURE 14.1: Notational illustration.

segment orthogonal to the canal, and $\Sigma_0 = \partial D_0 - \cup_{i=1}^m \Sigma_i$. Let $|D_0|$ denote the area of D_0 and b_i be the width of D_i, which is equal to the length of Σ_i. We describe the flow of water through the canals by Saint-Venant equations (or shallow water equations in one space dimension) involving scalar variables $H_i(x,t)$ (the average height of the water surface across the canal) and $V_i(x,t)$ (the average flow velocity across the canal in the canal direction), which are functions of the distance x along the canal measured from the interface Σ_i and of the time t (see Figure 14.1). Working in terms of averaged quantities across the canal allows one to eliminate the second spatial variable transverse to the canal.

We seek a simple model of the junction region that takes into account the geometry of the junction without using full shallow water equations over the region D_0. We do this in such a way that the junction is governed by a system of ODEs interacting with the boundary values of H_i and V_i on Σ_i. Assuming that the junction is small relative to the canals, we let $H_0(t)$ be the average height of the water surface over D_0.

The velocity vector field across D_0 will be given in terms of a finite basis containing sufficient functions to enable us to match fluxes over the m interfaces Σ_i between canals and the junction and to incorporate the fact that water enters and leaves the junction via only the canals.These requirements cannot simply be achieved by using averaged velocity fields across D_0. We take

$$\mathbf{V}_0(\mathbf{x},t) = \sum_{i=1}^{m} \alpha_i(t) \nabla u_i(\mathbf{x})$$

where $\mathbf{x} = (x_1, x_2)$ denotes the pair of coordinates of a point in D_0 and $u_i(\mathbf{x})$ is a solution (unique up to a constant) of the boundary value problem

$$\triangle u_i(\mathbf{x}) = \frac{b_i}{|D_0|}, \quad \frac{\partial u_i}{\partial \mathbf{n}} = \begin{cases} 1 & \text{on } \Sigma_i \\ 0 & \text{on } \partial D_0 - \Sigma_i. \end{cases} \tag{14.1}$$

We note that the following compatibility condition for the solvability of this Neumann problem is automatically satisfied:

$$\int_{D_0} \triangle u_i \, dx = \int_{D_0} \frac{b_i}{|D_0|} \, dx = \int_{\Sigma_i} 1 \, ds = \int_{\partial D_0} \frac{\partial u_i}{\partial \mathbf{n}} \, ds.$$

The conservation law for the fluid in D_0, namely $\frac{\partial H_0}{\partial t} + \nabla \cdot [H_0 V_0] = 0$, becomes

$$\frac{dH_0}{dt} + H_0 \sum_{i=1}^{m} \alpha_i(t) \frac{b_i}{|D_0|} = 0, \tag{14.2}$$

which is the first of the system of nonlinear ODEs to be satisfied by $H_0(t)$ and $\boldsymbol{\alpha}(t) = (\alpha_1(t).\dots, \alpha_m(t))$ at the junction. The flux across ∂D_0 is

$$H_0 \sum_{i=1}^{m} \alpha_i(t) \frac{\partial u_i}{\partial \mathbf{n}}(x) = \begin{cases} 0 & \text{on } \Sigma_0 \\ H_0 \alpha_l & \text{on } \Sigma_l. \end{cases} \tag{14.3}$$

We also require that across each Σ_i

$$\alpha_i H_0 = H_i V_i. \tag{14.4}$$

These last two conditions are "geometric constraints" that will have to be satisfied by our systems of variables when we use variational methods to derive dynamic equations. Together they imply, on multiplying Eq. (14.2) by $|D_0|$, that the rate of change of the volume of water in the junction is exactly equal to the flow rate into or out of the totality of canals meeting the junction.

We need to introduce some additional notation. We let L_i be the length of the i-th canal and e_i be the elevation of the bed of that canal. We introduce a Lagrangian function

$$\mathcal{L}(H_0, \boldsymbol{\alpha}, H_1, V_1, \dots, H_m, V_m) = \mathcal{L}_0(H_0, \mathbf{V}_0) + \sum_{i=1}^{m} \mathcal{L}_i(H_i, V_i). \tag{14.5}$$

where

$$\mathcal{L}_i(H_i, V_i) = b_i \int_0^T \int_0^{L_i} \left[\frac{1}{2} H_i V_i^2 - g\left(\frac{1}{2} H_i^2 + H_i e_i\right)\right] dx\, dt \tag{14.6}$$

and

$$\begin{aligned}\mathcal{L}_0(H_0, \boldsymbol{\alpha}) &= \int_0^T \iint_{D_0} \left[\frac{1}{2} H_0 |V_0|^2 - g\left(\frac{1}{2} H_0^2 + H_0 e_0\right)\right] dA\, dt \\ &= \int_0^T \left[\frac{1}{2} H_0 P\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} - |D_0| g\left(\frac{1}{2} H_0^2 + H_0 e_0\right)\right] dt \end{aligned} \tag{14.7}$$

with P the positive definite symmetric matrix defined by

$$P_{ij} = \iint_{D_0} \nabla u_i \cdot \nabla u_j dA = -\frac{b_i}{|D_0|} \iint_{D_0} u_j\, dA + \int_{\Sigma_i} u_j\, dS. \tag{14.8}$$

We note the dependence of P on the geometry of the junction region and its interfaces with the canals.

We perform variations on the state variables $H_0, \boldsymbol{\alpha}, H_1, V_1, \dots, H_m, V_m$. These have to respect the geometric constraints of Eq. (14.2) and Eq. (14.4) at the junction as well as the equation of continuity

$$\partial_t H_i + \partial_x [H_i V_i] = 0. \tag{14.9}$$

along each canal. That equation is the first of the Saint-Venant equations. Using standard notation one finds

$$\begin{aligned}\delta\mathcal{L} = \int_0^T \Bigg\{ & H_0 P\boldsymbol{\alpha} \cdot \delta\boldsymbol{\alpha} + \left[\frac{1}{2} P\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} - |D_0| g(H_0 + e_0)\right] \delta H_0 \\ & + \sum_{i=1}^{m} b_i \int_0^{L_i} \left[H_i V_i \delta V_i + \left(\frac{1}{2} V_i^2 - g(H_i + e_i)\right) \delta H_i\right] \Bigg\} dt. \end{aligned} \tag{14.10}$$

We can satisfy Eq. (14.9) by variations of the form

$$\delta H_i = \partial_x \phi_i, \quad \delta[H_i V_i] = -\partial_t \phi_i, \tag{14.11}$$

where ϕ_i is a smooth function of x and t, so that

$$H_i \delta V_i = -\partial_t \phi_i - V_i \partial_x \phi_i. \tag{14.12}$$

The conditions of Eq. (14.2) and Eq. (14.4) at the junction become

$$|D_0| \frac{d}{dt} \delta H_0 + \left(\sum_{i=1}^{m} \alpha_i b_i \right) \delta H_0 + H_0 \sum_{i=1}^{m} b_i \delta \alpha_i = 0; \tag{14.13}$$

$$\alpha_i \delta H_0 + H_0 \delta \alpha_i = \delta[H_i V_i(0,t)] = -\partial_t \phi_i(0,t). \tag{14.14}$$

These conditions imply

$$|D_0| \frac{d}{dt} \delta H_0 - \sum_{i=1}^{m} b_i \partial_t \phi_i(0,t) = 0$$

and hence, assuming that the variations all vanish at $t = 0$,

$$|D_0| \delta H_0(t) = \sum_{i=1}^{m} b_i \phi_i(0,t).$$

So, because of Eq. (14.14),

$$H_0 \delta \alpha_i = -\partial_t \phi_i(0,t) - \alpha_i \delta H_0(t) \tag{14.15}$$

$$= -\partial_t \phi_i(0,t) - \frac{\alpha_i}{|D_0|} \sum_{i=1}^{n} b_i \phi_i(0,t). \tag{14.16}$$

We substitute this back into the expression of Eq. (14.10) for $\delta\mathcal{L}$, using the vector notation $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, to get

$$\begin{aligned}
\delta\mathcal{L} &= \int_0^T \left\{ P\boldsymbol{\alpha} \cdot \left[-\partial_t \boldsymbol{\phi} - \frac{\boldsymbol{\alpha}}{|D_0|} \mathbf{b} \cdot \boldsymbol{\phi} \right] \right. \\
&\quad + \left[\frac{1}{2} P\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} - |D_0| g(H_0 + e_0) \right] \frac{1}{|D_0|} \mathbf{b} \cdot \boldsymbol{\phi}(0,t) \\
&\quad \left. + \sum_{i=1}^{n} \int_0^{L_i} \left[-V_i \partial_t \phi_i - S_i \partial_x \phi_i \right] dx \right\} dt \\
&= \int_0^T \left\{ -P\boldsymbol{\alpha} \cdot \partial_t \boldsymbol{\phi}(0,t) - S_0 \mathbf{b} \cdot \boldsymbol{\phi}(0,t) + \sum_{i=1}^{n} b_i S_i \phi_i(0,t) \right\} dt
\end{aligned} \tag{14.17}$$

with

$$S_i = \frac{1}{2} V_i^2 + g(H_i + e_i) \quad \text{and} \quad S_0 = \frac{1}{|D_0|} \frac{P\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}}{2} + g(H_0 + e_0).$$

Then, if we choose arbitrary test functions ϕ_i vanishing near $x = 0$, $\delta\mathcal{L} = 0$ yields the weak forms of the second Saint-Venant equation, which describes the individual canal dynamics:

$$\partial_t V_i + \partial_x S_i = 0, \tag{14.18}$$

and subsequently, using general admissible test functions, we get the system of equations

$$\sum_{j=1}^{m} P_{ij} \frac{d\alpha_j}{dt} = b_i (S_0 - S_i) \quad \text{for } i = 1, \ldots, m. \tag{14.19}$$

governing the dynamics at the junction.

14.3 Reflections on the Model

This modeling process has led us to the system consisting of the familiar partial differential Saint-Venant equations

$$\partial_t H_i + \partial_x [H_i V_i] = 0 \quad \text{and} \quad \partial_t V_i + \partial_x S_i = 0 \tag{14.20}$$

along each canal and the ordinary differential junction equations

$$\frac{dH_0}{dt} + H_0 \sum_{i=1}^{m} \alpha_i(t) \frac{b_i}{|D_0|} = 0 \quad \text{and} \quad \sum_{j=1}^{m} P_{ij} \frac{d\alpha_j}{dt} = b_i(S_0 - S_i) \quad \text{for } i = 1, \ldots, m. \tag{14.21}$$

These equations have to be supplemented by boundary conditions describing the flow of water through the ends of the canals not adjoining the junction. The variables $H_i(x, t)$, $V_i(x, t)$ along the canals and $H_0(t)$, $\boldsymbol{\alpha}(t)$ interact via the term $S_i(0, t) - S_0(t)$ in the right-hand side of the second equation at the junction and the conservation conditions

$$\alpha_i(t) H_0(t) = H_i(0, t) V_i(0, t) \quad \text{for } i = 1, \ldots, m. \tag{14.22}$$

One can certainly adapt this model to networks of canals with many junctions. This simply involves use of appropriate notation such as that used in Reference 4.

By rescaling the length parameter along each canal, one can assume that for all canals $x \in [0, L]$. In that case the junctions of a particular canal with other canals may occur at either or both endpoints. The equations for H_i, V_i can then be assembled into a single system of conservation laws on $[0, L]$, which has a quite transparent structure. However, the conditions of Eq. (14.2), Eq. (14.4), and Eq. (14.19) yield rather complicated boundary conditions at the endpoints involving the additional variables H_0 and $\boldsymbol{\alpha}$ corresponding to each junction. The theory of boundary conditions for systems of conservation laws is delicate and depends on the sub- or supercritical nature of the flows. Using results available in the monograph of Li Ta-tsien [6], we were able in Reference 3 to obtain existence and stabilizability results for a star configuration of canals by using a more primitive model of the junction and with certain feedback mechanisms operative at the other ends of the canals. The hybrid system derived here presents significant additional challenges!

The relationship between the positive definite matrix P and the geometry of a junction merits further analysis, and it would also be interesting to obtain an alternative derivation of the conditions by a limiting process starting with the Navier-Stokes equations.

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Chapter 15

On Uniform Null Controllability and Blowup Estimates

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Abstract The paper looks at some of the techniques (separation of variables, Fourier series, Carleman estimates) used to obtain observability and null-controllability results to determine qualitatively the asymptotic behavior of the estimates obtained with respect to relevant parameters. We are particularly concerned with blowup as the control time becomes short ($T \to 0$).

15.1 Introduction

Most of the work done on observability or null controllability for distributed parameter systems refers to single settings, but increasingly one is interested to consider the relation among a family of such problems depending on some parameter (as, e.g., the coefficients of the equations or the length of the time interval or a discretization mesh size). Our twin themes here—*uniformity* and *blowup*—are, of course, just two possible alternatives in an investigation of asymptotic behavior. Here, in looking at some of the techniques used to obtain observability and null-controllability results, we will examine them more carefully to determine the asymptotic behavior of the estimates obtained with respect to relevant parameters.

We will be particularly concerned with blowup as the control time becomes short ($T \to 0$), and so we will restrict our attention to problems, as for the heat equation, with no minimum control time. This question of blowup as $T \to 0$ was apparently introduced for distributed parameter systems in Reference 26 (and only later for finite dimensional systems in Reference 30) and has recently been the subject of greater interest (cf., e.g., References 33, 2, 3, 4, and 5) partly in response to Da Prato's observation [7] that this is relevant to the analysis of corresponding stochastic differential equations.

A few of the examples presented here are new, but many are relevant historical examples revisited. These have largely been drawn from my own earlier work, simply because of my greater familiarity with that. On the other hand, the central position of references to Reference 31 may be considered a shameless bit of advertising for a result in nonharmonic analysis specifically designed for its relation to the present concerns.

Consider an abstract linear autonomous system

$$u_t = \mathbf{A}u \tag{15.1}$$

where we include the homogeneous boundary conditions in specification of the domain of the operator **A**. Given observation of

$$y(t) = \mathbf{B}u(t,\cdot) \quad \text{for } 0 \le t \le T, \tag{15.2}$$

our principal concern here is with an *observability estimate*:

$$\|u(T,\cdot)\| \le C\|y(\cdot)\|. \tag{15.3}$$

We speak of *uniform observability* for a family $\{(\mathbf{A},\mathbf{B},T)\}$ of such problems if Eq. (15.3) holds in each of the instances with a fixed constant C used for all the problems considered.

It is well known that Eq. (15.1) and Eq. (15.2) are dual to the null-controllability problem for the adjoint equation (after a time reversal):

$$v_t = \mathbf{A}^*v + \mathbf{B}^*\varphi \quad \text{with } v|_{t=0} = \omega \tag{15.4}$$

> *Given ω, determine a null-control φ on $[0,T]$ so that the solution of Eq. (15.4) will satisfy: $v|_{t=T} = 0$.*

This duality means that one will have Eq. (15.3) if and only if for each ω one can choose such a null control with

$$\|\varphi\| \le C\|\omega\| \tag{15.5}$$

using the same constant C as in Eq. (15.3). Indeed, we will restrict our attention here to settings in which the relevant spaces are Hilbert spaces, and we can then choose each null-control φ to minimize the norm; the mapping $\mathbf{C} : \omega \mapsto \varphi$ for each of the problems is then linear and continuous with operator norm $\|\mathbf{C}\| \le C$. We now speak of *uniform null controllability* for a family of such problems if this holds in each instance with a fixed null-controllability bound C.

PROPOSITION 15.1
Suppose we have a weak continuity property for the sequence of problems $\{(15.4)_\nu : \nu = 1,2,\ldots;\infty\}$:

$$\begin{gathered}\text{using } \varphi_\nu \text{ in each } (15.4)_\nu \text{ to get a solution } v_\nu,\\ \text{if } \varphi_\nu \rightharpoonup \varphi_\infty \text{ weakly, then also } v_\nu(T) \rightharpoonup v_\infty(T).\end{gathered} \tag{15.6}$$

Then, if the sequence $\{(15.4)_\nu : \nu = 1,2,\ldots\}$ is uniformly null controllable, it follows that the limit problem of Eq. $(15.4)_\infty$ is also null controllable, with the same bound.

PROOF In a Hilbert space context, for any ω the uniform bound on the sequence of null controls $\{\varphi_\nu\}$ gives a subsequence weakly convergent to some φ_0, and by Eq. (15.6) this φ_0 must be a null control for ω in Eq. $(15.4)_\infty$. ∎

Complementary to this is an obvious blowup result: if the limit problem Eq. $(15.4)_\infty$ is *not* null controllable, then one must have blowup: $C = C_\nu \to \infty$ in Eq. (15.5). In particular, suppose one were to take $\mathbf{A}_\nu$, $\mathbf{B}_\nu$ fixed in Eq. $(15.4)_\nu$, but vary the control time—more precisely, keep the nominal time T fixed as there but restrict support of the control φ_ν to $t \in [0, T_\nu]$—then, if this can be done with $T_\nu \to 0$, any weakly convergent subsequence would necessarily give convergence to f_0 with empty support, clearly not a null control for any $\omega \neq 0$. Thus, if one does have null controllability for arbitrarily short control times, this bound must blow up as the time goes to 0.

The *spectral approach* to the observability estimate of Eq. (15.3) utilizes spectral decomposition of the operator $\mathbf{A}$ appearing in Eq. (15.1), making a spatial expansion in eigenfunctions so time dependence is given by an exponential series. For null controllability, this is also known as the *method of moments*. This has been a useful tool for treating distributed parameter systems since the earliest days of the subject; note the papers cited here, especially the survey paper [23] and the book [6].

Because the paper in Reference 31 was motivated precisely by our present concerns regarding uniformity and blowup in applying the spectral approach, we will devote the next section to describing the principal results obtained there. Section 15.3 is then devoted to discussing the spectral approach and some historical examples of how it works out, whereas Section 15.4 considers blowup, mostly utilizing the spectral approach but also noting the behavior of Carleman estimate techniques in relation to our thematic concerns. The final section then discusses some further recent results.

15.2 The "Window Problem" for Complex Exponential Series

Let Λ be a complex sequence $\{\lambda_k = \tau_k + i\sigma_k\}$ and consider functions of the form:

$$f(t) = \sum_k c_k e^{i\lambda_k t}. \tag{15.7}$$

We think of these as "observed through the time window $[0, T]$" and topologize this set of functions $\mathcal{M} = \mathcal{M}_T(\Lambda)$ as a subset of $L^2(0, T)$. The "window problem" we consider here is then to determine the sequence of terms (evaluated at $t = T$):

$$\mathbf{c}_T = \left\{c_k e^{i\lambda_k T}\right\} \tag{15.8}$$

from observation of $f(\cdot)$ on $(0, T)$.

We will impose the following conditions on the exponent sequence Λ:

$$\sigma_k \geq 0, \tag{15.9}$$

$$\text{uniform separation: for some } r_0 > 0, \quad |\lambda_j - \lambda_k| \geq r_0 \ (j \neq k), \tag{15.10}$$

$$\begin{gathered}\text{uniform sparsity: for some } a > 0 \text{ and uniformly for } \lambda_* \in \Lambda, \text{ one has}\\ \#\{\lambda \in \Lambda : 0 < |\lambda - \lambda_*| \leq r\} \leq \nu(r) \equiv a\sqrt{r}.\end{gathered} \tag{15.11}$$

The principal result of Reference 31, somewhat specialized to this form of ν, is then Theorem 15.1.

THEOREM 15.1
If the complex sequence Λ satisfies Eqs. (15.9) to (15.11)*, then there is a constant $C = C(T, \Lambda)$ such that*

$$\sum_k \left|c_k e^{i\lambda_k T}\right|^2 \leq C^2 \int_0^T \left|\sum_k c_k e^{i\lambda_k t}\right|^2 dt \tag{15.12}$$

for all f, $\mathbf{c}_T$ as in Eq. (15.7) *and Eq.* (15.8). *For the special form $\nu(r) = a\sqrt{r}$ used here in Eq.* (15.11), *we have*

$$C = C(T, \Lambda) \leq Ae^{B/T} \tag{15.13}$$

with positive constants A and B depending only on r_0 and a.

Thus, we necessarily have uniformity of the estimate over families $\{\Lambda\}$ of such exponent sequences for which we can use fixed r_0, a, and we have blowup in the estimate exponential to the order of $1/T$ as $T \to 0$.

We note that the heart of the proof in Reference 31 is a technical lemma.

LEMMA 15.1

For any $T > 0$ and any $\nu(r) = a\sqrt{r}$, there exists an entire function $P(\cdot)$ such that

- *$|P(z)| \leq 1$ on the upper half-plane $\mathbb{C}_+$ and is real and positive on the imaginary axis, with a somewhat technical a-dependent lower bound for $P(is)$ when $s \geq 0$.*
- *P is of the exponential type with*

$$\left|e^{-i(T/2)z} P(z)\right| \leq Ke^{(T/2)|z|} \quad (z \in \mathbb{C}). \tag{15.14}$$

- *For real r one has a bound*

$$\left|P(r)e^{\nu(|r|)}\right| \leq C = C(a, T). \tag{15.15}$$

The constant C in Eq. (15.15) *satisfies*

$$C(a, T) \leq Ae^{B/T} \tag{15.16}$$

for T near 0 *(with a-dependent A, B).*

Reference 31 actually considers f as observed on an interval $[0, \delta]$, although the terms are evaluated at $t = T$ without necessarily taking $\delta = T$ as here. In this, as in some other respects, we are simplifying the description here of the results of Reference 31 for our present convenience. In Reference 31 the admissible functions $\nu(\cdot)$ are, more generally,

> *continuous and unboundedly increasing, but with $\nu(s)/s^2$ decreasing and integrable on* $[r_0, \infty)$.

Obviously, the statements of Theorem 15.1 and Lemma 15.1 become more complicated with more general admissible functions $\nu(\cdot)$ for Eq. (15.11). For full details, of course, see Reference 31.

This approach is particularly effective for one-dimensional problems. We can restrict ourselves here to taking $\nu(r) = a\sqrt{r}$ in Eq. (15.11) because the sequences involved for our applications typically come from eigenvalues of Sturm-Liouville problems and so are quadratically distributed (i.e., asymptotically like ck^2). We note, for example, that for the exponent sequence $\{\lambda_k = ck^2\}$, one has an easy computation to obtain Eq. (15.11) with $\nu(r) = \sqrt{2r/|c|}$.

It should be noted that with the restriction to $\nu(r) = a\sqrt{r}$, results like Lemma 15.1 and Theorem 15.1 had been obtained earlier (cf., e.g., References 20, 17, 19, 9)—except for consideration of the asymptotics of Eq. (15.16) and Eq. (15.13). As compared, for example, with Reference 29, the particular innovation of Theorem 15.1 is the treatment of more general complex exponent sequences, used here for Theorems 15.4 and 15.8.

15.3 Spectral Methods; Some History

We have already noted that spectral methods have long provided a useful tool for treating control-theoretic questions for partial differential equations, and we sketch here some historical examples. Although the original treatments referred to a variety of background results on Dirichlet series and nonharmonic analysis by Schwartz [24], Redheffer [20], Luxemburg and Korevaar [17], etc.—and developed some additional theory themselves—it will be sufficient and more convenient here to refer only to Theorem 15.1 as described in the preceding section.

The "spectral approach" to the observability estimate of Eq. (15.3) for Eq. (15.1) and Eq. (15.2) utilizes a spatial expansion in eigenfunctions so the time dependence is given by an exponential series. Thus, we assume the set of eigenfunctions $\{e_k(\cdot)\}$ of $\mathbf{A}$ is a Riesz basis so the solution u of Eq. (15.1) has an expansion:

$$u(t,\cdot) = \sum_k a_k e^{\alpha_k t} e_k(\cdot) \tag{15.17}$$

where $\{\alpha_k\}$ is the corresponding set of eigenvalues: $\mathbf{A}e_k = \alpha_k e_k$. Using Eq. (15.17) in Eq. (15.2) then gives an exponential series for the observation as a function of t:

$$y(t) = \sum_k c_k e^{\alpha_k t} \tag{15.18}$$

where

$$c_k = \beta_k a_k \quad (\beta_k = \mathbf{B}e_k). \tag{15.19}$$

Suppose, now, $y(\cdot)$ is scalar and one has a uniform lower bound

$$|\beta_k| \geq \kappa > 0 \text{ so } K_1 = \sup_k\{1/|\beta_k|\} \leq 1/\kappa < \infty \tag{15.20}$$

and also suppose the sequence $\{\alpha_k = i\lambda_k\}$ is such that Theorem 15.1 applies to the exponential series of Eq. (15.18). Using Eq. (15.17) for $t = T$ and recalling Eq. (15.19) (so $a_k = c_k/\beta_k$) and that $\{e_k(\cdot)\}$ is a Riesz basis, we then have

$$\begin{aligned} \|u(T,\cdot)\|^2 &\leq K^2 \sum_k \left|a_k e^{\alpha_k T}\right|^2 = K^2 \sum_k \left|\frac{c_k}{\beta_k} e^{i\lambda_k T}\right|^2 \\ &\leq K^2 [\sup_k\{1/|\beta_k|\}]^2 \sum_k \left|c_k e^{i\lambda_k T}\right|^2 \\ &\leq (CKK_1)^2 \|y(\cdot)\|^2_{L^2(0,T)} \end{aligned} \tag{15.21}$$

which is just Eq. (15.3) with the bound as given by Eq. (15.12), apart from the fixed constants K, K_1.

Although our original consideration was the observability map: $y \mapsto u(T,\cdot)$, we note that the spectral method has factored this as

$$y \mapsto \mathbf{c}_T = \left(c_k e^{-\alpha_k T} : k = 1, \ldots\right) \mapsto \sum_k (1/\beta_k) c_k e^{-\alpha_k T} e_k = u(T,\cdot).$$

This use of Theorem 15.1 for Eq. (15.18) requires the sparsity condition of Eq. (15.11), which is plausible only for one-dimensional settings. We do note, however, that a separable setting (e.g., for a cylindrical [product] region $\Omega = (0,1) \times \Omega_*$) can reduce the problem to a collection of one-dimensional problems. Suppose the eigenfunctions of $\mathbf{A}$ were to have the form of products

$\{e_k(x) f_\ell(x_*)\}$ (with $\{f_\ell\}$ orthonormal for simplicity. We then can replace Eq. (15.17) and Eq. (15.18) by

$$\begin{aligned} u(t,\cdot) &= \sum_{k,\ell} a_{k,\ell}\, e^{\alpha_{k,\ell} t}\, e_k(\cdot) f_\ell(\cdot) \\ y(t,\cdot) &= \sum_{\ell} y_\ell(t) f_\ell(\cdot) \\ \text{with } y_\ell(t) &= \sum_k c_{k,\ell}\, e^{\alpha_{k,\ell} t} \end{aligned} \tag{15.22}$$

where we assume **B** acts only at the base $\Gamma = \{0\} \times \Omega_*$ with $\mathbf{B}[e_k f_\ell] = [\mathbf{B} e_k] f_\ell = \beta_k f_\ell$ so Eq. (15.19) becomes $c_{k,\ell} = \beta_k a_{k,\ell}$, and we continue to assume Eq. (15.20). We may then consider each problem$_\ell$ separately: as in Eq. (15.21) we get

$$\|u_\ell(T,\cdot)\|^2 \le (C_\ell K K_1)^2 \|y_\ell(\cdot)\|^2_{L^2(0,T)}$$

with

$$u_\ell(t,\cdot) = \langle f_\ell, u(t,\cdot)\rangle = \sum_k a_{k,\ell} e^{\alpha_{k,\ell} t} e_k(\cdot)$$

so

$$\begin{aligned} \|u(T,\cdot)\|^2 &= \sum_\ell \|u_\ell(T,\cdot)\|^2 \\ &\le [(K K_1) \max_\ell \{C_\ell\}]^2 \sum_\ell \|y_\ell\|^2 \\ &= \hat{C}^2 \|y\|^2_{L^2([0,T]\times\Gamma)} \end{aligned} \tag{15.23}$$

with existence of $\max_\ell\{C_\ell\}$ giving $\hat{C}$ corresponding to a requirement of uniform observability for the family {problem$_\ell$}.

Going back to the 1960s, we begin by following Reference 18 in considering boundary observation at $x = 0$ for the one-dimensional heat equation

$$u_t = u_{xx} \text{ on } (0,1) \quad \text{with } u = 0 \text{ at } x = 0, 1. \tag{15.24}$$

In the absence of information about the initial state $(u|_{t=0})$, we observe the endpoint heat flux, $y(t) = u_x(t,0)$, for an interval $0 \le t \le T$ and wish to determine the terminal state $(u|_{t=T})$. More specifically, we seek an estimate

$$\int_0^1 |u(T,x)|^2\, dx \le C^2 \int_0^T |u_x(t,0)|^2\, dt \tag{15.25}$$

for solutions of Eq. (15.24), because that estimate ensures the observability. To obtain Eq. (15.25), we use the spectral approach as in the preceding section.

The operators here corresponding to **A**, **B** in Eq. (15.1) and Eq. (15.2) are the Sturm-Liouville operator $\mathbf{A} : z \mapsto z''$ with homogeneous Dirichlet boundary conditions (so the eigenvalue sequence is $\{\alpha_k = -\pi^2 k^2\}$ with the orthonormal basis of corresponding eigenfunctions $\{e_k(x) = (1/\sqrt{2}) \sin k\pi x\}$) and $\mathbf{B} : z \mapsto z'(0)$ (giving $\beta_k = k\pi/\sqrt{2}$ so we obviously have Eq. (15.20) with $K_1 = 1$). As was remarked in the previous section, the exponent sequence $\{\lambda_k = i\pi^2 k^2\}$ satisfies Eqs. (15.19) to (15.21) so it follows immediately from Theorem 15.1 that we have Eq. (15.12). Thus, we have Eq. (15.21) and Eq. (15.3)—that is, Eq. (15.25).[1]

[1] This is so much easier now than it seemed in the late 1960s! Indeed, at that time the corresponding null-controllability result was obtained by an independent direct argument (cf., Reference 8, solving a moment problem to construct the control) rather than by an appeal to the now-standard duality.

Already in Reference 18 the argument via Eq. (15.22) was noted for the heat equation on a cylindrical region $\Omega = (0, 1) \times \Omega_*$— although the uniformity of the component one-dimensional problems was neither noted nor even noticed, because one had eigenvalues $\alpha_{k,\ell} = -\pi^2 k^2 - \hat{\alpha}_\ell$ where $\{-\hat{\alpha}_\ell\}$ is the eigenvalue sequence for the Laplacian on the cross-sectional region Ω_*, which permitted simple absorption of the factors $e^{-\hat{\alpha}_\ell T} < 1$ in the estimation. The first problem that explicitly required[2] consideration of uniformity for the estimates of a family of quadratically distributed exponent sequences was the treatment of the heat equation on a sphere [9, 19] by way of separation of variables for the spherical Laplacian. This is along the lines of Eq. (15.22) above but required some concern for the zeroes of Bessel functions to verify the necessary uniformity in Eq. (15.10) and Eq. (15.11).

A rather different kind of spectral approach was used by Russell, avoiding the necessity to use spatial separation of variables to decompose into one-dimensional problems by deriving observability and null-controllability results for the heat equation from similar results for the corresponding wave equation (which were then obtained by using scattering theory, etc.).

THEOREM 15.2

On a spatial domain Ω, consider a second-order (wave) equation for $w = w(\tilde{t}, x)$:

$$w_{\tilde{t}\tilde{t}} + \mathbf{A}^2 w = 0 \quad (0 \le \tilde{t} \le \tilde{T}) \tag{15.26}$$

where $\mathbf{A}$ is a positive operator [as, e.g., $(-\Delta)^{1/2}$) and assume that, for a suitable observation operator $\mathbf{B}$, one has an observability estimate]

$$\|\mathbf{A}w(\tilde{T})\|^2 + \|w_t(\tilde{T})\|^2 \le \tilde{C}^2 \int_0^{\tilde{T}} \|\mathbf{B}w(t)\|^2 \, dt \tag{15.27}$$

for solutions of Eq. (15.26). *If one considers the corresponding heat equation for the same Ω:*

$$u_t + \mathbf{A}^2 u = 0 \quad (0 \le t \le T), \tag{15.28}$$

then one also has observability, using the same $\mathbf{B}$, for arbitrarily small $T > 0$ with a corresponding observability estimate

$$\|u(T)\|^2 \le C^2 \int_0^T \|\mathbf{B}u(t)\|^2 \, dt. \tag{15.29}$$

PROOF See References 21 and 22; compare also Reference 27. One notes that the Fourier transforms in t of Eq. (15.26) and Eq. (15.28) are

$$-\tilde{\tau}^2 \hat{w} + \mathbf{A}^2 \hat{w} = 0, \quad i\tau\hat{u} + \mathbf{A}^2 \hat{u} = 0$$

so, formally, the equations can be related in the Fourier domain by substituting $\tau = \tilde{\tau}^2$. This permitted Russell to use Fourier transform techniques, especially the Paley-Wiener theorem, to obtain Eq. (15.29) from Eq. (15.27). The technical lemma needed to justify the formal procedure was a version of Lemma 15.1, above, although without Eq. (15.16). Essentially, Eq. (15.15) justifies that the relevant functionals will transform legitimately (in L^2), whereas Eq. (15.14) ensures that the kernels have support in $[0, T]$. ∎

[2] In a sense this was not required but was only an artifact of proofs by way of separation of variables for the Laplacian on a sphere along the lines of Eq. (15.22) above. At the time of Reference 19, for example, one had available neither the deep arguments of References 21 and 22 nor even the more elementary observation of Reference 25 that (in the null-controllability context) one could obtain boundary null controllability for a general bounded region Ω by embedding Ω in a large box or cylinder (for which the expansion could be treated as here) and then using as control the trace on $\partial\Omega$ of that null-controlled solution.

The spectral approach (with reduction to one-dimensional problems) has also been used for partial differential equations other than the heat equation.

THEOREM 15.3
For the Euler plate equation

$$\begin{aligned} u_{tt} + \Delta^2 u &= 0 \quad \textit{on } \Omega = (0,1)^2, \\ u_\nu &= 0 = (\Delta u)_\nu \quad \textit{on } \partial\Omega. \end{aligned} \tag{15.30}$$

one has boundary observability—and so controllability, by duality—for arbitrarily short times $T > 0$, using the observation of $y(t, x_2) = u(t, 0, x_2)$ for $0 < x_2 < 1$ and $0 < t < T$.

PROOF See References 14 and 29. Note also Reference 13 for the one-dimensional case. The boundary conditions selected have the considerable advantage of making the problem separable and permitting explicit computation so this problem can be treated as in the double expansion of Eq. (15.22), with

$$e_k(x_1) = (1/\sqrt{2}) \cos \pi k x_1, \quad f_\ell(x_2) = (1/\sqrt{2}) \cos \pi \ell x_2.$$

We then obtain the expansions

$$\begin{aligned} u(t,\cdot) &= \sum_\ell u_\ell(t,\cdot)\varphi_\ell(\cdot) \quad \text{with } u_\ell(t,\cdot) = \sum_{k,\pm} a_{[k,\pm],\ell} e^{\pm i\pi^2[k^2+\ell^2]t} e_k(\cdot) \\ y(t,\cdot) &= \sum_\ell y_\ell(t) f_\ell(\cdot) \quad \text{with } y_\ell(t) = \sum_{k,\pm} c_{[k,\pm],\ell} e^{\pm i\pi^2[k^2+\ell^2]t}. \end{aligned}$$

where, for this observation, $c_{[k,\pm],\ell} = a_{[k,\pm],\ell}$. For each ℓ one has the exponent sequence

$$\Lambda_\ell = \left\{ \lambda^\ell_{k,\pm} = \pm\pi^2[k^2 + \ell^2] \right\},$$

and one easily verifies Eq. (15.10) and Eq. (15.11) for these sequences, uniformly in ℓ. Thus, Theorem 15.1 applies to give Eq. (15.23) and so the desired observability. ∎

15.4 Blowup

We have already noted as a corollary to Proposition 15.1 that null controls associated to control times $T \to 0+$ cannot remain bounded. The question of determining the asymptotic *blowup rate* (e.g., for boundary control of the one-dimensional heat equation) was raised as far back as the mid-1970s in Reference 26—although with the wildly optimistic conjecture that this blowup rate was $\mathcal{O}(1/\sqrt{T})$ as $T \to 0$. (It is interesting that this question of blowup rates was considered for distributed parameter systems before the corresponding question had been raised for finite dimensional control problems.) The finite dimensional case, however, now seems quite well understood (see References 30 and 32; see also Reference 33). The infinite dimensional case remains fertile ground for further investigation.

By the mid-1980s, the incorrect conjecture in Reference 26 had been somewhat corrected: the paper [28] kept track of the relevant "constants" in the treatment in Reference 20 and obtained an $e^{\mathcal{O}(1/T)}$ upper bound on the blowup rate for the one-dimensional heat equation (15.24). This was complemented by Güichal's computation (Reference 11) of a lower bound with the same asymptotic behavior. Thus, at least for Eq. (15.24), it is now known that the correct asymptotics as $T \to 0$ are precisely *"exponential to the order of* $1/T$*."* We may note that use of Theorem 15.1 in the analysis in the previous section already provides, through Eq. (15.23), an upper bound exponential to the order of $1/T$ for each of the problems discussed there: for example, again for Eq. (15.24), we may note that the constant C appearing in Eq. (15.25) = Eq. (15.3) was exactly the constant obtained in Eq. (15.12) and so satisfies Eq. (15.13) by Theorem 15.1. The same use of Eq. (15.13) applies also to the other heat equation problems considered there. In particular, we note that, Although Theorem 15.2 did not include any consideration of the asymptotics as $T \to 0$, the inclusion of Eq. (15.16) in Lemma 15.1 now provides the blowup rate

$$C(a,T) \leq Ae^{B/T}\tilde{C} \tag{15.31}$$

in (15.19) (with a-dependent A, B). As an example other than the heat equation, we also recall the treatment above of the Euler plate equation (15.30). Apart from the necessary uniformity of the one-dimensional problems, the treatment above and in Reference 29 gives the now-familiar $e^{\mathcal{O}(1/T)}$ blowup as $T \to 0$.

We might next consider observability for the equation

$$\begin{aligned} u_{tt} - 2\kappa\Delta u_t + \Delta^2 u = 0 \quad &\text{on } \Omega = (0,1)^2 \\ u_\nu = 0 = (\Delta u)_\nu \quad &\text{on } \partial\Omega \end{aligned} \tag{15.32}$$

describing a structurally damped Euler plate; one can also write this as a first-order system

$$U_t = \mathbf{A}U \quad \text{with } \mathbf{A} = (-\Delta)M,\ M = \begin{bmatrix} 0 & 1 \\ -1 & -2\kappa \end{bmatrix}. \tag{15.33}$$

Boundary control for this plate model problem was considered by Hansen (Reference 12)—and is also described in Section 6 of Reference 31, specifically devoted to some applications to distributed parameter system theory, because (apart from the facility of obtaining the required observability estimate from a general result) the use of Theorem 15.1 automatically provides the blowup estimate of Eq. (15.5) for this problem as for Eq. (15.30).

THEOREM 15.4

For the structurally damped Euler plate model of Eq. (15.32) *with*[3] *damping coefficient* $0 < \kappa < 1$, *consider observation of* $y = u|_{x_1=0}$ *for* $0 < t < T$. *Then we have observability with an observability estimate*

$$\left[\int_0^1\int_0^1 \left(|\Delta u|^2 + |u_t|^2\right)\, dx_1 dx_2\right]^{1/2} \leq Ae^{B/T}\left[\int_0^T\int_0^1 |u(t,0,x_2)|^2\, dx_2 dt\right]^{1/2}. \tag{15.34}$$

PROOF Note that the eigenvalues of $\mathbf{A}$ are $\xi_\pm\pi^2[k^2+\ell^2]$ where the eigenvalues of M are $\xi_\pm = -\kappa \pm i\sqrt{1-\kappa^2}$. Much as for the treatment of Eq. (15.30), one gets a family of exponent sequences for Eq. (15.7)

$$\Lambda_\ell = \left\{\lambda_{k,\pm}^\ell = -i\xi_\pm\pi^2[k^2+\ell^2]\right\},$$

but here this sequence no longer lies on the real axis; instead, each of the resulting one-dimensional problems here involves a complex exponent sequence with each Λ_ℓ the union of two copies of a

[3]There is no new difficulty as $\kappa \to 0$, for which one just gets Eq. (15.30) in the limit. We do note, however, that one must have blowup as $\kappa \to 1$, when the eigenvalues and eigenvectors of M degenerate, but we will discuss here only the blowup rate as $T \to 0$ and not for this.

quadratically distributed sequence placed along the rays $\{[\pm\sqrt{1-\kappa^2}+i\kappa]\tau : \tau > 0\}$. That geometry is discussed in Reference 31 and one easily verifies Eq. (15.9) and that such a union continues to satisfy the conditions of Eq. (15.10) and Eq. (15.11) and that the relevant shifts leave the same $\nu(\cdot)$ uniformly applicable. Thus, Theorem 15.1 gives the desired uniform estimate. That estimate was, of course, obtained by Hansen. Section 6 of Reference 31 observes that use of Theorem 15.1 automatically provides the blowup estimate of Eq. (15.13) for this problem as was noted earlier for Eq. (15.30). □

OTHER APPROACHES We have been considering above applications of the spectral expansion approach, using Theorem 15.1 to obtain blowup estimates. We will not discuss this here, but note that weighted energy estimates have also been effectively used to this end (cf. References 2–5). We note, at this point, that finite dimensional results of References 30 and 33 have also proved directly applicable for distributed parameter systems (cf., e.g., Reference 16).

There are also, of course, a variety of situations in which observability results seem unavailable by any use of spectral expansions and have been obtained only through the use of Carleman estimates. For the heat equation $u_t = \Delta u$ alone, these situations include most of the known results about observability and null-controllability with interaction restricted to a small patch as well as for variants with variable coefficients. Three things become clear from, for example, a look at Reference 10:

- This is a powerful approach to these observability-controllability problems.
- The dependence of the estimate on the coefficients is through only certain bounds on coefficients and their derivatives and so is uniform over relevant classes of equations.
- The calculations of these Carleman estimates is messy enough[4] to make it difficult to track any time dependence so as to obtain an estimation of the blowup rate.

Apparently no such estimation has previously been done, but, at least for the heat equation, we have verified in Reference 15 the blowup rate for the patch control settings that rely on Carleman estimates.

THEOREM 15.5
Consider the patch null-controllability problem for the heat equation

$$\begin{aligned} u_t - \Delta u = \varphi = \textit{control} \quad & \textit{on } \mathcal{Q} = [0,T]\times\Omega \\ u = 0 \quad & \textit{on } \Sigma = [0,T]\times\partial\Omega \\ u = u_0 \quad & \textit{on } \Omega \textit{ at } t = 0 \end{aligned} \tag{15.35}$$

with the spatial support of the control φ restricted to a specified patch ω (so $\omega \neq \emptyset$ is open with compact closure in the bounded, connected, open domain Ω). There are then constants A, B depending only on Ω, ω, T_* such that for each $0 < T < T_*$ and each initial state $u_0 \in L^2(\Omega)$ there exists a null-control function $\varphi \in L^2([0,T]\times\omega)$—that is, the solution of Eq. (15.35) *satisfies $u(T,\cdot) \equiv 0$—such that*

$$\|\varphi\|_{L^2([0,T]\times\omega)} \le Ae^{B/T}\|u_0\|_{L^2(\Omega)}. \tag{15.36}$$

PROOF See Reference 15. As usual, one works with the dual observability problem. Following the Carleman calculations in Reference 1, one can track the T-dependence with a careful T-dependent scaling of the parameters to obtain an estimate of Carleman type for the observability problem, with a constant independent of $T > 0$. It is in going from that bound to the observability estimate that we obtain the anticipated estimate of the now familiar form: $e^{\mathcal{O}(1/T)}$. □

[4]The forthcoming treatment in Reference 1 makes the Carleman calculations somewhat more transparent in the case of the heat equation, but these still remain formidable.

15.5 Some More Recent Results

We begin this section by mentioning another result presented in Section 6 of Reference 31. The intent here was to be able to consider homogenization of a control problem—so we are interested in considering $q = q(x, x/\varepsilon)$ with $\varepsilon \to 0+$. Such a rapidly varying q might correspond physically to heat dissipation by a closely spaced array of fins. One expects, in this setting, that $q(\cdot, \cdot/\varepsilon) \rightharpoonup q_0(\cdot)$ so, as the weak continuity hypothesis is easy to verify here, Theorem 15.1 would apply. (Eventually one might wish to use techniques like those of Reference 27 to show a stronger continuity for the controls as $\varepsilon \to 0$ so as to justify use of the limit control as a good approximation to the control associated with a problem involving a rapidly varying coefficient, but at present we inquire only as to uniform observability.)

THEOREM 15.6
Consider the observation problems

$$\begin{aligned} u_t &= u_{xx} - qu \quad (0 < x < \ell) \\ u_x\Big|_{x=0} &\equiv 0, \quad u\Big|_{x=\ell} \equiv 0 \end{aligned} \tag{15.37}$$

with unspecified initial data. We now observe $z(t) = u(t, 0)$ for $0 \le t \le T$ and seek to determine $u(T, \cdot)$. With spatially varying coefficients $q(\cdot)$ (subject to a uniform bound: $|q| \le M$), we have a uniformly observable family of observation problems—whence, also, we have uniform null controllability for the corresponding family of dual boundary null-control problems.

PROOF See Reference 31. Most of the effort consists of showing, by use of the Courant Minmax Theorem, that the bound $|q| \le M$ ensures that the condition of Eq. (15.11) holds uniformly and then showing (by a compactness argument) that Eq. (15.10) also holds uniformly. ∎

Next we note a new result in the same spirit about finite difference approximations for the problem considered earlier for Eq. (15.24). We take an equally spaced mesh on [0, 1] with $N - 1$ interior nodes $\{x_j = x_j^N = j/N : j = 1, \dots, N-1\}$ spaced $h = 1/N$ apart and let $\mathbf{u} = \mathbf{u}^N$ be the vector in $\mathbb{R}^{N-1}$ with entries $u_j = u_j^N$ intended to approximate the values $u(\cdot, x_j^N)$. Using the standard central difference approximation to the (spatial) second derivative but keeping time continuous, the partial differential equation (15.24) becomes a finite dimensional system of ordinary differential equations

$$\dot{u}_j = [u_{j-1} - 2u_j + u_{j+1}]/h^2 \quad (j = 1, \dots, N-1) \tag{15.38}$$

where, corresponding to the boundary conditions $u(t, 0) = 0 = u(t, 1)$, we are taking $u_0 = 0 = u_N$, that is, in Eq. (15.38) we take $u_{j-1} = 0$ for $j = 1$ and $u_{j+1} = 0$ for $j = N - 1$.

Although ultimately we might seek to show convergence for the controls, our present concern is only to show uniformity for the relevant family of dual observability problems:

> *Observe the "boundary flux" $y = y^N = (u_1 - u_0)/h = Nu_1^N$ for $0 \le t \le T$ and, without knowledge of the initial data, reconstruct the terminal state $\mathbf{u}^N(T)$.*

As earlier (compare Eq. (15.25)), we seek an estimate

$$(1/N) \sum_{j=1}^{N-1} \left|u_j^N\right|^2 \le C^2 \int_0^T \left|Nu_1^N(t)\right|^2 dt. \tag{15.39}$$

THEOREM 15.7

For the finite difference approximations of Eq. (15.38) *to the observability problem for Eq.* (15.24)*, the estimate of Eq.* (15.39) *holds uniformly, so C is independent of N, that is, as the mesh spacing $h \to 0$.*

PROOF The relevant exponent sequence Λ^N, here, is the finite sequence of eigenvalues of the standard tridiagonal matrix corresponding to the central difference scheme used in Eq. (15.38). It is not too difficult to obtain these eigenvalues and the corresponding eigenvectors explicitly: much as for the continuous problem of Eq. (15.24) we have

$$\begin{aligned} \left(e_k^N\right)_j &= \alpha_N \sin k\pi j/N \quad (j = 1, \ldots, N-1) \\ \sigma_k^N &= 2N^2(1 - \cos k\pi/N) \end{aligned} \tag{15.40}$$

for $k = 1, \ldots, N-1$ (with the normalizing constant $\alpha \approx 1/\sqrt{2}$).

Because Λ^N is a finite sequence, it is trivial that Eqs. (15.9) to (15.11) will hold for each N, and, indeed, we would have the stronger finite dimensional blowup results of References 30 and 32. Our concern is to verify that the separation condition of Eq. (15.10) and the sparsity condition of Eq. (15.11) hold *uniformly*. For Eq. (15.10) one need only bound $2N^2(\cos 2\pi/N - \cos\pi/N)$ away from 0, which is easy. For any choice of r in Eq. (15.11), we separately consider the two cases: $N^2 \leq 2r$ and $2r < N^2$. For the first case, because there are only $N-1$ eigenvalues altogether, we have

$$\nu^N(r) = \#\left[\sigma_j^N \in (\sigma_* - r, \sigma_* + r)\right] < N \leq \sqrt{2r}.$$

For the second case we see that $\nu^N(r) = k$ means that $\sigma_k^N \approx 2r$, so, setting $s = \sqrt{r}/N < 1/\sqrt{2}$, we have $(1 - \cos k\pi/N) \approx s^2$ and $\cos^{-1}(1 - s^2) \approx k\pi/N = k\pi s/\sqrt{r}$, which gives

$$\nu^N(r) = k \approx \frac{\cos^{-1}(1 - s^2)}{\pi s}\sqrt{r}.$$

Because $(1/s)\cos^{-1}(1 - s^2)$ is bounded on $(0, 1/\sqrt{2}]$, we have a uniform bound on a appearing in Eq. (15.11) for either of the cases, and our result then follows from Theorem 15.1.

Finally, we announce a new result [16] for boundary observability of a thermoelastic plate, here taken to be governed by the system of coupled partial differential equations on $\mathcal{Q} = [0, T] \times \Omega$

$$\begin{aligned} w_{tt} + \Delta^2 w - \alpha\Delta\vartheta &= 0 \\ \vartheta_t - \Delta\vartheta + \alpha\Delta w_t &= 0 \end{aligned} \quad \text{on } \mathcal{Q} = [0, T] \times \Omega$$
$$w, \Delta w, \vartheta = 0 \quad \text{on } \Sigma = [0, T] \times \partial\Omega \tag{15.41}$$

with coupling constant $\alpha > 0$. Much as for Eq. (15.33), this can be put in first-order form as

$$U_t = \mathbf{A}U \quad \text{with } \mathbf{A} = (-\Delta)M, \quad M = M(\alpha) = \begin{bmatrix} -1 & 0 & -\alpha \\ 0 & 0 & -1 \\ 1 & -\alpha & 0 \end{bmatrix} \tag{15.42}$$

for $U = [\vartheta, -\Delta w, w_t]^{\mathrm{T}}$, embedding the boundary conditions in specification of the Laplacian as an operator on $L^2(\Omega)$. We consider a cylindrical region $\Omega = [0, 1] \times \Omega_*$ and, as with Eq. (15.24), wish to observe the boundary flux at the base of the cylinder $\Gamma = \{0\} \times \Omega_*$

$$y(t, \cdot) = [-\partial\vartheta(t, \cdot)/\partial x_1]\Big|_{x=0} = [-1, 0, 0] \cdot U_{x_1}(t, 0, \cdot) \tag{15.43}$$

and use this to determine the full state U. This determination is clearly impossible when the equations are uncoupled ($\alpha = 0$), so we must have blowup both when $\alpha \to 0$ and when $T \to 0$. ∎

THEOREM 15.8

For the coupled thermoelastic plate model of Eq. (15.41) *and Eq.* (15.42) *with observation of Eq.* (15.43)*, one has observability with an estimate*

$$\left[\int_0^1 \int_0^1 (|\vartheta(T,\cdot)|^2 + |[\Delta w](T,\cdot)|^2 + |w_t(T,\cdot)|^2)\, dx_1 dx_2\right]^{1/2}$$

$$\leq Ae^{B/T}\left[\int_0^T \int_0^1 |\vartheta_{x_1}(t,0,x_2)|^2\, dx_2 dt\right]^{1/2}. \tag{15.44}$$

where $A = A(\alpha)$, $B = B(\alpha)$ are bounded for α in compact subsets of $(0,\infty)$ *with B bounded and $A = A(\alpha) = \mathcal{O}(1/\alpha)$ as $\alpha \to 0$.*

PROOF See Reference 16. We briefly sketch the argument, much along the lines noted above for Theorem 15.4. We here obtain the expansions

$$\begin{aligned} U(t,\cdot) &= \sum_\ell U_\ell(t,\cdot)\varphi_\ell(\cdot) \quad \text{with } U_\ell(t,\cdot) = \sum_{j,k} a_{[j,k],\ell}\, e^{\xi_j[\pi^2k^2+\mu_\ell]t} V_j e_k(\cdot) \\ y(t,\cdot) &= \sum_\ell y_\ell(t) f_\ell(\cdot) \quad \text{with } y_\ell(t) = \sum_{j,k} c_{[j,k],\ell}\, e^{\xi_j[\pi^2k^2+\mu_\ell]t} \end{aligned} \tag{15.45}$$

where $\{(\xi_j, V_j) : j = 0,1,2\}$ are the eigenpairs for $M = M(\alpha)$, and $\{\mu_\ell\}$ is the spectrum of the cross-sectional Laplacian (i.e., $-\Delta$ for Ω_* so $\mu_\ell > 0$). Note that Eq. (15.43) gives, much as for Eq. (15.19),

$$c_{[j,k],\ell} = \beta_{[j,k]} a_{[j,k],\ell} \quad \beta_{[j,k]} = \pi k[-1,0,0]\cdot V_j. \tag{15.46}$$

To proceed, one first shows that the eigenvalues of M are distinct—one real ($j = 0$) and a conjugate pair—with negative real parts. There is thus a basis of $\mathbb{C}^3$ consisting of eigenvectors of M, not orthonormal, because the matrix M is not normal, but a controllability-compactness argument gives uniformity in α of the norm equivalence of these coordinates to the usual Euclidean norm for $\mathbb{C}^3$. Because spectral asymptotics show the first components of V_1, V_2 are roughly proportional to α for α near 0, Eq. (15.46) gives

$$K_1 = K_1(\alpha) = \max_{[j,k]}\{1/|\beta_{[j,k]}|\} = \mathcal{O}(1/\alpha) \tag{15.47}$$

as $\alpha \to 0$. Each of the relevant exponent sequences now is the union of quadratically distributed sequences placed along the three rays in $\mathbb{C}$ given by $\{-i\tau\xi_j : \tau > 0\}$. By following the discussion in Reference 31, it is again not difficult to verify Eq. (15.9) and that the conditions of Eq. (15.10) and Eq. (15.11) hold uniformly in ℓ. As with our previous results, this suffices for applicability of Theorem 15.1 to obtain Eq. (15.44), with the blowup estimate: $A(\alpha) = \mathcal{O}(1/\alpha)$ as $\alpha \to 0$ following from Eq. (15.47). ∎

In connection with Eq. (15.41) it is also interesting to consider the use of observation of ϑ (or other of the state components) in all of Ω, rather than only the boundary, for determination of the full state. This turns out to be much closer in its nature ($\mathcal{O}[T^{-[k+1/2]}]$ estimate) to the finite dimensional analysis in Reference 30. A direct application of that analysis also appears in Reference 16 (including discussion of the α asymptotics) along with a parallel application of the weighted energy approach introduced in References 2 to 5 (which has the advantage here of applicability to more general boundary condition); note also the treatment of this problem in Reference 33.

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Chapter 16

Poroelastic Filtration Coupled to Stokes Flow

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Abstract We report on some recent progress in the mathematical theory of fluid transport and poromechanics, specifically, the exchange of fluid between the Biot model of an elastic porous structure saturated with a slightly compressible viscous fluid coupled to the Stokes flow in an adjacent open channel. The coupled system is resolved by semigroup methods by developing appropriate variational formulations. These lead to either a standard weak formulation or a mixed formulation for the resolvent equation.

16.1 Introduction

Consider the flow of a single-phase, slightly compressible viscous fluid through a system composed of two regions, the first being an elastic and porous structure and the second being an adjacent open channel, possibly a macropore, an isolated cavity, or a connected fracture system. Both regions are saturated with the fluid, and we need to prescribe the stress and flow couplings on the interface between the Biot filtration flow through the deforming porous medium and the Stokes flow in the open channel. Our objective is to formulate a model of this composite hydromechanical system that accurately characterizes the depletion history and transient response of the fluid exchange and stress balance between the saturated elastic porous medium and the contiguous fluid-filled chamber and to show that this model leads to a mathematically well-posed problem that is amenable to analysis and computation.

Suppose that the disjoint pair of regions Ω_1 and Ω_2 in $\mathbb{R}^3$ share the common *interface*, $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. The first region Ω_1 is the fully saturated *elastic porous matrix* structure, and the second region Ω_2 is the fluid-filled *macro-void system,* which is adjacent to Ω_1. Here we denote by $\mathbf{n}$ the

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unit normal vector on the boundaries, directed *out* of Ω_1 and *into* Ω_2. The derivative with respect to time will be denoted by a superscript dot, so $\mathbf{v}^1(x,t) = \dot{\mathbf{u}}^1(x,t)$ denotes the *velocity* corresponding to a *displacement* $\mathbf{u}^1(x,t)$ of the *porous structure* at $x \in \Omega_1$. Also, we let $\mathbf{v}^2(x,t)$ be the velocity of the *fluid* at $x \in \Omega_2$. The *pressure* of the fluid in the pores of Ω_1 is given by $p^1(x,t)$ and the pressure of the fluid in the adjacent channel system Ω_2 by $p^2(x,t)$.

16.1.1 The Conservation Equations

The laminar flow of a slightly compressible viscous fluid through the deformable porous medium Ω_1 is described by the *Biot system* [11–13]

$$c_1\dot{p}^1 - \partial_i k_{ij}\partial_j p^1 + c_0\nabla\cdot\dot{\mathbf{u}}^1 = h_1(x,t), \tag{16.1a}$$

$$\rho_1\ddot{\mathbf{u}}^1 - (\lambda_1+\mu_1)\nabla(\nabla\cdot\mathbf{u}^1) - \mu_1\Delta\mathbf{u}^1 + c_0\nabla p^1 = \mathbf{f}_1(x,t), \tag{16.1b}$$

consisting of the *diffusion equation* for conservation of fluid mass and the *momentum equation* for the balance of forces, respectively. The porosity of the matrix and the compressibility of the fluid or the solid material on the meso-scale are incorporated in c_1. The *conductivity* k_{ij} combines the permeability of the structure and the viscosity of the fluid to provide a measure of the *filtration velocity* or *fluid flux* $\mathbf{q}=(q_1,q_2,q_3)$, given by *Darcy's law*, $q_i=-k_{ij}\partial_j p^1$. The density of the saturated porous matrix is denoted by ρ_1, and the positive Lamé constants λ_1 and μ_1 represent the *dilation* and *shear* moduli of elasticity, respectively. The first accounts for *compression* and the second for *distortion* of the medium [23, 19]. The dilation $c_0\nabla\cdot\mathbf{u}^1(t)$ provides the additional *pore fluid content* because of the local volume change, and the term $c_0\nabla p^1(t)$ is the *pressure stress* of the pore fluid on the structure. The *Biot-Willis* constant c_0 is the *pressure-storage coupling* coefficient [14]. See References 9, 20, 28, 21, 62, 22, 18, 55, 56, and 57 for background and recent results. We shall include here the situation of consolidation problems in which the inertial effects of the matrix are negligible, hence, $\rho_1 = 0$.

The slow flow of a slightly compressible viscous fluid in the adjacent open channel Ω_2 is described by the *compressible Stokes system* [59, 53]

$$c_2(x)\dot{p}^2 + \nabla\cdot\mathbf{v}^2 + c_2(x)\rho_2(x)\mathbf{g}(x)\cdot\mathbf{v}^2 = 0\,, \tag{16.2a}$$

$$\rho_2(x)\dot{\mathbf{v}}^2 - (\lambda_2+\mu_2)\nabla(\nabla\cdot\mathbf{v}^2) - \mu_2\Delta\mathbf{v}^2 + \nabla p^2 = c_2(x)\rho_2(x)\mathbf{g}(x)\,p^2. \tag{16.2b}$$

The constants λ_2 and μ_2 represent dilation and shear viscosity of the fluid, respectively. We also include the limiting case of an *incompressible* fluid (see p. 147 of Reference 59, p. 269 of Reference 52) for which $c_2 = 0$ and the flow in the channel is the classical *Stokes flow*,

$$\nabla\cdot\mathbf{v}^2 = 0, \quad \rho_2(x)\dot{\mathbf{v}}^2 - \mu_2\Delta\mathbf{v}^2 + \nabla p^2 = \mathbf{0}.$$

The system is obtained by linearization about a steady situation in which ρ_2 is the density of the fluid at the reference pressure. The coefficient $c_2(\cdot)$ arises from the compressibility of the fluid, and the terms with $\mathbf{g}(\cdot)$ are the gravitational contribution to momentum and to convection.

16.1.2 Interface Conditions

The objectives below are to identify a physically consistent set of interface conditions that couple these systems together and to formulate a variational statement of the resulting problem that leads to a mathematically well-posed initial-boundary-value problem. The interface coupling conditions must recognize the conservation of mass and total momentum. Thus, they will include the continuity of the normal fluid flux and of stress. The two additional constitutive relations concern the dependence of the Darcy flux at the interface on the pressure increment and the effect of the tangential component of stress on the velocity increment at the interface. The former is the classical *Robin* boundary condition, and the latter is the slip condition of *Beavers-Joseph-Saffman*.

16.2 The Biot-Stokes System

We assume the mechanical behavior of the porous solid is determined by classical small-strain elasticity. To describe this, we denote hereafter by Σ the space of *symmetric second-order tensors*. Boldface letters will be used to indicate vectors in $\mathbb{R}^3$ and Greek letters to indicate second-order tensors in Σ. We denote by $\delta = \{\delta_{ij}\}$ the identity consisting of ones on the diagonal and zeros elsewhere. We adopt the convention that repeated indices are summed. In particular, the scalar product of two vectors is $\mathbf{v} \cdot \mathbf{w} = v_i w_i$ and that of two second-order tensors is $\sigma : \tau = \sigma_{ij} \tau_{ij}$.

Standard function spaces will be used [1, 59]. Let Ω be a smoothly bounded region in $\mathbb{R}^3$, and denote its boundary by $\Gamma = \partial\Omega$. Let $H^1(\Omega)$ be the *Sobolev space* consisting of those functions in $L^2(\Omega)$ having each of their partial derivatives also in $L^2(\Omega)$. The *trace* map or restriction to the boundary is the linear map $\gamma : H^1(\Omega) \to L^2(\Gamma)$ defined by $\gamma(w) = w|_\Gamma$. Corresponding spaces of vector-valued functions will be denoted by boldface symbols. For example, we denote the product space $L^2(\Omega)^3$ by $\mathbf{L}^2(\Omega)$ and the corresponding triple of Sobolev spaces by $\mathbf{H}^1(\Omega) \equiv H^1(\Omega)^3$. We shall also use the space $\mathbf{L}^2_{\text{div}}(\Omega)$ of vector functions $\mathbf{L}^2(\Omega)$ whose divergence belongs to $L^2(\Omega)$. Recall that for the functions $\mathbf{r} \in \mathbf{L}^2_{\text{div}}(\Omega)$ there is a *normal trace* on the interface, and this is denoted by $\mathbf{r} \cdot \mathbf{n}$, because it takes this value on the smooth functions $\mathbf{r}$ in $\mathbf{L}^2_{\text{div}}(\Omega)$. Finally, we denote by $L^2(\Omega; \Sigma)$ the indicated space of Σ-valued functions on Ω.

Let $\mathbf{n} = \{n_i\}$ be the unit normal vector on a surface. For a vector $\mathbf{w}$, we denote the normal projection $w_n = \mathbf{w} \cdot \mathbf{n}$ and the tangential component $\mathbf{w}_T = \mathbf{w} - w_n \mathbf{n}$. Likewise for the tensor τ in Σ, we have its value at $\mathbf{n}$, $\tau(\mathbf{n}) = \{\tau_{ij} n_i\} \in \mathbb{R}^3$ and its normal and tangential parts $\tau(\mathbf{n})(\mathbf{n}) = \tau_n = \tau_{ij} n_i n_j$, $\tau_T = \tau(\mathbf{n}) - \tau_n \mathbf{n}$.

16.2.1 The System

We shall write the constitutive equations together with the partial differential equations for mass and momentum balance as a system of first-order partial differential equations in each of the two regions. Recall that $\mathbf{v}^1 = \dot{\mathbf{u}}^1$ denotes the *velocity* corresponding to a *displacement* $\mathbf{u}^1$ of the *porous structure* in Ω_1, and $\mathbf{v}^2$ is the velocity of the *fluid* in Ω_2. The symmetric derivative of a vector function $\mathbf{u}(x)$ is the tensor $\varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i) \in \Sigma$. The constitutive laws take the forms $\sigma^1(\mathbf{u}^1)_{ij} = \lambda_1 \delta_{ij} \varepsilon(\mathbf{u}^1)_{kk} + 2\mu_1 \varepsilon(\mathbf{u}^1)_{ij}$ in Ω_1 for the *elastic stress* corresponding to the *strain* $\varepsilon(\mathbf{u}^1)$ in the homogeneous and isotropic structure and $\sigma^2(\mathbf{v}^2)_{ij} = \lambda_2 \delta_{ij} \varepsilon(\mathbf{v}^2)_{kk} + 2\mu_2 \varepsilon(\mathbf{v}^2)_{ij}$ in Ω_2 for the *viscous stress* corresponding to the *strain rate* $\varepsilon(\mathbf{v}^2)$ of the *Newtonian fluid*. Note that $\sigma^1(\mathbf{u}^1) - c_0\, p^1\, \delta$ is the *total stress* due to elastic deformation and pore pressure p^1 within the matrix, and $\sigma^2(\mathbf{v}^2) - p^2\, \delta$ is the combined viscous and pressure stress of the fluid. Here both p^1 and p^2 are the thermodynamic pressure of the barotropic fluid in the respective regions. The *Biot-Stokes system* takes the form

$$c_1 \dot{p}^1 + \boldsymbol{\nabla} \cdot \mathbf{q} + c_0 \boldsymbol{\nabla} \cdot \mathbf{v}^1 = h_1(x, t), \tag{16.3a}$$

$$\mathcal{Q}\mathbf{q} + \boldsymbol{\nabla} p^1 = 0, \tag{16.3b}$$

$$\rho_1 \dot{\mathbf{v}}^1 - \boldsymbol{\nabla} \cdot \sigma^1 + c_0 \boldsymbol{\nabla} p^1 = \mathbf{f}_1(x, t), \tag{16.3c}$$

$$\mathcal{C}^1 \sigma^1 - \varepsilon(\mathbf{u}^1) = 0 \text{ in } \Omega_1, \quad \text{and} \tag{16.3d}$$

$$c_2(x) \dot{p}^2 + \boldsymbol{\nabla} \cdot \mathbf{v}^2 + c_2(x) \rho_2(x) \mathbf{g}(x) \cdot \mathbf{v}^2 = h_2(x, t), \tag{16.3e}$$

$$\rho_2(x) \dot{\mathbf{v}}^2 - \boldsymbol{\nabla} \cdot \sigma^2 + \boldsymbol{\nabla} p^2 - c_2(x) \rho_2(x)\, p^2\, \mathbf{g}(x) = \mathbf{f}_2(x, t), \tag{16.3f}$$

$$\mathcal{C}^2 \sigma^2 - \varepsilon(\mathbf{v}^2) = 0 \text{ in } \Omega_2. \tag{16.3g}$$

Eq. (16.3a) is the *storage equation* for the fluid mass conservation in the pores of the matrix in which the *flux* $\mathbf{q}$ is the relative velocity of the fluid within the porous structure given by *Darcy's law*. This is written in the form of Eq. (16.3b) of a force balance in which the flow resistance tensor $\mathcal{Q}$ is the inverse of the conductivity tensor k_{ij}. Eq. (16.3c) is the standard *Navier* system for the conservation of momentum of the elastic matrix structure, the constitutive relation (16.3d) is *Hookes law* for the stress-strain relationship, and the *compliance tensor* $\mathcal{C}^1$ is just the inverse of elasticity. These first four equations are equivalent to the Biot system (16.1). The last three equations are just the compressible Stokes system (16.2) for pressure $p^2(x,t)$ and velocity $\mathbf{v}^2(x,t)$ of the fluid. Eq. (16.3e) accounts for the fluid mass conservation in the channel, and Eq. (16.3f) is the momentum conservation equation. The gravitational force $\mathbf{g}$ contributes to both of these. The *Newtonian fluid* is described by the constitutive relation of Eq. (16.3g) in which the tensor $\mathcal{C}^2$ is the inverse to the viscosity tensor.

16.2.2 Boundary and Interface Conditions

We choose the *boundary conditions* on $\partial\Omega_1 \cup \partial\Omega_2 - \Gamma_{12}$ in a classical simple form, because they play no essential role here. On the exterior boundary of the porous medium, $\partial\Omega_1 - \Gamma_{12}$, we shall impose *drained conditions* $p_1 = 0$ on fluid pressure and the *clamped condition* $\mathbf{v}_1 = \mathbf{0}$ on velocity of the structure. On the exterior boundary of the free fluid, $\partial\Omega_2 - \Gamma_{12}$, we shall impose the *no-slip condition* $\mathbf{v}_2 = \mathbf{0}$ on fluid velocity.

To complete a well-posed problem, additional *interface conditions* must be imposed across the interior boundary Γ_{12}. Let us begin by reviewing the interface conditions that have been used previously to couple various models of fluid and solid composites.

16.2.2.1 Fluid-Solid Contact

The natural transmission conditions at the interface of a free fluid and an impervious elastic solid consist of the continuity of displacement and of stress [52]. The effective flow through a rigid microporous and permeable matrix is described by *Darcy's law*, $q_i = -k_{ij}\partial_j p^1$, where $\mathbf{q}$ is the filtration velocity or flux of fluid driven by a pressure gradient, and k_{ij} is the *conductivity*. In fact, Darcy's law can be realized as the upscaled limit by averaging or *homogenization* of a fine-scale periodic array of a rigid solid and intertwined fluid. See References 58, 2, 27. Similar results are obtained when the solid is permitted to be *elastic*, and then various scalings of the viscosity lead to a *viscous solid* or to the *Biot model of poroelasticity* of Eq. (16.1). See References 6, 51, 53, 16, 61, 7, 24, 8, 60.

16.2.2.2 Fluid-Porous Medium

The description of a free fluid in contact with a rigid but porous solid matrix requires a means to couple the slow flow to the upscaled Darcy filtration. Because a Stokes system is used for the free fluid, we have two distinct scales of hydrodynamics, and these are represented by two completely different systems of partial differential equations. Fluid conservation is a natural requirement at the interface, and other classically assumed conditions such as continuity of pressure or vanishing tangential velocity of the viscous fluid have been investigated [25, 43], but these issues have been controversial. See the discussion on p. 157 of Reference 53. In fact, one can even question the *location* of the interface, because the porous medium itself is already a mixture of fluid and solid. Moreover, Beavers and Joseph [10] discovered that fluid in contact with a porous medium flows faster along the interface than a fluid in contact with a solid surface; there is a substantial *slip* of the fluid at the interface with a porous medium. They proposed that the normal derivative of the tangential component of fluid velocity $\mathbf{v}_T$ satisfy

$$\frac{\partial}{\partial n}\mathbf{v}_T = \frac{\gamma}{\sqrt{K}}(\mathbf{v}_T - \mathbf{q}_T)$$

where K is the permeability of the porous medium, and γ is the *slip rate coefficient*. This condition was developed further in References 49 and 31, and a substantial rigorous analysis of such interface conditions was given in References 29 and 30. See References 47 and 44 for an excellent discussion; References 50, 26, 42, 4, and 3 for numerical work; Reference 48 for dependence on the slip parameter; and Reference 5 for homogenization results on related problems.

16.2.2.3 Fluid-Elastic Porous Medium

Any model of free fluid in contact with a *deformable* and porous medium contains the upscaled filtration velocity in addition to the displacement and stress variations of the porous matrix. These must be coupled to the Stokes flow, so all of the previous issues are present in the interface conditions. See References 45 and 46.

We begin with the mass-conservation requirement that the normal fluid flux be continuous across the interface. For this purpose, we introduce the parameter β, which represents the surface fraction of the interface on which the diffusion paths of the structure are *sealed*. The remaining fraction $1-\beta$ is the *contact surface* along Γ_{12}, where the diffusion paths of the porous medium are exposed to the fluid in the open channel, and so the motion of the structure contributes to the interfacial fluid mass flux. Thus, the solution is required to satisfy the *admissability constraint*

$$\big(c_0(1-\beta)\mathbf{v}^1 + \mathbf{q}\big)\cdot\mathbf{n} = \mathbf{v}^2\cdot\mathbf{n} \tag{16.4a}$$

for the conservation of fluid mass across the interface. We shall assume that the Darcy flow across Γ_{12} is driven by the difference between the total normal stress of the fluid and the pressure internal to the porous medium according to

$$\sigma_n^2 - p^2 + p^1 = \alpha\mathbf{q}\cdot\mathbf{n}. \tag{16.4b}$$

The constant $\alpha \geq 0$ is the *fluid entry resistance*. The conservation of momentum requires that the total stress of the porous medium is balanced by the total stress of the fluid. For the normal component this means

$$\sigma_n^1 - c_0 p^1 = c_0(1-\beta)\big(\sigma_n^2 - p^2\big), \tag{16.4c}$$

and for the tangential component we have

$$\sigma_T^1 = \sigma_T^2. \tag{16.4d}$$

Finally, this common tangential stress is assumed to be proportional to the *slip rate* according to the Beavers-Joseph-Saffman condition

$$\sigma_T^2 = \gamma\sqrt{\mathcal{Q}}\big(\mathbf{v}_T^2 - \mathbf{v}_T^1\big). \tag{16.4e}$$

We shall show next that the *interface conditions* (16.4) suffice precisely to couple the Biot system (16.1) in Ω_1 to the Stokes system (16.2) in Ω_2.

16.2.3 The Weak Formulation

Our objective is to construct an appropriate *variational formulation* of the Biot-Stokes system (16.3) coupled by the interface conditions (16.4). We seek a solution in spaces

$$p^1(t)\in V_1,\quad p^2(t)\in L^2(\Omega_2),\quad \mathbf{q}(t)\in\mathbf{W},\quad \mathbf{v}^1(t)\in\mathbf{V}_1,\quad \mathbf{v}^2(t)\in\mathbf{V}_2,$$

which are determined by boundary conditions. To focus on the interface conditions, we have chosen here the simplest classical examples, clamped and drained conditions on the appropriate boundaries, so the corresponding spaces are given by

$$\mathbf{V}_j = \big\{\mathbf{v}\in\mathbf{H}^1(\Omega_j) : \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega_j - \Gamma_{12}\big\},\quad j=1,2,$$
$$V_1 = \big\{p\in H^1(\Omega_1) : p = 0 \text{ on } \partial\Omega_1 - \Gamma_{12}\big\},\quad \mathbf{W} = \mathbf{L}^2_{\text{div}}(\Omega_1).$$

Those functions of V_1, $\mathbf{V}_1$, or $\mathbf{V}_2$ have a well-defined trace on the external boundary and on the interface Γ_{12}, and those from $\mathbf{W}$ have a normal trace, as noted above.

We want an appropriate weak form of the initial-boundary-value problem for the system of Eqs. (16.3) to (16.4). Multiply the momentum equations by test functions $\mathbf{w}^j \in \mathbf{V}_j$ and Darcy's law by $\mathbf{r} \in \mathbf{W}$, integrate the spatial derivatives and add to obtain

$$\begin{aligned}
&\int_{\Omega_1} \big(\rho_1 \dot{\mathbf{v}}^1 \cdot \mathbf{w}^1 + \big(\sigma^1 - c_0 p^1 \delta\big) : \varepsilon(\mathbf{w}^1) + \mathcal{Q}\mathbf{q} \cdot \mathbf{r} - p^1 \delta : \varepsilon(\mathbf{r})\big)\, dx \\
&\quad + \int_{\Omega_2} \big(\rho_2 \dot{\mathbf{v}}^2 \cdot \mathbf{w}^2 + (\sigma^2 - p^2 \delta) : \varepsilon(\mathbf{w}^2)\big)\, dx \\
&\quad + \int_{\Gamma_{12}} \big(-\sigma^1(\mathbf{n})(\mathbf{w}^1) + \sigma^2(\mathbf{n})(\mathbf{w}^2) + \big(c_0 \mathbf{w}^1 + \mathbf{r}\big) \cdot \mathbf{n} p^1 - \mathbf{w}^2 \cdot \mathbf{n} p^2\big)\, dS \\
&= \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{w}^1\, dx + \int_{\Omega_2} \mathbf{f}_2 \cdot \mathbf{w}^2\, dx. \qquad (16.5)
\end{aligned}$$

For each triple of test functions satisfying the admissibility constraint of Eq. (16.4a), the interface integral reduces to

$$\int_{\Gamma_{12}} \big(c_0 \beta p^1 \mathbf{n} \cdot \mathbf{w}^1 - \sigma^1(\mathbf{n})(\mathbf{w}^1) + \sigma^2(\mathbf{n})(\mathbf{w}^2) + (p^1 - p^2)\mathbf{n} \cdot \mathbf{w}^2\big)\, dS.$$

Moreover, decomposing the stress terms into their normal and tangential components, we obtain

$$\int_{\Gamma_{12}} \big(\big(c_0 \beta p^1 - \sigma_n^1\big) w_n^1 - \sigma_T^1 \cdot \mathbf{w}_T^1 + \sigma_T^2 \cdot \mathbf{w}_T^2 + \big(\sigma_n^2 + p^1 - p^2\big) w_n^2\big)\, dS,$$

and then the interface conditions of Eqs. (16.4b) to (16.4e) yield

$$\begin{aligned}
&\int_{\Gamma_{12}} \big(\alpha \mathbf{q} \cdot \mathbf{n} \big(w_n^2 - c_0(1-\beta) w_n^1\big) + \gamma \sqrt{\mathcal{Q}} \big(\mathbf{v}_T^2 - \mathbf{v}_T^1\big)\big(\mathbf{w}_T^2 - \mathbf{w}_T^1\big)\big) dS \\
&\quad = \int_{\Gamma_{12}} \big(\alpha (\mathbf{q} \cdot \mathbf{n})\, (\mathbf{r} \cdot \mathbf{n}) + \gamma \sqrt{\mathcal{Q}} \big(\mathbf{v}_T^2 - \mathbf{v}_T^1\big)\big(\mathbf{w}_T^2 - \mathbf{w}_T^1\big)\big) dS.
\end{aligned}$$

Finally, multiply the fluid conservation equations by test functions $\varphi^1 \in V_1$, $\varphi^2 \in L^2(\Omega_2)$ and the constitutive equations by $\tau^1 \in L^2(\Omega_1; \Sigma)$ and $\tau^2 \in L^2(\Omega; \Sigma)$, integrate over the corresponding regions, and add to the above to obtain the *variational statement*

$$\begin{aligned}
&\int_{\Omega_1} \big(\rho_1 \dot{\mathbf{v}}^1(t) \cdot \mathbf{w}^1 + \big(\sigma^1(t) - c_0 p^1(t) \delta\big) : \varepsilon(\mathbf{w}^1) + \mathcal{Q}\mathbf{q}(t) \cdot \mathbf{r} - p^1(t) \delta : \varepsilon(\mathbf{r}) \\
&\quad + \mathcal{C}^1 \dot{\sigma}^1(t) : \tau^1 - \varepsilon(\mathbf{v}^1(t)) : \tau^1 + c_1 \dot{p}^1(t) \varphi^1 + \delta : \varepsilon(\mathbf{q}(t)) \varphi^1 + c_0 \delta : \varepsilon(\mathbf{v}^1(t)) \varphi^1\big)\, dx \\
&\quad + \int_{\Omega_2} \big(\rho_2 \dot{\mathbf{v}}^2(t) \cdot \mathbf{w}^2 + (\sigma^2(t) - p^2(t) \delta) : \varepsilon(\mathbf{w}^2) + \mathcal{C}^2 \sigma^2(t) : \tau^2 - \varepsilon(\mathbf{v}^2(t)) : \tau^2 \\
&\quad - c_2 \rho_2 p^2(t) \mathbf{g} \cdot \mathbf{w}^2 + c_2 \dot{p}^2(t) \varphi^2 + \delta : \varepsilon(\mathbf{v}^2(t)) \varphi^2 + c_2 \rho_2 \mathbf{g} \cdot \mathbf{v}^2(t) \varphi^2\big)\, dx \\
&\quad + \int_{\Gamma_{12}} \big(\alpha (\mathbf{q}(t) \cdot \mathbf{n})\, (\mathbf{r} \cdot \mathbf{n}) + \gamma \sqrt{\mathcal{Q}} \big(\mathbf{v}_T^2(t) - \mathbf{v}_T^1(t)\big)\big(\mathbf{w}_T^2 - \mathbf{w}_T^1\big)\big)\, dS \\
&= \int_{\Omega_1} \mathbf{f}_1(t) \cdot \mathbf{w}^1\, dx + \int_{\Omega_2} \mathbf{f}_2(t) \cdot \mathbf{w}^2\, dx + \int_{\Omega_1} h_1(t) \varphi^1\, dx + \int_{\Omega_2} h_2(t) \varphi^2\, dx. \qquad (16.6)
\end{aligned}$$

Note that we have carefully written the operators on the *stress variables* as dual operators that contain an interior differential operator and boundary conditions, whereas the operator on *displacement* *variables* is the *local* differential operator. In summary, we define the product space

$$\mathbb{V} = \{[\varphi^1, \mathbf{r}, \mathbf{w}^1, \tau^1, \varphi^2, \mathbf{w}^2, \tau^2] \in V_1 \times \mathbf{W} \times \mathbf{V}_1 \times L^2(\Omega_1, \Sigma) \times L^2(\Omega_2) \times \mathbf{V}_2 \\ \times L^2(\Omega_2, \Sigma) : \big(c_0(1-\beta)\mathbf{w}^1 + \mathbf{r}\big) \cdot \mathbf{n} = \mathbf{w}^2 \cdot \mathbf{n} \text{ on } \Gamma_{12}\},$$

and then the *weak formulation* of the problem is to find the vector-valued functions

$$\mathbf{v}(t) \equiv [p^1(t), \mathbf{q}(t), \mathbf{v}^1(t), \sigma^1(t), p^2(t), \mathbf{v}^2(t), \sigma^2(t)] \in \mathbb{V}, \quad t > 0,$$

such that Eq. (16.6) holds for every $[\varphi^1, \mathbf{r}, \mathbf{w}^1, \tau^1, \varphi^2, \mathbf{w}^2, \tau^2] \in \mathbb{V}$, and we have the *initial conditions*

$$\rho_1\mathbf{v}^1(0) = \rho_1\mathbf{v}_0^1, \quad c_1 p^1(0) = c_1 p_0^1 \text{ in } \Omega_1, \tag{16.7a}$$

$$\rho_2\mathbf{v}^2(0) = \rho_2\mathbf{v}_0^2, \quad c_2 p^2(0) = c_2 p_0^2 \text{ in } \Omega_2. \tag{16.7b}$$

Of course, $\sigma^1(0)$ is also determined from Eq. (16.3d) and the initial displacement, $\mathbf{u}^1(0)$.

16.3 The Evolution Dynamics

The equations in the system are to hold in the product space

$$\mathbb{H} = L^2(\Omega_1) \times \mathbf{L}^2(\Omega_1) \times \mathbf{L}^2(\Omega_1) \times L^2(\Omega_1, \Sigma) \times L^2(\Omega_2) \times \mathbf{L}^2(\Omega_2) \times L^2(\Omega_2, \Sigma),$$

and the solution will be sought in the space $\mathbb{V}$. Note that we have the continuous inclusions $\mathbb{V} \hookrightarrow \mathbb{H} = \mathbb{H}' \hookrightarrow \mathbb{V}'$. The (explicit) *divergence operator* $\delta : \varepsilon$ is a map of each of $\mathbf{W} \to \mathbf{L}^2(\Omega_1)$, $\mathbf{V}_1 \to \mathbf{L}^2(\Omega_1)$, and $\mathbf{V}_2 \to \mathbf{L}^2(\Omega_2)$, and then the corresponding *dual* operator $-\boldsymbol{\nabla} \cdot \delta$ mapping $\mathbf{L}^2(\Omega_1) \to \mathbf{W}'$, $\mathbf{L}^2(\Omega_1) \to \mathbf{V}_1'$, or $\mathbf{L}^2(\Omega_2) \to \mathbf{V}_2'$, respectively, consists of the gradient and a boundary condition. Note that $\mathbf{V}_1 \hookrightarrow \mathbf{W} \hookrightarrow \mathbf{L}^2(\Omega_1)$. Similar remarks hold for $\varepsilon : \mathbf{V}_j \to L^2(\Omega_j, \Sigma)$ and its dual $-\boldsymbol{\nabla}\cdot : L^2(\Omega_j, \Sigma) \to \mathbf{V}_j'$. We have two *interface operators* in the variational formulation of Eq. (16.6). These are the *normal trace* $\gamma_n(\mathbf{q}) = \mathbf{q} \cdot \mathbf{n}$ and the *tangential trace* $\gamma_T(\mathbf{v}) = \mathbf{v}_T$, which define linear maps $\gamma_n : \mathbf{W} \to L^2(\Gamma_{12})$, $\gamma_T : \mathbf{V}_j \to \mathbf{L}^2(\Gamma_{12})$, for $j = 1, 2$.

16.3.1 The Initial-Value Problem

With the operators so defined, the system has the form

$$\mathbf{v}(t) \in \mathbb{V} : \frac{d}{dt}(\mathcal{A}\mathbf{v}(t)) + \mathcal{B}\mathbf{v}(t) = \mathbf{f}(t) \text{ in } \mathbb{H}', \quad t > 0, \tag{16.8a}$$

where the matrix of operators and variables are denoted by

$$\mathcal{A} = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{v}(t) = \begin{bmatrix} p^1(t) \\ \mathbf{q}(t) \\ \mathbf{v}^1(t) \\ \sigma^1(t) \\ p^2(t) \\ \mathbf{v}^2(t) \\ \sigma^2(t) \end{bmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} 0 & \delta:\varepsilon & c_0\delta:\varepsilon & 0 & 0 & 0 & 0 \\ \nabla\cdot\delta & \mathcal{Q}+\gamma_n'\alpha\gamma_n & 0 & 0 & 0 & 0 & 0 \\ c_0\nabla\cdot\delta & 0 & \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & -\nabla\cdot & 0 & -\gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta:\varepsilon + c_2\rho_2\mathbf{g} & 0 \\ 0 & 0 & -\gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & 0 & \nabla\cdot\delta - c_2\rho_2\mathbf{g} & \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T & -\nabla\cdot \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon & \mathcal{C}^2 \end{pmatrix}.$$

The evolution equation (16.8a) is to be solved subject to the *initial condition*

$$\mathcal{A}\mathbf{v}(0) = \mathcal{A}\mathbf{v}_0, \tag{16.8b}$$

where $\mathcal{A}\mathbf{v}_0$ is determined from Eq. (16.7). Note that $\mathcal{A} : \mathbb{H} \to \mathbb{H}'$ is degenerate but symmetric and nonnegative, and it is easy to see that $\mathcal{B} : \mathbb{V} \to \mathbb{V}'$ is monotone. Equation (16.8a) is an example of an *implicit evolution equation* with *degenerate* operators as coefficients, sometimes known as a degenerate *Sobolev equation*. We recall that Jack Lagnese was a major contributor to the development of the theory of these abstract Sobolev equations, especially the singular perturbation theory and dependence of the solution on the operators. See References 32–41.

Because $\mathcal{A}+\mathcal{B}$ is $\mathbb{H}$-coercive in our situation, *uniqueness* for the initial value problem of Eq. (16.8) is easy to establish. According to the general theory of such equations [54, 17], for *existence* of a solution it suffices to establish the *range condition* $\mathrm{Rg}(\lambda\mathcal{A} + \mathcal{B}) \supset \mathrm{Rg}(\mathcal{A})$ for $\lambda > 0$. For this, we consider the *resolvent system*

$$\mathbf{v} = [p^1(t), \mathbf{q}(t), \mathbf{v}^1(t), \sigma^1(t), p^2(t), \mathbf{v}^2(t), \sigma^2(t)] \in \mathbb{V} :$$

$$\lambda c_1 p^1 + \delta:\varepsilon(\mathbf{q}) + c_0\delta:\varepsilon(\mathbf{v}^1) = c_1 h_1, \tag{16.9a}$$

$$(\mathcal{Q} + \gamma_n'\alpha\gamma_n)\mathbf{q} + \nabla\cdot\delta p^1 = \mathbf{s}, \tag{16.9b}$$

$$\lambda\rho_1\mathbf{v}^1 - \nabla\cdot\sigma^1 + c_0\nabla\cdot\delta p^1 + \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T(\mathbf{v}^1 - \mathbf{v}^2) = \rho_1\mathbf{f}_1, \tag{16.9c}$$

$$\lambda\mathcal{C}^1\sigma^1 - \varepsilon(\mathbf{v}^1) = \xi_1, \tag{16.9d}$$

$$\lambda c_2(x) p^2 + \delta:\varepsilon(\mathbf{v}^2) + c_2(x)\rho_2(x)\mathbf{g}(x)\cdot\mathbf{v}^2 = c_2 h_2, \tag{16.9e}$$

$$\lambda\rho_2(x)\mathbf{v}^2 - \nabla\cdot\sigma^2 + \nabla\cdot\delta p^2 - c_2(x)\rho_2(x)\, p^2\,\mathbf{g} + \gamma_T'\gamma\sqrt{\mathcal{Q}}\gamma_T(\mathbf{v}^2 - \mathbf{v}^1) = \rho_2\mathbf{f}_2, \tag{16.9f}$$

$$\mathcal{C}^2\sigma^2 - \varepsilon(\mathbf{v}^2) = \xi_2, \tag{16.9g}$$

where the right side of this system is given as $[c_1h_1, \mathbf{s}, \rho_1\mathbf{f}_1, \xi_1, c_2h_2, \rho_2\mathbf{f}_2, \xi_2] \in \mathbb{H}'$. Note that Eq. (16.9) contains the interface conditions of Eq. (16.4).

The means by which we establish the solvability of the resolvent system will depend critically on how much degeneracy occurs in the operators. For example, in the *least degenerate* case in which all the constants $c_1,\ \rho_1,\ c_2,\ \rho_2$ are strictly positive, the resolution of Eq. (16.9) is straightforward. In the mathematically more interesting and practically more relevant situations, some of these coefficients will vanish. In many of these cases, we can eliminate appropriate variables, thereby obtaining elliptic terms in the system and then solve the reduced higher-order system. We shall indicate briefly how one can establish the solvability by means of the *mixed formulation* of the resolvent system in which it is regarded as a *saddlepoint problem* from convex analysis [15].

16.3.2 The Mixed Formulation

Here we shall consider the resolvent system of Eq. (16.9), but instead of writing a single operator equation in the space $\mathbb{V}'$ with 7 unknowns, we shall reorder the variables according to their role in the *physics* of the model, not in the *geometry* of the problem. Thus, we write the resolvent system on a product of two spaces so that it is realized as a *saddlepoint problem*. The first space $\mathbf{X}$ consists of the *displacement* variables,

$$\mathbf{X} = \{[\mathbf{q}, \mathbf{v}_1, \mathbf{v}_2] \in \mathbf{W} \times \mathbf{V}_1 \times \mathbf{V}_2 : \ (c_0(1-\beta)\mathbf{v}^1 + \mathbf{q}) \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n}\},$$

and the second space $\mathbf{Y}$ contains the *generalized stress* variables,

$$\mathbf{Y} = \left\{[p_1, \sigma_1, p_2, \sigma_2] \in L^2(\Omega_1) \times L^2(\Omega_1, \Sigma) \times L^2(\Omega_2) \times L^2(\Omega_2, \Sigma)\right\}.$$

If we define the operators

$$A : \mathbf{X} \to \mathbf{X}' \quad B : \mathbf{X} \to \mathbf{Y}' \quad C : \mathbf{Y} \to \mathbf{Y}'$$

by means of the matrices

$$A = \begin{pmatrix} \mathcal{Q} + \gamma_n' \alpha \gamma_n & 0 & 0 \\ 0 & \lambda\rho_1 + \gamma_T' \gamma \sqrt{\mathcal{Q}} \gamma_T & -\gamma_T' \gamma \sqrt{\mathcal{Q}} \gamma_T \\ 0 & -\gamma_T' \gamma \sqrt{\mathcal{Q}} \gamma_T & \lambda\rho_2 + \gamma_T' \gamma \sqrt{\mathcal{Q}} \gamma_T \end{pmatrix},$$

$$B = \begin{pmatrix} \delta : \varepsilon & c_0 \delta : \varepsilon & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & \delta : \varepsilon + c_2 \rho_2 \mathbf{g} \cdot \\ 0 & 0 & -\varepsilon \end{pmatrix}, \quad C = \begin{pmatrix} \lambda c_1 & 0 & 0 & 0 \\ 0 & \lambda \mathcal{C}^1 & 0 & 0 \\ 0 & 0 & \lambda c_2 & 0 \\ 0 & 0 & 0 & \mathcal{C}^2 \end{pmatrix},$$

then the system of Eq. (16.9) is obtained in the form

$$\begin{aligned} A\mathbf{x} - B'\mathbf{y} &= \mathbf{f} \\ B\mathbf{x} + C\mathbf{y} &= \mathbf{g} \end{aligned}$$

for the unknowns $\mathbf{x} \equiv [\mathbf{q}, \mathbf{v}_1, \mathbf{v}_2] \in \mathbf{X}$, $\mathbf{y} \equiv [p_1, \sigma_1, p_2, \sigma_2] \in \mathbf{Y}$. This formulation requires a *closed-range condition* on the operator B, and it provides a natural and well-established approach to the *numerical approximation* of such a problem. In addition, the analysis of this formulation provides a means to establish the relation with the *singular limits* such as the incompressible case $c_2 = 0$ of the Stokes flow and the *quasistatic* case $\rho_1 = 0$ of consolidation processes. However, we can work directly with the original formulation of Eq. (16.9) to obtain these limits and the corresponding existence results. These issues will be developed for nonlinear extensions of these models in forthcoming works.

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Chapter 17

Operator-Valued Analytic Functions Generated by Aircraft Wing Model (Subsonic Case)

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Abstract The present paper is devoted to our recent results on an aircraft wing model in a subsonic airflow. The model is governed by a system of two coupled integro-differential equations and a two-parameter family of boundary conditions modeling the action of the self-straining actuators. The system of equations of motion is equivalent to a single operator evolution-convolution equation in the energy space. The Laplace transform of the solution involves the so-called generalized resolvent operator, which is a finite meromorphic operator–valued function of the spectral parameter defined on the complex plane with a branchcut along the negative real semiaxis. The main findings in this study are (a) asymptotic distribution of the poles (called *aeroelastic modes*) of the generalized resolvent and (b) the *Riesz basis property* of the mode shapes.

17.1 Introduction

In the present chapter, we summarize our results obtained for the initial boundary-value problem arising in modeling the vibrations of aircraft wing in a subsonic air flow. An ultimate goal of an aircraft wing modeling is to find an approach to flutter control (see Reference 6 and references therein). Flutter is a physical phenomenon that occurs in a solid elastic structure interacting with a flow of gas or fluid. It is a structural dynamic instability consisting of violent vibrations of the solid structure with rapidly increasing amplitude and usually leads either to serious damage of the structure or to its complete destruction. The physical reason for this phenomenon is that under special conditions, the energy of the flow is rapidly absorbed by the structure and transformed into the energy of mechanical vibrations. The most well-known cases of flutter are related to the flutter in aircraft wings, tails, and control surfaces. Flutter is an in-flight event that happens beyond some speed-altitude combinations. High-speed aircrafts are most susceptible for flutter, although no speed regimen is truly immune from flutter.

Flutter instabilities occur in a variety of different engineering and even biomedical situations. Namely in aeronautic engineering, flutter of helicopter, propeller, and turbine blades is a serious problem. It also affects electric transmission lines, high-speed hard disk drives, and long-spanned suspension bridges. Flutter of cardiac tissue and blood vessel walls is of a special concern to medical practitioners. Flutter is an extremely complex physical phenomenon whose complete theoretical explanation is *an open problem*. At the present moment, there exists only a few models of fluid–structure interaction involving flutter development for which precise mathematical formulations are available. We believe that analytical treatment of the flutter problem can provide insights not available from purely computational or experimental results. It is certainly important for designing flutter-control mechanisms.

Ideally, a complete picture of a fluid-structure interaction should be described by a system of partial differential equations, a system that contains both the equations governing the vibrations of an elastic structure and the hydrodynamic equations governing the motion of gas or fluid flow. The system of equations of motion should be supplied with appropriate boundary and initial conditions. The structural and hydrodynamic parts of the system must be coupled in the following sense. The hydrodynamic equations define a pressure distribution on the elastic structure. This pressure distribution in turn defines the so-called aerodynamic loads, which appear as forcing terms in structural equations. On the other hand, the parameters of the elastic structure enter the boundary conditions for the hydrodynamic equations. The above picture is mathematically very complicated, and to make a particular problem tractable, it is necessary to introduce simplifying assumptions. We have assumed that the model describes a wing of high-aspect ratio (i.e., the length of a wing is much greater than its width, though both quantities are finite) in a *subsonic, inviscid, incompressible airflow*. In this model, the hydrodynamic equations have been solved explicitly, and aerodynamic loads are represented as forcing terms in the structural equations as time convolution-type integrals with complicated kernels. Thus, the model is described by a system of integro-differential equations. The very notion of spectral analysis for such systems is a new mathematical challenge. We treat the system of equations of motion of the model as a single evolution-convolution equation in the Hilbert state space of the model. (The integral convolution part of this equation vanishes if a speed of an air stream is equal to zero, and we obtain the equation of motion for the so-called ground vibrations [7–10, 13].) Our results on this model include the following [7–14]:

1. We represent the solution of the initial boundary-value problem in the frequency domain in terms of the generalized resolvent, which is an analytic finite-meromorphic, operator-valued function of the spectral parameter. We define the aeroelastic modes as the poles of the generalized resolvent; the corresponding mode shapes are defined in terms of the residues at the poles.

2. We have found explicit asymptotic formulae for the aeroelastic modes. (To the best of our knowledge, these are the first such formulae in the literature on aeroelasticity.) The entire set of aeroelastic modes splits asymptotically into two branches, which are asymptotically close to the eigenvalues of the structural part of the system [7–12]. The fact that the aeroelastic modes are asymptotically close to torsional and bending modes was believed to be true for years. However, no rigorous proof was available in the literature.

3. We have derived the asymptotic formulae for the aeroelastic mode shapes [8, 12].

4. We have shown that the set of the mode shapes forms a nonorthogonal basis (Riesz basis) in the state space of the system. (The set of the generalized eigenvectors of the structural part of the system has a similar property [9, 11, 12].)

5. Using *the Riesz basis property* of the mode shapes, we have presented the solution of the original problem in the form of expansion with respect to the mode shapes and the

integral along the negative real semiaxis, which is formally associated with *the continuous spectrum.*

Statement of problem. Operator setting in energy space Let us introduce the following dynamic variables [1, 2]:

$$X(x,t) = \begin{bmatrix} h(x,t) \\ \alpha(x,t) \end{bmatrix}, \quad -L \le x \le 0,\ t \ge 0, \tag{17.1}$$

where $h(x,t)$ is the bending and $\alpha(x,t)$ is the torsion angle. The model, which we will investigate, can be described by the following linear system:

$$(M_s - M_a)\ddot{X}(x,t) + (D_s - uD_a)\dot{X}(x,t) + (K_s - u^2 K_a)X = \begin{bmatrix} f_1(x,t) \\ f_2(x,t) \end{bmatrix}. \tag{17.2}$$

From now on, we will use an "overdot" to denote the differentiation with respect to t. We use the subscripts s and a to distinguish the structural and aerodynamical parameters, respectively. All 2×2 matrices in Eq. (17.2) are given by the following formulae:

$$M_s = \begin{bmatrix} m & S \\ S & I \end{bmatrix}, \quad M_a = (-\pi\rho)\begin{bmatrix} 1 & -a \\ -a & (a^2 + 1/8) \end{bmatrix}, \tag{17.3}$$

where m is the density of the flexible structure (mass per unit length), S is the mass moment, I is the moment of inertia, ρ is the density of air, u is the stream speed, a is the linear parameter of the structure $-1 \le a \le 1$, (a is the relative distance between the elastic axis of a model wing and its line of center of gravity), and

$$D_s = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_a = -\pi\rho\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K_s = \begin{bmatrix} E\frac{\partial^4}{\partial x^4} & 0 \\ 0 & -G\frac{\partial^2}{\partial x^2} \end{bmatrix}, \quad K_a = -\pi\rho\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \tag{17.4}$$

where E is the bending stiffness and G is the torsion stiffness. The right-hand side of the system in Eq. (17.2) can be represented as the following system of two convolution-type integral operations:

$$\begin{aligned} f_1(x,t) &= -2\pi\rho\int_0^t [uC_2(t-\sigma) - \dot{C}_3(t-\sigma)g](x,\sigma)d\sigma \equiv \int_0^t \tilde{C}_1(t-\sigma)g(x,\sigma)\,d\sigma, \\ f_2(x,t) &= -2\pi\rho\int_0^t [1/2C_1(t-\sigma) - auC_2(t-\sigma) + a\dot{C}_3(t-\sigma) \\ &\quad + uC_4(t-\sigma) + 1/2\dot{C}_5(t-\sigma)]g(x,\sigma)d\sigma \equiv \int_0^t \tilde{C}_2(t-\sigma)g(x,\sigma)\,d\sigma, \\ g(x,t) &= u\dot{\alpha}(x,t) + \ddot{h}(x,t) + (1/2 - a)\ddot{\alpha}(x,t). \end{aligned} \tag{17.5}$$

The aerodynamic functions $C_i,\ i = 1 \ldots 5$, are defined in the following ways [3]:

$$\begin{aligned} \hat{C}_1(\lambda) &\equiv \int_0^\infty e^{-\lambda t}C_1(t)\,dt = \frac{u}{\lambda}\,\frac{e^{-\lambda/u}}{K_0(\lambda/u) + K_1(\lambda/u)}, \quad \Re\,\lambda > 0, \\ C_3(t) &= \int_0^t C_1(t-\sigma)\big(u\sigma - \sqrt{u^2\sigma^2 + 2u\sigma}\big)d\sigma, \\ C_2(t) &= \int_0^t C_1(\sigma)\,d\sigma, \quad C_4(t) = C_2(t) + C_3(t), \\ C_5(t) &= \int_0^t C_1(t-\sigma)\big[(1+u\sigma)\sqrt{u^2\sigma^2 + 2u\sigma} - (1+u\sigma)^2\big]d\sigma, \end{aligned} \tag{17.6}$$

where K_0 and K_1 are the modified Bessel functions [3]. The self-straining control actuator action can be modeled by the following boundary conditions [1, 2, 7–14]:

$$Eh''(0,t) + \beta\dot{h}'(0,t) = 0, \quad h'''(0,t) = 0, \tag{17.7}$$

$$G\alpha'(0,t) + \delta\dot{\alpha}(0,t) = 0, \quad \beta, \delta \in \mathbb{C}^+ \cup \{\infty\}, \tag{17.8}$$

where $\mathbb{C}^+$ is the closed right half-plane. The boundary conditions at $x = -L$ are

$$h(-L,t) = h'(-L,t) = \alpha(-L,t) = 0. \tag{17.9}$$

Let the initial state of the system be given as follows:

$$h(x,0) = h_0(x), \quad \dot{h}(x,0) = h_1(x), \quad \alpha(x,0) = \alpha_0(x), \quad \dot{\alpha}(x,0) = \alpha_1(x). \tag{17.10}$$

We will consider the solution of the problem given by Eq. (17.2) and conditions in Eqs. (17.7) to (17.10) in the energy space $\mathcal{H}$, when the parameters satisfy the following two conditions:

$$det \begin{bmatrix} m & S \\ S & I \end{bmatrix} > 0, \quad 0 < u \le \frac{\sqrt{2G}}{L\sqrt{\pi\rho}}. \tag{17.11}$$

The second condition in Eq. (17.11) has clear physical interpretation: it means that the flow speed must be below the "divergence" or static aeroelastic instability speed for the system. Let $\mathcal{H}$ be the set of 4-component vector-valued functions $\Psi = (h, \dot{h}, \alpha, \dot{\alpha})^T \equiv (\psi_0, \psi_1, \psi_2, \psi_3)^T$ (T means the transposition) obtained as a closure of smooth functions satisfying the conditions

$$\psi_0(-L) = \psi_0'(-L) = \psi_2(-L) = 0 \tag{17.12}$$

in the following energy norm:

$$\begin{aligned} \|\Psi\|^2_{\mathcal{H}} = \frac{1}{2}\int_{-L}^{0} \big\{ & E|\psi_0''(x)|^2 + G|\psi_2'(x)|^2 + \tilde{m}|\psi_1(x)|^2 + \tilde{I}|\psi_3(x)|^2 \\ & + \tilde{S}[\psi_3(x)\bar{\psi}_1(x) + \bar{\psi}_3(x)\psi_1(x)] - \pi\rho u^2|\psi_2(x)|^2 \big\}\, dx. \end{aligned} \tag{17.13}$$

Under conditions (17.11), the norm of Eq. (17.13) is well defined. In what follows, we need the notations

$$\tilde{m} = m + \pi\rho, \quad \tilde{S} = S - a\pi\rho, \quad \tilde{I} = I + \pi\rho(a^2 + 1/8), \quad \Delta = \tilde{m}\tilde{I} - \tilde{S}^2. \tag{17.14}$$

The initial boundary–value problem of Eq. (17.2) and Eqs. (17.10) to (17.13) can be represented in the form

$$\dot{\Psi} = i\mathcal{L}_{\beta\delta}\Psi + \tilde{\mathcal{F}}\dot{\Psi}, \quad \Psi = (\psi_0, \psi_1\psi_2, \psi_3)^T, \quad \Psi|_{t=0} = \Psi_0. \tag{17.15}$$

$\mathcal{L}_{\beta\delta}$ is the following matrix differential operator in $\mathcal{H}$:

$$\mathcal{L}_{\beta\delta} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{E\tilde{I}}{\Delta}\frac{d^4}{dx^4} & -\frac{\pi\rho u\tilde{S}}{\Delta} & -\frac{\tilde{S}}{\Delta}\left(G\frac{d^2}{dx^2} + \pi\rho u^2\right) & -\frac{\pi\rho u\tilde{I}}{\Delta} \\ 0 & 0 & 0 & 1 \\ \frac{E\tilde{S}}{\Delta}\frac{d^4}{dx^4} & \frac{\pi\rho u\tilde{m}}{\Delta} & \frac{\tilde{m}}{\Delta}\left(G\frac{d^2}{dx^2} + \pi\rho u^2\right) & \frac{\pi\rho u\tilde{S}}{\Delta} \end{bmatrix} \tag{17.16}$$

defined on the domain

$$\mathcal{D}(\mathcal{L}_{\beta\delta}) = \{\Psi \in \mathcal{H}:\ \psi_0 \in H^4(-L,0),\ \psi_1,\ \psi_2 \in H^2(-L,0), \psi_3 \in H^1(-L,0); \psi_0'''(0) = 0;$$
$$\psi_1(-L) = \psi_1'(-L) = \psi_3(-L) = 0; E\psi_0''(0) + \beta\psi_1'(0) = 0, G\psi_2'(0) + \delta\psi_3(0) = 0\}. \tag{17.17}$$

$\tilde{\mathcal{F}}$ is a linear integral operator in $\mathcal{H}$ given by the following formula:

$$\tilde{\mathcal{F}} = \frac{1}{\Delta}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & [\tilde{I}(\tilde{C_1}*) - \tilde{S}(\tilde{C_2}*)] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & [-\tilde{S}(\tilde{C_1}*) + \tilde{m}(\tilde{C_2}*)] \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & u & (1/2 - a) \\ 0 & 0 & 0 & 0 \\ 0 & 1 & u & (1/2 - a) \end{bmatrix}. \tag{17.18}$$

In Eq. (17.18), $*$ stands for the convolution, and the kernels $\tilde{C_1}$ and $\tilde{C_2}$ are defined in Eq. (17.5).

REMARK 17.1 Equation (17.15) is not an *evolution equation*. It does not have a dynamics generator and does not define any semigroup in the standard sense. However, the notion of the spectral analysis is now well understood. Namely, the aircraft wing model can be described by the evolution-convolution equation of the form

$$\dot{\Psi}(t) = i\,A\,\Psi(t) + \int_0^t F(t-\tau)\dot{\Psi}(\tau)\,d\tau. \tag{17.19}$$

Here $\Psi(\cdot) \in \mathcal{H}$, with $\mathcal{H}$ being the state space of the system; A ($A = \mathcal{L}_{\beta\delta}$) is a matrix differential operator and $F(t)$ is a matrix-valued function. A formal solution of Eq. (17.19) in the Laplace representation can be given by the formula

$$\hat{\Psi}(\lambda) = [\lambda I - iA - \lambda\hat{F}(\lambda)]^{-1}[I - \hat{F}(\lambda)]\Psi_0, \tag{17.20}$$

where Ψ_0 is the initial state (i.e., $\Psi(0) = \Psi_0$) and the symbol ˆ is used to denote the Laplace transform. It is an extremely nontrivial problem "to calculate" the inverse Laplace transform of Eq. (17.19) to have the space-time domain representation of the solution. To do this, it is necessary to investigate the "generalized resolvent operator"

$$\mathcal{R}(\lambda) = [\lambda I - iA - \lambda\hat{F}(\lambda)]^{-1}. \tag{17.21}$$

In our case, $\mathcal{R}(\lambda)$ is an operator-valued meromorphic function on the complex plane with a branchcut along the negative real semiaxis. The poles of $\mathcal{R}(\lambda)$ are called *the aeroelastic modes*. The branchcut corresponds to the continuous spectrum. The Laplace transform representation for the solution of the problem in Eq. (17.15) corresponding to Eq. (17.19) has the form

$$\hat{\Psi}(\lambda) = [\lambda I - i\mathcal{L}_{\beta\delta} - \lambda\hat{\mathfrak{F}}(\lambda)]^{-1}\,[I - \hat{\mathfrak{F}}(\lambda)]\Psi_0. \tag{17.22}$$

To calculate the inverse Laplace transform of $\hat{\Psi}$, one has to accomplish the contour integration in the complex λ-plane. In this connection, the properties of the "generalized resolvent operator"

$$\mathcal{R}(\lambda) = [\lambda I - i\mathcal{L}_{\beta\delta} - \lambda\hat{\mathfrak{F}}(\lambda)]^{-1} \tag{17.23}$$

are of crucial importance for us. In our work [10–13], we have shown that the convolution part does not "destroy" the main characteristics of the discrete spectrum, which is produced by the differential part of the problem. Namely, we have proven that the aeroelastic modes are asymptotically close to the discrete spectrum of the operator $i\mathcal{L}_{\beta\delta}$ and have calculated the rate at which the aeroelastic modes approach that spectrum. Second, we have also proven that there may be only a finite number of the aeroelastic modes having positive real parts. The latter means that for a given stream speed u, there exists at most a finite number of *unstable aeroelastic mode shapes* (eigenfunctions of the problem). Third, we have shown that the set of the mode shapes forms a Riesz basis in the energy space, which is crucial for "calculation" of the inverse Laplace transform of the function $\hat{\Psi}$ from Eq. (17.19).

17.2 Properties of Matrix Differential and Integral Operators

Now we summarize asymptotic and spectral results obtained in our papers [7–14].

THEOREM 17.1

1. *$\mathcal{L}_{\beta\delta}$ is a closed linear operator in $\mathcal{H}$, whose resolvent is compact, and therefore the spectrum is discrete.*

2. *Operator $\mathcal{L}_{\beta\delta}$ is non–Self-adjoint unless $\Re\,\beta = \Re\,\delta = 0$. If $\Re\,\beta \geq 0$ and $\Re\,\delta \geq 0$, then this operator is dissipative, that is, $\Im(\mathcal{L}_{\beta\delta}\Psi, \Psi) \geq 0$ for $\Psi \in \mathcal{D}(\mathcal{L}_{\beta\delta})$. The adjoint operator $\mathcal{L}^*_{\beta\delta}$ is given by the matrix differential expression of Eq.* (17.16) *on the domain obtained from Eq.* (17.17) *by replacing the parameters β and δ with $(-\bar{\beta})$ and $(-\bar{\delta})$, respectively.*

3. *When $\mathcal{L}_{\beta\delta}$ is dissipative, then it is maximal, that is, it does not admit dissipative extensions.*

DEFINITION 17.1 *A vector Φ in a Hilbert space H is an* associate vector *of a non–Self-adjoint operator A of an order m corresponding to an eigenvalue λ if $\Phi \neq 0$ and*

$$(A - \lambda I)^m \Phi \neq 0 \quad and \quad (A - \lambda I)^{m+1}\Phi = 0. \tag{17.24}$$

If $m = 0$, then Φ is an eigenvector. The set of all associate vectors and eigenvectors together will be called the set of root vectors *[4].*

THEOREM 17.2

1. *The operator $\mathcal{L}_{\beta\delta}$ has a countable set of complex eigenvalues. If*

$$\delta \neq \sqrt{G\tilde{I}}, \tag{17.25}$$

then the set of eigenvalues is located in a strip parallel to the real axis.

2. *The entire set of eigenvalues asymptotically splits into two different subsets. We call them the β–branch and the δ-branch and denote these branches by $\{\nu_n^\beta\}_{n\in\mathbb{Z}}$ and $\{\nu_n^\delta\}_{n\in\mathbb{Z}}$, respectively. If $\Re\,\beta \geq 0$ and $\Re\,\delta > 0$, then each branch is asymptotically close to its own horizontal line in the closed upper half-plane. If $\Re\,\beta > 0$ and $\Re\,\delta = 0$, then both horizontal lines coincide with the real axis. If $\Re\,\beta = \Re\,\delta = 0$, then the operator $\mathcal{L}_{\beta\delta}$ is self-adjoint, and, thus, its spectrum is real. The entire set of eigenvalues may have only two points of accumulation: $+\infty$ and $-\infty$ in the sense that $\Re\,\nu_n^{\beta(\delta)} \longrightarrow \pm\infty$ and $|\Im\ \nu_n^{\beta(\delta)}| \leq const$ as $n \longrightarrow \pm\infty$. See formulae in Eq.* (17.26) *and Eq.* (17.27) *below. There can be only a finite number of multiple eigenvalues of a finite multiplicity each.*

3. *The following asymptotic representation is valid for the β-branch of the spectrum as $|n| \longrightarrow \infty$:*

$$\nu_n^\beta = (sgn\ n)(\pi^2/L^2)\sqrt{E\tilde{I}/\Delta}\ (|n| - 1/4)^2 + \kappa_n(\omega), \quad \omega = |\delta|^{-1} + |\beta|^{-1}, \tag{17.26}$$

with Δ *being defined in Eq.* (17.14). *A complex-valued sequence* $\{\kappa_n\}$ *is bounded above in the following sense:* $\sup_{n\in\mathbb{Z}}\{|\kappa_n(\omega)|\} = C(\omega)$, $C(\omega) \longrightarrow 0$ *as* $\omega \longrightarrow 0$.

4. *The following asymptotic representation is valid for the* δ*-branch of the spectrum:*

$$\nu_n^\delta = \frac{\pi n}{L\sqrt{\tilde{I}/G}} + \frac{i}{2L\sqrt{\tilde{I}/G}} \ln \frac{\delta + \sqrt{G\tilde{I}}}{\delta - \sqrt{G\tilde{I}}} + O(|n|^{-1/2}), \quad |n| \longrightarrow \infty. \tag{17.27}$$

In representation (17.27)*, ln means the principal value of the logarithm. If* β *and* δ *stay away from zero, that is,* $|\beta| \geq \beta_0 > 0$ *and* $|\delta| \geq \delta_0 > 0$*, then the estimate* $O(|n|^{-1/2})$ *in Eq.* (17.27) *is uniform with respect to both parameters.*

DEFINITION 17.2 *Two sequences of vectors* $\{\phi_n\}$ *and* $\{\chi_n\}$ *in a Hilbert space* H *are said to be* biorthogonal *if for every m and n, we have* $(\phi_m, \chi_n)_H = \delta_{mn}$.

For a given sequence $\{\phi_n\}$, a biorthogonal sequence $\{\chi_n\}$ exists if and only if $\{\phi_n\}$ is minimal (i.e., each element of the sequence does not belong to the closed linear span of the others). If the biorthogonal sequence $\{\chi_n\}$ exists and is unique, then the sequence $\{\phi_n\}$ is minimal and complete. The set which is biorthogonal to the root vectors of the main differential operator $\mathcal{L}_{\beta\delta}$ is formed by the root vectors of the adjoint operator $\mathcal{L}^*_{\beta\delta}$.

DEFINITION 17.3 *A basis in a Hilbert space is a* Riesz basis *if it is linearly isomorphic to an orthonormal basis, that is, if it is obtained from an orthonormal basis by means of a bounded and boundedly invertible operator [4].*

THEOREM 17.3

The set of root vectors of the operator $\mathcal{L}_{\beta\delta}$ *forms a Riesz basis in the energy space* $\mathcal{H}$.

The Riesz basis property of the root vectors has been proven in [9] based on the *Nagy-Foias [15] functional model* for non–Self-adjoint operators. The next statement is one of the main results obtained in References 11 and 12.

THEOREM 17.4

1. *The set of the aeroelastic modes (which are the poles of the generalized resolvent operator) is countable and does not have accumulation points on the complex plane* $\mathbb{C}$. *There might be only a finite number of multiple poles of a finite multiplicity each. There exists a sufficiently large* $R > 0$ *such that all aeroelastic modes, whose distance from the origin is greater than R, are simple poles of the generalized resolvent. The value of R depends on the speed u of an airstream, that is,* $R = R(u)$.

2. *The set of the aeroelastic modes splits asymptotically into two series, which we call the* β*-branch and the* δ*-branch. Asymptotical distribution of the* β*- and the* δ*-branches of the aeroelastic modes can be obtained from asymptotical distribution of the spectrum of the operator* $\mathcal{L}_{\beta\delta}$. *Namely, if* $\{\lambda_n^\beta\}_{n\in\mathbb{Z}}$ *is the* β*-branch of the aeroelastic modes, then* $\lambda_n^\beta = i\hat{\lambda}_n^\beta$ *and the asymptotics of the set* $\{\hat{\lambda}_n^\beta\}_{n\in\mathbb{Z}}$ *is given by the right-hand side of the formula in representation* (17.26). *Similarly, if* $\{\lambda_n^\delta = i\hat{\lambda}_n^\delta\}_{n\in\mathbb{Z}}$ *is the* δ*-branch of the aeroelastic modes, then the asymptotical distribution of the set* $\{\hat{\lambda}_n^\delta\}_{n\in\mathbb{Z}}$ *is given by the right-hand side of the formula in representation* (17.27).

Structure and properties of the matrix integral operator Asymptotical information (Theorem 17.4) is not sufficient to derive such "geometric" properties of the mode shapes as *minimality*, *completeness,* and *the Riesz basis property*. To address the aforementioned properties, we have to analyze the integral part of the problem.

LEMMA 17.1

Let $\hat{\mathfrak{F}}$ *be the Laplace transform of the kernel of the matrix integral operator of Eq.* (17.18). *The following formula is valid for* $\hat{\mathfrak{F}}$:

$$\hat{\mathfrak{F}}(\lambda) = -\frac{2\pi\rho}{\Delta}\frac{u}{\lambda}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}(\lambda) & u\mathcal{L} & (1/2-a)\mathcal{L}(\lambda) \\ 0 & 0 & 0 & 0 \\ 0 & \mathcal{N}(\lambda) & u\mathcal{N}(\lambda) & (1/2-a)\mathcal{N}(\lambda) \end{bmatrix}, \tag{17.28}$$

where

$$\mathcal{L}(\lambda) = \left\{-\frac{\tilde{S}}{2} + [\tilde{I} + (a+1/2)\tilde{S}]T(\lambda/u)\right\}, \quad \mathcal{N}(\lambda) = \left\{\frac{\tilde{m}}{2} - [\tilde{S} + (a+1/2)\tilde{m}]T(\lambda/u)\right\}, \tag{17.29}$$

and T *is the Theodorsen function defined by the formula* $T(z) = K_1(z)[K_0(z) + K_1(z)]^{-1}$, K_0 *and* K_1 *are the modified Bessel functions [3].*

The following asymptotic representations hold for the Theodorsen function:

$$T(z) = 1 + O(z\ln z), \quad z \to 0; \qquad T(z) = \frac{1}{2} + \frac{1}{16z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \tag{17.30}$$

which are valid on the complex plane with the branchcut along the negative real semi axis (the branchcut is necessary to make the functions single-valued). Thus, the Theodorsen function is a bounded analytic function on the complex plane with the aforementioned branchcut. Using representations (17.30), we give a new form of the generalized resolvent. Let V be defined by the formula

$$V(z) = T(z) - 1/2; \qquad V(z) = O\left(\frac{1}{1+|z|}\right), \quad \text{as} \quad |z| \to \infty. \tag{17.31}$$

Taking into account that $z = \lambda/u$, we can write $\lambda\hat{\mathfrak{F}}(\lambda)$ as the following sum:

$$\lambda\hat{\mathfrak{F}}(\lambda) = \mathfrak{M} + \mathfrak{N}(\lambda), \tag{17.32}$$

where the matrices $\mathfrak{M}$ and $\mathfrak{N}(\lambda)$ are defined by the formulae

$$\mathfrak{M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A & uA & (1/2-a)A \\ 0 & 0 & 0 & 0 \\ 0 & B & uB & (1/2-a)B \end{bmatrix}, \quad \mathfrak{N}(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1(\lambda) & uA_1(\lambda) & (1/2-a)A_1(\lambda) \\ 0 & 0 & 0 & 0 \\ 0 & B_1(\lambda) & uB_1(\lambda) & (1/2-a)B_1(\lambda) \end{bmatrix}, \tag{17.33}$$

with A, B, $A_1(\lambda)$, and $B_1(\lambda)$ being given by

$$\begin{aligned} A &= -\pi\rho u\Delta^{-1}[\tilde{I} + (a-1/2)\tilde{S}], \quad B = \pi\rho u\Delta^{-1}[\tilde{S} + (a-1/2)\tilde{m}]. \\ A_1(\lambda) &= -2\pi\rho u\Delta^{-1}V(z)[\tilde{I} + (a+1/2)\tilde{S}] \equiv -2\pi\rho u\Delta^{-1}V(z)\,d_1, \\ B_1(\lambda) &= 2\pi\rho u\Delta^{-1}V(z)[\tilde{S} + (a+1/2)\tilde{m}] \equiv 2\pi\rho u\Delta^{-1}V(z)\,d_2, \quad z = \lambda/u. \end{aligned} \tag{17.34}$$

Therefore, the generalized resolvent of Eq. (17.23) can be written in the form

$$\mathcal{R}(\lambda) = \mathfrak{S}^{-1}(\lambda), \quad \text{where} \quad \mathfrak{S}(\lambda) = \lambda I - i\mathcal{L}_{\beta\delta} - \mathfrak{M} - \mathfrak{N}(\lambda). \tag{17.35}$$

THEOREM 17.5

1. *$\mathfrak{M}$ is a bounded linear operator in $\mathcal{H}$. The operator $\mathcal{K}_{\beta\delta}$ defined by the formula*

$$\mathcal{K}_{\beta\delta} = \mathcal{L}_{\beta\delta} - i\,\mathfrak{M} \tag{17.36}$$

is an unbounded non–Self-adjoint operator in $\mathcal{H}$ with compact resolvent. This operator has a purely discrete spectrum. The spectral asymptotics of the operator $\mathcal{K}_{\beta\delta}$ coincide with the spectral asymptotics of the operator $\mathcal{L}_{\beta\delta}$, and, therefore, those asymptotics are presented in Theorem 17.2. In contrast to the operator $\mathcal{L}_{\beta\delta}$, the operator $\mathcal{K}_{\beta\delta}$ is not dissipative for any boundary control gains. ***However, $\mathcal{K}_{\beta\delta}$ is also a Riesz spectral operator, that is, the set of the root vectors of $\mathcal{K}_{\beta\delta}$ forms a Riesz basis of $\mathcal{H}$.***

2. *$\mathfrak{N}(\lambda)$ is an analytic matrix-valued function on the complex plane with the branchcut along the negative real semiaxis. For each λ, $\mathfrak{N}(\lambda)$ is a bounded operator in $\mathcal{H}$ with the following estimate for its norm:*

$$\|\mathfrak{N}\|_{\mathcal{H}} \le C(1 + |\lambda|)^{-1}, \tag{17.37}$$

where C is an absolute constant, the precise value of which is immaterial for us.

The main result, which is crucial for the "calculation" of the inverse Laplace transform and writing the solution of the initial-boundary value problem in the space-time domain representation, is given in the next statement [14].

THEOREM 17.6
The set of mode shapes forms a Riesz basis in the energy space.

17.3 Analytic Fredholm Operator-Valued Functions in Wing Model

The purpose of the next three sections is to outline the main steps in the proof of Theorem 17.6 (full proof is contained in Reference 14). First, we recall the definitions and general properties of Fredholm operator–valued functions (see, e.g., Reference 5). Let $\mathfrak{L}(H)$ be the set of all bounded operators in a Hilbert space H. We say that $\mathfrak{A} : \Omega \to \mathfrak{L}(H)$ is an operator-valued function defined for $\lambda \in \Omega \subset \mathbb{C}$ if for each λ, $\mathfrak{A}(\lambda)$ is a closed linear operator in H. The resolvent set of $\mathfrak{A}$ is the set of $\lambda \in \Omega$ such that $\mathfrak{A}(\lambda)$ has a bounded inverse operator. The spectrum $\sum(\mathfrak{A})$ of $\mathfrak{A}$ is the complement of the resolvent set in Ω. If $\sum_\lambda[\mathfrak{A}(\lambda)]$ is the spectrum of the fixed operator $\mathfrak{A}(\lambda)$, then $\lambda \in \sum(\mathfrak{A})$ if and only if $0 \in \sum_\lambda[\mathfrak{A}(\lambda)]$. The point spectrum $\sum_p(\mathfrak{A})$, is the set of $\lambda \in \Omega$ such that $Ker\,\mathfrak{A}(\lambda) \neq \{0\}$. Such points are called *eigenvalues*, nontrivial vectors in the kernel are called the corresponding *eigenvectors*, and the dimension of the kernel is called the *geometric multiplicity*. If the domain of $\mathfrak{A}(\lambda)$ is independent of λ, and if for each f in this domain the function $\lambda \mapsto \mathfrak{A}(\lambda)f$ is analytic in the usual strong sense, then we can define the *chains of* and *canonical systems of eigen- and associated vectors* as well as a concept of *algebraic multiplicity* of an eigenvalue λ_0.

DEFINITION 17.4 *A bounded linear operator* **A** *in a Hilbert space H is called a Fredholm operator if its range* Im **A** *is closed and the numbers*

$$n(\mathbf{A}) = dim\ Ker\{\mathbf{A}\}, \quad d(\mathbf{A}) = dim\ Ker\ \{\mathbf{A}^*\} \tag{17.38}$$

are finite. The number ind $\mathbf{A} \equiv n(\mathbf{A}) - d(\mathbf{A})$ *is called the* index *of* **A**.

DEFINITION 17.5 *Let* $\mathfrak{A} : \Omega \mapsto \mathfrak{L}(H)$ *be an analytic operator-valued function.* $\lambda_0 \in \Omega$ *is said to be an* eigenvalue of a finite type *of* $\mathfrak{A}(\cdot)$ *if (a)* $\mathfrak{A}(\lambda_0)$ *is a Fredholm operator, (b)* $\mathfrak{A}(\lambda_0)f = 0$ *for some nonzero* $f \in H$*, and (c)* $\mathfrak{A}(\lambda)$ *is an invertible operator for all* λ *in some punctured disk* $0 < |\lambda - \lambda_0| < \epsilon$ *around* λ_0*. In that case* ind $\mathfrak{A}(\lambda_0) = 0$.

THEOREM 17.7 (See Reference 5)
Assume that for some $\lambda_0 \in \Omega$*, an analytic function* $\mathfrak{A}(\cdot)$ *is a Fredholm operator with* $ind\,\mathfrak{A}(\lambda_0) = 0$*. Then there exists* $\epsilon > 0$ *such that for all* $\lambda : 0 < |\lambda - \lambda_0| < \epsilon$*, the operator-valued function* $\mathfrak{A}(\cdot)$ *can be factored as* $\mathfrak{A}(\lambda) = E(\lambda)D(\lambda)F(\lambda)$*, where* $E(\cdot)$ *and* $F(\cdot)$ *are Fredholm functions having bounded inverses at each point of the disk* $|\lambda - \lambda_0| \le \epsilon$*, and* $D(\cdot)$ *has the form* $D(\lambda) = P_0 + (\lambda - \lambda_0)^{k_1} P_1 + (\lambda - \lambda_0)^{k_2} P_2 + \cdots + (\lambda - \lambda_0)^{k_r} P_r$*, where* $P_0, P_1, \ldots P_r$ *are mutually disjoint projections of rank one; the projection* $(I - P_0)$ *has a finite rank and* $k_1 \le k_2 \le \cdots \le k_r$ *are positive integers. The sum*

$$m[\lambda_0; \mathfrak{A}(\cdot)] = k_1 + k_2 + \cdots + k_r \tag{17.39}$$

is called the algebraic multiplicity *of* $\mathfrak{A}$ *at* λ_0*. (*$\mathfrak{A}(\cdot)$ *has only one normalized eigenvector* φ_0 *corresponding to the eigenvalue* λ_0 *if and only if the projection* $(I - P_0)$ *is one-dimensional.)*

The next result is of a special importance for us [5].

THEOREM 17.8
Let $\mathfrak{A} : \Omega \mapsto \mathfrak{L}(H)$ *be an analytic Fredholm operator-valued function. If for some* $\lambda \in \Omega$*, the operator* $\mathfrak{A}(\lambda)$ *has bounded inverse, then the spectrum* $\sum(\mathfrak{A})$ *is discrete, that is, it contains at most countably many eigenvalues with no accumulation points in* Ω*; each eigenvalue has a finite algebraic multiplicity. Furthermore,* $\lambda \mapsto [\mathfrak{A}(\lambda)]^{-1}$ *is a meromorphic function on* Ω *whose poles are the eigenvalues of* $\mathfrak{A}$*. The principal part of the Laurent expansion around each pole has only finitely many nonzero terms, the coefficients in these terms being of finite rank operators.*

To formulate the generalization of the Rouche's theorem to the analytic operator-valued functions, we need the following definitions.

DEFINITION 17.6

1. A Cauchy domain *is a disjoint union in* $\mathbb{C}$ *of a finite number of non-empty open connected sets* $\Delta_1, \Delta_2, \ldots, \Delta_r$*, such that* $\bar{\Delta}_i \cap \bar{\Delta}_j = \{0\}$*,* $i \ne j$*, and for each* j*, the boundary of the set* Δ_j *consists of a finite number of nonintersecting closed rectifiable Jordan curves, which are oriented in such a way that* Δ_j *belongs to the inner domains of the curves.*
2. Γ *is called a* Cauchy contour *if* Γ *is the oriented boundary of a bounded Cauchy domain.*
3. *Let* Δ *be a Cauchy domain with* $\bar{\Delta} \subset \Omega$ *and let* $\mathfrak{A}$ *be a bounded operator-valued analytic function of* λ*,* $\lambda \in \Omega$*. Then* $\mathfrak{A}$ *is said to be* normal with respect *to* Γ*,* $\Gamma := \partial\Delta$*, if* $\mathfrak{A}(\lambda)$ *has a bounded inverse for* $\lambda \in \Gamma$ *and* $\mathfrak{A}(\lambda)$ *is a Fredholm operator for* $\lambda \in \Delta$.

The following generalization of the Rouche's theorem holds [5].

THEOREM 17.9

1. *Let* $W(\cdot)$ *and* $Z(\cdot) : \Omega \mapsto \mathfrak{L}(H)$ *be analytic operator-valued functions, and let* W *be normal with respect to a contour* Γ*. If*

$$\|Z(\lambda)[W(\lambda)]^{-1}\| < 1, \quad \lambda \in \Gamma, \tag{17.40}$$

then the operator-valued function

$$U(\lambda) = Z(\lambda) + W(\lambda) \tag{17.41}$$

is also normal with respect to Γ *and*

$$m[\Gamma; U(\cdot)] = m[\Gamma; W(\cdot)], \tag{17.42}$$

where $m[\Gamma; U(\cdot)]$ *and* $m[\Gamma; W(\cdot)]$ *are algebraic multiplicities of* U *and* W *relative to* Γ*, that is,*

$$\begin{aligned} m[\Gamma; U(\cdot)] &= m[\lambda_1; U(\cdot)] + m[\lambda_2; U(\cdot)] + \cdots + m[\lambda_p; U(\cdot)], \\ m[\Gamma; W(\cdot)] &= m[\hat{\lambda}_1; W(\cdot)] + m[\hat{\lambda}_2; W(\cdot)] + \cdots + m[\hat{\lambda}_q; W(\cdot)], \end{aligned} \tag{17.43}$$

where $\{\lambda_j\}_{j=1}^{p}$ *and* $\{\hat{\lambda}_j\}_{j=1}^{q}$ *are the eigenvalues of finite types of* U *and* W *inside* Γ.

2. *If "tr" denotes the trace of the operator, and the multiplicities of the eigenvalues are taken into account, then the following formula for total multiplicity holds:*

$$m[\Gamma; \mathfrak{A}(\cdot)] = tr\left\{\frac{1}{2\pi i}\oint \mathfrak{A}'(\lambda)[\mathfrak{A}(\lambda)]^{-1} d\lambda\right\}. \tag{17.44}$$

DEFINITION 17.7 *An operator-valued function* G *is* finitely meromorphic *at* λ_0 *if* G *has a pole at* λ_0 *and the coefficients of the principal part of its Laurent expansion at* λ_0 *are finite-rank operators, that is, in a punctured neighborhood of* λ_0*, the following expansion is valid:* $G(\lambda) = \sum_{\nu=-q}^{\infty}(\lambda - \lambda_0)^{\nu} G_{\nu}$*, which converges in the operator norm on* $\mathfrak{L}(H)$. $G_{-1}, \ldots, G_{-q}$ *are finite-rank operators;* $\Xi G(\lambda) \equiv \sum_{\nu=-q}^{-1}(\lambda - \lambda_0)^{\nu} G_{\nu}, \lambda \neq \lambda_0$*, is the principal part of* G *at* λ_0*. Thus,* ΞG *is analytic on* $\mathbb{C} \setminus \{\lambda_0\}$*, and its values are finite-rank operators.*

In what follows, to be able to use the theory of analytic Fredholm operator-valued functions, we have to replace the operator-valued functions $\mathfrak{S}(\lambda)$ and $(\lambda I - i\mathcal{K}_{\beta\delta})$, which are unbounded, with some Fredholm operator-valued functions, which have to be bounded. Let $\mathbf{K}_{\beta\delta} = i\mathcal{K}_{\beta\delta}$ and let W and $Z : \Omega \mapsto \mathfrak{L}(\mathcal{H})$ defined by

$$W(\lambda) = I - \lambda \mathbf{K}_{\beta\delta}^{-1}, \quad Z(\lambda) = I - \lambda \mathbf{K}_{\beta\delta}^{-1} - \mathfrak{N}(\lambda)\mathbf{K}_{\beta\delta}^{-1}. \tag{17.45}$$

In the next statement, we establish the relationship between the spectral characteristics of the operators $\mathbf{K}_{\beta\delta}$ and $\mathfrak{S}(\lambda)$ and the operator-valued function $W(\lambda)$ and $Z(\lambda)$ (see Reference 14).

THEOREM 17.10

1. λ_0 *is an eigenvalue of* $Z(\lambda)$*, and* g *is the corresponding eigenvector if and only if* λ_0 *is an aeroelastic mode and* $\varphi = \mathbf{K}_{\beta\delta}^{-1} g$ *is the corresponding mode shape.*

2. *The spectrum of the operator-valued function* $W(\lambda)$ *coincides with the spectrum of the operator* $\mathbf{K}_{\beta\delta}$*, and the corresponding root spaces are identical.*

17.4 Minimality and Normalization of Adjoint Mode Shapes

Recall that the eigenvectors of $Z(\lambda)$ are connected to the mode shape through the relation $\tilde{\Phi}_n^{\beta(\delta)} = \mathbf{K}_{\beta\delta}^{-1}\Phi_n^{\beta(\delta)}$, where $\tilde{\Phi}_n^{\beta(\delta)}$ is the eigenvector of $Z(\lambda)$ and $\Phi_n^{\beta(\delta)}$ is the mode shape. Now let us take, for example, the δ-branch simple aeroelastic mode. The adjoint operator-valued function $[Z(\lambda)]^*$ has its eigenvalue at the point $\bar{\lambda}_n^\delta$, (i.e., $Z^*(\bar{\lambda}_n^\delta)\tilde{\Psi}_n^\delta = 0$). The operator-valued function $[Z(\lambda)]^{-1}$ admit the following representation when $\lambda \in U(\lambda_n^\delta)$:

$$[Z(\lambda)]^{-1} = \frac{1}{\lambda - \lambda_n^\delta}\left(\cdot, \tilde{\Psi}_n^\delta\right)\tilde{\Phi}_n^\delta + H(\lambda). \tag{17.46}$$

Recall that $U(\lambda_n^\delta)$ is an open neighborhood of the point λ_n^δ such that for all $\lambda \in U(\lambda_n^\delta)$, $\lambda \neq \lambda_n^\delta$, $Z(\lambda)^{-1}$ exists. In Eq. (17.46), $H(\lambda)$ is a holomorphic operator-valued function in $U(\lambda_n^\delta)$. Combining the representation in Eq. (17.46) with Theorem 17.9, we obtain the following result.

THEOREM 17.11 (Normalization condition [14])
Let

$$\left\{\hat{\lambda}_n^\delta\right\}_{n=1}^{n_0} \cup \left\{\hat{\lambda}_m^\beta\right\}_{m=1}^{m_0} \cup \left\{\lambda_n^\delta\right\}_{n\in\mathbb{Z}} \cup \left\{\lambda_n^\beta\right\}_{n\in\mathbb{Z}} \tag{17.47}$$

be the set of aeroelastic modes, where $\{\hat{\lambda}_n^\delta\}_{n=1}^{n_0} \cup \{\hat{\lambda}_m^\beta\}_{m=1}^{m_0}$ *are the aeroelastic modes corresponding to multiple poles of the generalized resolvent and* $\{\lambda_n^\delta\}_{n\in\mathbb{Z}} \cup \{\lambda_m^\beta\}_{m\in\mathbb{Z}}$ *are the aeroelastic modes corresponding to the simple poles of the generalized resolvent. Let* $\{\tilde{\Phi}_n^\delta\}_{n\in\mathbb{Z}} \cup \{\tilde{\Phi}_m^\beta\}_{m\in\mathbb{Z}}$ *be the set of eigenvectors of the function* $Z(\lambda)$ *corresponding to the set* $\{\lambda_n^\delta\}_{n\in\mathbb{Z}} \cup \{\lambda_m^\beta\}_{m\in\mathbb{Z}}$*, and let* $\{\tilde{\Psi}_n^\delta\}_{n\in\mathbb{Z}} \cup \{\tilde{\Psi}_m^\beta\}_{m\in\mathbb{Z}}$ *be the corresponding set of the eigenvectors of the adjoint operator-valued function* $[Z(\lambda)]^*$*. Then the following normalization conditions hold:*

$$\left(Z'\left(\lambda_n^\delta\right)\tilde{\Phi}_n^\delta, \tilde{\Psi}_n^\delta\right) = 1, \quad \left(Z'\left(\lambda_m^\beta\right)\tilde{\Phi}_m^\beta, \tilde{\Psi}_m^\beta\right) = 1, \quad n \in \mathbb{Z}, \quad m \in \mathbb{Z}, \tag{17.48}$$

where "prime" denotes the derivative with respect to λ.

THEOREM 17.12 (Minimality [12])
The set of aeroelastic mode shapes is minimal in $\mathcal{H}$*.*

To prove this result, we use the notion of the *Riesz integral* [4, 5]. Let Γ be a simple rectifiable contour, which encloses some region $G_\Gamma \subset \mathbb{C}$ and lies entirely within a resolvent set of $Z(\lambda)$. Thus, the inverse $Z(\lambda)^{-1}$ is also an analytic operator-valued function on Γ. Assuming a positive orientation of Γ with respect to the region G_Γ, let us form an integral

$$\mathfrak{P}_\Gamma = -\frac{1}{2\pi i}\oint_\Gamma [Z(\lambda)]^{-1} Z'(\lambda)\, d\lambda. \tag{17.49}$$

$\mathfrak{P}_\Gamma$ is the projection ($\mathfrak{P}_\Gamma^2 = \mathfrak{P}_\Gamma$). Let λ_1 and λ_2 be two different modes and Γ_1 and Γ_2 be two boundaries for the domains G_1 and G_2 containing λ_1 and λ_2, respectively. The following orthogonality relation holds: $\mathfrak{P}_{\Gamma_1}\mathfrak{P}_{\Gamma_2} = \mathfrak{P}_{\Gamma_2}\mathfrak{P}_{\Gamma_1} = 0$.

Now we briefly outline the proof. If λ is not an aeroelastic mode, then $\mathcal{R}(\lambda)$ is a compact operator in $\mathcal{H}$. Each aeroelastic mode can be surrounded by a small circle that does not contain other modes. Let us assume that the entire set of aeroelastic modes is *not minimal*, that is, there exists a sequence $\{c_n\}_{n\in\mathbb{Z}}$ of complex numbers such that $\Sigma_{n\in\mathbb{Z}} c_n \Phi_n = 0$. Clearly, there exists $c_m \neq 0$, such that $\Phi_m = \sum_{n\in\mathbb{Z}, n\neq m} c_n/c_m \Phi_n$. Let us apply the Riesz projection $\mathfrak{P}_{\Gamma_m}$ to both parts of the latter equation. We have

$$\mathfrak{P}_{\Gamma_m}\Phi_m = \sum_{\substack{n\in\mathbb{Z}\\ n\neq m}} c_n/c_m \mathfrak{P}_{\Gamma_m}\Phi_n = \sum_{\substack{n\in\mathbb{Z}\\ n\neq m}} c_n/c_m \mathfrak{P}_{\Gamma_m}\mathfrak{P}_{\Gamma_n}\Phi_n = 0, \tag{17.50}$$

which yields the proof.

17.5 Riesz Basis Property of Mode Shapes

First we describe the properties of the β-branch mode shapes.

THEOREM 17.13 (Riesz basis property of β-branch mode shapes [14])
The β-branch of the mode shapes $\{\Phi_n^\beta\}_{n\in\mathbb{Z}}$ forms a Riesz basis in its closed linear span in $\mathcal{H}$.

The proof of this theorem uses the facts that (a) the β-branch root vectors form a *minimal set* in $\mathcal{H}$ and (b) that the β-branch mode shapes $\{\Phi_n^\beta\}_{n\in\mathbb{Z}}$ are quadratically close to the β-branch of the eigenfunctions $\{\varphi_n^\beta\}_{n\in\mathbb{Z}}$ of the operator $\mathcal{K}_{\beta\delta}$, *that is,* $\Sigma_{n\in\mathbb{Z}}\|\Phi_n^\beta - \varphi_n^\beta\|^2 < \infty$.

Refined asymptotic estimate for aeroelastic modes From Statement (2) of Theorem 17.4 and the formula of Eq. (17.27), we can conclude that $|\mu_n^\delta - \lambda_n^\delta| = O(|n|^{-1/2})$ as $|n| \to \infty$. The latter result is probably the best that can be obtained based on the techniques of the asymptotic analysis. However, it is not enough for our goal for proving the Riesz basis of the mode shapes. Using the techniques of operator theory and of complex analysis, we can refine those formulae.

THEOREM 17.14
For the δ-branch aeroelastic modes, the following asymptotic estimate is valid when $|n| \to \infty$:

$$\left|\mu_n^\delta - \lambda_n^\delta\right| \le C|n|^{-1}, \tag{17.51}$$

with $\{\mu_n^\delta\}$ being the δ-branch of the eigenvalues of the operator $\mathbf{K}_{\beta\delta}$. The precise value of the constant C in Eq. (17.51) *is immaterial for us.*

Comments to Proof As follows from the asymptotic formulae for the spectrum of the operator $\mathbf{K}_{\beta\delta}$, there could be the following cases: either $\inf_{n,\ m\in\mathbb{Z}}|\mu_n^\delta - \mu_m^\beta| > 0$ or $\inf_{n,\ m\in\mathbb{Z}}|\mu_n^\delta - \mu_m^\beta| = 0$. In the first case, we would say that the δ- and β-branches are separated; in the second case, the branches are not separated. Clearly, in the second case, there exist two subsequences of the β- and the δ-branches of the operator $\mathbf{K}_{\beta\delta}$ such that $\text{dist}\{\mu_{n_k}^\delta, \mu_{m_k}^\beta\} \to 0$ as $k \to \infty$. In our proof [14], we deal with the second case, which is technically much more difficult than the first one, and at the end we briefly address the proof for the first case.

Quadratic closeness of mode shapes and eigenfunctions of operator $\mathbf{K}_{\beta\delta}$ We now formulate the statement that is crucial for the proof of the main result—Theorem 17.6. Namely, we show that the set of the mode shapes is quadratically close to the set of the eigenfunctions of the operator $\mathbf{K}_{\beta\delta}$.

THEOREM 17.15
If $\{\lambda_n\}_{n=1}^N$ is the set of multiple modes, then the entire set of the aeroelastic modes can be represented in the form

$$\Lambda = \hat{\Lambda} \cup \Lambda^\delta \cup \Lambda^\beta; \qquad \hat{\Lambda} = \{\lambda_n\}_{n=1}^N, \quad \Lambda^\delta = \left\{\lambda_n^\delta\right\}_{n\in\mathbb{Z}}, \quad \Lambda^\beta = \left\{\lambda_n^\beta\right\}_{n\in\mathbb{Z}}. \tag{17.52}$$

The in the sets, Λ^β and Λ^δ correspond to the simple roots of the generalized resolvent operator. Let the set of the eigenvalues of the operator $\mathbf{K}_{\beta\delta}$ be represented in the following way:

$$\mathcal{U} = \hat{\mathcal{U}} \cup \mathcal{U}^\delta \cup \mathcal{U}^\beta; \qquad \hat{\mathcal{U}} = \{\mu_n\}_{n=1}^N, \quad \mathcal{U}^\delta = \left\{\mu_n^\delta\right\}_{n\in\mathbb{Z}}, \quad \mathcal{U}^\beta = \left\{\mu_n^\beta\right\}_{n\in\mathbb{Z}}. \tag{17.53}$$

The numeration in Eq. (17.53) *has been done in such a way that*

$$\left|\lambda_n^\delta - \mu_n^\delta\right| \le C(1+|n|)^{-1}, \quad \left|\lambda_n^\beta - \mu_n^\beta\right| \le C(1+|n|)^{-1}. \tag{17.54}$$

Then the sets of the corresponding mode shapes and eigenvectors of the operator $\mathbf{K}_{\beta\delta}$ are quadratically close, that is,

$$\sum_{n\in\mathbb{Z}} \left\|\Phi_n^\delta - \varphi_n^\delta\right\|^2 < \infty, \quad \sum_{n\in\mathbb{Z}} \left\|\Phi_n^\beta - \varphi_n^\beta\right\|^2 < \infty. \tag{17.55}$$

REMARK 17.2 We notice that though the set $\hat{\Lambda}$ contains the aeroelastic modes corresponding to the multiple roots of the generalized resolvent operator, the same may not be true for the set $\hat{\mathcal{U}}$. Namely, the set $\hat{\mathcal{U}}$ may contain both multiple and simple eigenvalues of $\mathbf{K}_{\beta\delta}$. It may also happen that several multiple eigenvalues should be added to the δ- or β-branch. In the latter case, a multiple eigenvalue should be repeated as many times as its multiplicity. The splitting of Eq. (17.53) has to be done in such a way that relations of Eq. (17.55) hold.

Riesz basis property of the mode shapes follows from the facts that the estimate of Eq. (5.5) holds, and the set of the generalized mode shapes is minimal in the space $\mathcal{H}$. Those two facts yield the Riesz basis property through the application of the Bari's theorem.

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Chapter 18

Optimal Design of Mechanical Structures[1]

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Abstract We prove new properties for the linear isotropic elasticity system and for thickness minimization problems. We also present very recent results concerning shape optimization problems for three-dimensional curved rods and for shells. The questions discussed in this paper are related to the control variational method and to control into coefficient problems.

18.1 Introduction

The analysis and the computation of various optimal mechanical structures has a long history and many applications. We quote the recent books by Bendsoe [3], Cherkaev [6], Allaire [1], Zolesio and Delfour [19], where such topics are studied from various points of view and where numerous references may be found.

In this chapter, we shall consider structures like plates, curved rods, and shells under low regularity assumptions with respect to their geometry. In the first section we analyze the application of the control variational method, introduced by the authors [11, 15, 16, 18] to the general linear elasticity system and to linear elastic plates. Variational inequalities are also considered. It turns out that the approach is advantageous from the numerical point of view because the solution is reduced to sequential applications of Laplace's equation. In Section 18.2, we discuss thickness minimization problems for plates. The last section contains a presentation of very recent results in shape optimization problems for curved rods and shells, obtained by the authors.

18.2 The Linear Elasticity System

We consider in $\Omega \subset I\!R^3$ the weak formulation of the isotropic linear elasticity system,

$$\int_\Omega [\lambda e_{pp}(u) e_{qq}(v) + 2\mu e_{ij}(u) e_{ij}(v)]\, dx = \int_\Omega f_i v_i \, dx, \tag{18.1}$$
$$u = (u_1, u_2, u_3) \in V(\Omega), \quad \forall v = (v_1, v_2, v_3) \in V(\Omega) = \{v \in H^1(\Omega)^3, \quad v|_{\Gamma_0} = 0\}.$$

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Above, it is assumed that the smooth boundary of Ω, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, consists of two nonoverlapping open parts and Eq. (18.1) corresponds to homogeneous mixed boundary conditions, imposed for simplicity. The constants $\lambda \geq 0$, $\mu > 0$ are the Lamé coefficients, $e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $i, j = \overline{1, 3}$, the summation convention is used, and $f = (f_1, f_2, f_3)$ gives the body forces. The existence of a unique solution $u = (u_1, u_2, u_3) \in V(\Omega)$ for Eq. (18.1) is well known see Ciarlet [7, 8]. We prove here that Eq. (18.1) admits an advantageous treatment via control theory. To this end, we consider the following problem:

$$\text{Min}\left\{\frac{1}{2}\int_\Omega \left\{\mu|w|^2_{R^9} + \lambda[\text{div}\,(u)]^2 + \mu\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right] + 2\mu\left(\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right)\right\} dx\right\}, \tag{18.2}$$

subject to $w \in L^2(\Omega)^9$ and to

$$\int_\Omega \nabla u : \nabla v\, dx = \int_\Omega w : \nabla v\, dx + \frac{1}{\mu}\int_\Omega f \cdot v\, dx, \quad \forall v \in V(\Omega), \tag{18.3}$$

where ∇u is the Jacobian of u and

$$\nabla u : \nabla v = \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}.$$

The relation in Eq. (18.3) is just the weak formulation of the system of the three decoupled Poisson equations

$$-\Delta u = -\text{div}\, w + \frac{1}{\mu} f, \quad \text{in } \Omega, \tag{18.4}$$

with homogeneous mixed boundary conditions. The divergence operator in Eq. (18.4) is applied to the rows of the 3×3 "matrix" $w \in L^2(\Omega)^9$.

We study briefly the problem of Eqs. (18.2) and (18.3), and we show that it provides exactly the solution of Eq. (18.1). The two problems are in fact equivalent.

PROPOSITION 18.1

Assume that $[u^*, w^*] \in V(\Omega) \times L^2(\Omega)^9$ *is an optimal pair for the problem in Eqs.* (18.2) *and* (18.3). *Then it holds*

$$\int_\Omega \left[\mu w^* : q + \lambda \text{div}\,(u^*)\text{div}\,(z) + \mu\left(\frac{\partial u_1^*}{\partial x_1}\frac{\partial z_1}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2}\frac{\partial z_2}{\partial x_2} + \frac{\partial u_3^*}{\partial x_3}\frac{\partial z_3}{\partial x_3} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial z_2}{\partial x_1} + \frac{\partial z_1}{\partial x_2}\frac{\partial u_2^*}{\partial x_1} + \frac{\partial u_1^*}{\partial x_3}\frac{\partial z_3}{\partial x_1} + \frac{\partial u_3^*}{\partial x_1}\frac{\partial z_1}{\partial x_3} + \frac{\partial u_2^*}{\partial x_3}\frac{\partial z_3}{\partial x_2} + \frac{\partial u_3^*}{\partial x_2}\frac{\partial z_2}{\partial x_3}\right)\right] dx = 0, \tag{18.5}$$

for any $z \in V(\Omega)$ *and for* $q \in L^2(\Omega)^9$ *with* $q = \nabla z$.

PROOF This is the usual Euler equation associated with Eqs. (18.2) and (18.3). As the control problem is unconstrained, we can take arbitrary variations of the form $u^* + sz$, $s \in \mathbb{R}$, around u^*, which correspond to variations $w^* + sq$ around the optimal control w^*, because z is the solution of the "equation in variations" corresponding to q:

$$\int_\Omega \nabla z : \nabla v\, dx = \int_\Omega q : \nabla v\, dx, \quad \forall v \in V(\Omega).$$

One then writes that the cost corresponding to w^* is lower than the cost corresponding to $w^* + sq$, then subtracts, divides by s (for $s > 0$ or $s < 0$), and takes the limit $s \to 0$ to obtain the result. □

REMARK The relation in Eq. (18.5) is a characterization of optimality. If w is any control, then $w = q + l$ (unique orthogonal decomposition) where q has the form in Eq. (18.5). Then Eq. (18.3) shows that the state is independent of l, whereas q also provides a lower cost than $w = q + l$. The optimal pair, if it exists, is unique, by the strict convexity of Eq. (18.2).

Next, we define the adjoint system for $p \in V(\Omega)$:

$$\int_\Omega \nabla p : \nabla z = \int_\Omega \Bigg[\lambda \operatorname{div}(u^*) \operatorname{div}(z) + \mu \Bigg(\frac{\partial u_1^*}{\partial x_1}\frac{\partial z_1}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2}\frac{\partial z_2}{\partial x_2} + \frac{\partial u_3^*}{\partial x_3}\frac{\partial z_3}{\partial x_3} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial z_2}{\partial x_1} + \frac{\partial u_2^*}{\partial x_1}\frac{\partial z_1}{\partial x_2} + \frac{\partial u_1^*}{\partial x_3}\frac{\partial z_3}{\partial x_1} + \frac{\partial u_3^*}{\partial x_1}\frac{\partial z_1}{\partial x_3} + \frac{\partial u_2^*}{\partial x_3}\frac{\partial z_3}{\partial x_2} + \frac{\partial u_3^*}{\partial x_2}\frac{\partial z_2}{\partial x_3}\Bigg)\Bigg] dx = 0, \quad \forall z \in V(\Omega). \tag{18.6}$$

Existence and uniqueness of the solution $p \in V(\Omega)$ are obvious.

PROPOSITION 18.2

The optimality conditions for the problem in Eqs. (18.2) *and* (18.3) *are given by Eqs.* (18.3) *and* (18.6) *and the Pontryagin maximum principle:*

$$\int_\Omega (\mu w^* + \nabla p) : \nabla z \, dx = 0, \quad \forall z \in V(\Omega). \tag{18.7}$$

Moreover, $p = \mu h - \mu u^*$ *in* Ω *with* h *defined in Eq.* (18.8) *below.*

PROOF By Eq. (18.6) and Eq. (18.5), we get

$$0 = \int_\Omega [\mu w^* : q + \nabla p : \nabla z]\, dx = \int_\Omega [\mu w^* : q + \nabla p : q]\, dx,$$

which is exactly the relation in Eq. (18.7), as $q = \nabla z$. Notice that, by virtue of Eq. (18.3) and Eq. (18.7), we have

$$\int_\Omega \nabla u^* : \nabla z \, dx = \int_\Omega w^* : \nabla z \, dx + \frac{1}{\mu}\int_\Omega f \cdot z \, dx = -\frac{1}{\mu}\left[\int_\Omega \nabla p : \nabla z \, dx - \int_\Omega f \cdot z \, dx\right].$$

That is, if we denote by $h \in V(\Omega)$ the (weak) solution to the problem:

$$\int_\Omega \nabla h : \nabla z \, dx = \frac{1}{\mu}\int_\Omega f \cdot z \, dx, \quad \forall z \in V(\Omega), \tag{18.8}$$

then we obtain

$$\int_\Omega \nabla u^* : \nabla z \, dx = -\frac{1}{\mu}\int_\Omega \nabla p : \nabla z \, dx + \int_\Omega \nabla h : \nabla z \, dx, \quad \forall z \in V(\Omega).$$

As u^*, p, h satisfy the same boundary conditions, the unique solvability of Laplace's problem concludes the proof. □

Again, by Eq. (18.3), and by the definition of q in Proposition 18.1, we can write

$$\int_\Omega \mu w^* : q\,dx = \mu \int_\Omega w^* : \nabla z\,dx = \mu \int_\Omega \nabla u^* : \nabla z\,dx - \int_\Omega f \cdot z\,dx, \quad \forall z \in V(\Omega).$$

By replacing this in Eq. (18.5), we have

$$\begin{aligned}\int_\Omega \Bigg[&\mu \nabla u^* : \nabla z\,dx + \lambda\,\text{div}\,(u^*)\,\text{div}\,(z) + \mu\bigg(\frac{\partial u_1^*}{\partial x_1}\frac{\partial z_1}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2}\frac{\partial z_2}{\partial x_2}\\ &+ \frac{\partial u_3^*}{\partial x_3}\frac{\partial z_3}{\partial x_3} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial z_2}{\partial x_1} + \frac{\partial u_2^*}{\partial x_1}\frac{\partial z_1}{\partial x_2} + \frac{\partial u_1^*}{\partial x_3}\frac{\partial z_3}{\partial x_1} + \frac{\partial u_3^*}{\partial x_1}\frac{\partial z_1}{\partial x_3}\\ &+ \frac{\partial u_2^*}{\partial x_3}\frac{\partial z_3}{\partial x_2} + \frac{\partial u_3^*}{\partial x_2}\frac{\partial z_2}{\partial x_3}\bigg)\Bigg]\,dx = \int_\Omega f \cdot z\,dx, \quad \forall z \in V(\Omega).\end{aligned} \tag{18.9}$$

Regrouping the terms in Eq. (18.9) conveniently, we have thus proved:

COROLLARY 18.1

$u^* \in V(\Omega)$ *is the unique solution to Eq.* (18.1).

REMARK The relations in Eqs. (18.3), (18.6), and (18.7) provide a nonstandard decomposition of Eq. (18.1).

REMARK Corollary 18.1 provides a simple convenient method to solve Eq. (18.1) via Eqs. (18.2) and (18.3). In the setting of this control problem, we have to solve the state system of Eq. (18.3) and the adjoint system of Eq. (18.6) (both associated with the Laplace operator). Then, the gradient of the cost functional may be computed by Proposition 18.2, and gradient methods may be used for the numerical approximation. Notice also that the existence in Eqs. (18.2) and (18.3) follows from the result for Eq. (18.1), by Proposition 18.1 and Corollary 18.1.

Let us now consider the example of a linear elastic plate ($\Omega \subset I\!R^2$!) submitted to unilateral restrictions:

$$\begin{aligned} a(y, v) = \int_\Omega e^3[&y_{,11}v_{,11} + \tau y_{,11}v_{,22}\\ &+ \tau y_{,22}v_{,11} + y_{,22}v_{,22} + 2(1-\tau)y_{,12}v_{,12}]\,dx,\end{aligned} \tag{18.10}$$

$$\forall y \in H_0^2(\Omega), \quad \forall v \in H_0^2(\Omega).$$

$$a(y, y - v) \le \int_\Omega f(y - v)\,dx, \quad y \in \mathcal{K}, \quad \forall v \in \mathcal{K}, \tag{18.11}$$

where, for $y \in H^2(\Omega)$, $y_{,ij} = \frac{\partial^2 y}{\partial x_i \partial x_j}$, $i, j = 1, 2$.

Here $\mathcal{K} \subset H_0^2(\Omega)$ is a non-empty closed and convex set. The scalar functions $y \in H_0^2(\Omega)$, $e \in L^\infty(\Omega)_+$, $f \in L^2(\Omega)$, represent, respectively, the deflection, the positive thickness, and the load of the plate, whereas $0 < \tau < \frac{1}{2}$ is the Poisson coefficient (see Duvaut and Lions [9, Chap. 4]). We replace Eqs. (18.10) and (18.11) by the following optimal control problem:

$$\text{Min}\left\{\frac{1}{2}\int_\Omega e^3\left[w^2 + 2(1-\tau)y_{,12}^2 + 2(\tau - 1)y_{,11}y_{,22}\right]dx\right\} \tag{18.12}$$

subject to the state equation and constraints

$$\Delta y = w + e^{-3} g \quad \text{in } \Omega, \tag{18.13}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{18.14}$$

$$y \in \mathcal{K}. \tag{18.15}$$

Above, $g \in H^2(\Omega) \cap H_0^1(\Omega)$ is the solution to the Poisson problem with $\Delta g = f$ in Ω. We shall prove that the solution of Eq. (18.11) may be obtained via the control variational method given by Eqs. (18.12) to (18.15). Notice the differences between Eq. (18.13) and Eq. (18.3) that show the flexibility of our approach. It is also clear that a numerical solution of Eqs. (18.12) to (18.15) may be obtained by using first-order finite elements, which provides a simple way for the solution of Eq. (18.11).

Any pair $[y, w]$, $y \in \mathcal{K} \subset H_0^2(\Omega)$, $w = \Delta y - e^{-3} g$ is admissible for the problem of Eqs. (18.12) to (18.15).

In this special situation, one can prove directly the existence of optimal pairs:

PROPOSITION 18.3
The problem of Eqs. (18.12) *to* (18.15) *has a unique optimal pair* $[y^*, w^*]$.

PROOF Let $[y^n, w_n]$ be a minimizing sequence for Eq. (18.12). Then $y^n_{,11} + y^n_{,22} = w_n + e^{-3} g$, and the cost functional is bounded from above:

$$c \geq \int_\Omega e^3 \left\{ w_n^2 + 2(1-\tau)\left(y^n_{,12}\right)^2 + 2(\tau - 1)\left[y^n_{,11} w_n + e^{-3} g y^n_{,11} - \left(y^n_{,11}\right)^2\right]\right\} dx. \tag{18.16}$$

As $0 < \tau < \frac{1}{2}$, the relation of Eq. (18.16) shows that $\{w_n\}$, $\{y^n_{,12}\}$, $\{y^n_{,11}\}$ are bounded in $L^2(\Omega)$, and Eq. (18.13) yields that also $\{y^n_{,22}\}$ is bounded in $L^2(\Omega)$. That is, $\{y^n\}$ is bounded in $H_0^2(\Omega)$. One can take weakly convergent subsequences $y^n \to y^*, 0, w_n \to w^*$ in $H_0^2(\Omega)$, $L^2(\Omega)$, respectively, pass to the limit in Eqs. (18.13) to (18.15) as $\mathcal{K}$ is weakly closed, and end the proof by the weak lower semicontinuity of the cost functional of Eq. (18.12). Uniqueness is a clear consequence of the strict convexity of Eq. (18.12). ∎

REMARK Notice that, in this proof, $\Omega \subset \mathbb{R}^2$ plays an essential role.

The characterization of $[y^*, w^*]$ via the Euler (in)equation has to take the state constraints into account. We perform admissible variations of the form $y^* + s(z - y^*)$, $w^* + s(l - w^*)$, $s \in [0, 1]$, $\forall z \in \mathcal{K}$, $l = \Delta z - e^{-3} g \in L^2(\Omega)$ to obtain that

$$\begin{aligned} 0 \leq & \int_\Omega e^3 \{ w^*(l - w^*) + 2(1-\tau) y^*_{,12}(z_{,12} - y^*_{,12}) \\ & + 2(\tau - 1)[y^*_{,11}(z_{,22} - y^*_{,22}) + y^*_{,22}(z_{,11} - y^*_{,11})]\} \, dx, \end{aligned} \tag{18.17}$$

for any $z \in \mathcal{K}$.

Using the fact that $w^* = \Delta y^* - e^{-3} g, l = \Delta z - e^{-3} g$, a convenient grouping of the terms in Eq. (18.17) and the partial integration

$$\int_\Omega e^3 (\Delta z - \Delta y^*) e^{-3} g \, dx = \int_\Omega f(z - y^*) \, dx$$

yield:

COROLLARY 18.2
$y^* \in H_0^2(\Omega)$ *is the unique solution to Eqs.* (18.10) *and* (18.11).

REMARK It is possible to compute directional derivatives and to write necessary conditions as in the previous case. Other boundary conditions may be studied as well, for instance partially clamped plates. Then, another artificial control has to be introduced in Eq. (18.14), which becomes $y = v \in H^{3/2}(\partial\Omega)$, $v = 0$ on the "clamped" part of $\partial\Omega$. A weak penalization $\varepsilon|v|^2_{H^{3/2}(\partial\Omega)}$, $\varepsilon > 0$, has to be added to Eq. (18.12). The analysis involves a limiting process for $\varepsilon \to 0$, and it is more technical. Finally, let us underline that the cost functionals of Eq. (18.2) or Eq. (18.12) represent the usual energy (up to a constant), after the substitution of the control by the state.

18.3 Thickness Optimization of Plates with Unilateral Conditions

We study the optimal design problem

$$\mathrm{Min}\,\{J(e, y), \quad e \in E_{ad}\}, \tag{18.18}$$

subject to Eq. (18.10) and Eq. (18.11), and with $J : L^\infty(\Omega) \times H_0^2(\Omega) \to I\!R$, a lower semicontinuous functional;

$$E_{ad} = \left\{e \in L^\infty(\Omega);\ 0 < \alpha \le e \le \beta \quad \text{a.e. } \Omega;\ |e|_{W^{1,t}(\Omega)} \le \gamma\right\}. \tag{18.19}$$

Here, $\alpha,\ \beta, \gamma,\ t > 2$ are some given positive real numbers. One can also include other constraints in the definition of E_{ad}. For instance, the constant volume constraint

$$\int_\Omega e\, dx = const$$

may be considered. Concerning the possible state constraints, as $\Omega \subset I\!R^2$, the solution y of Eq. (18.11) belongs to $C(\bar\Omega)$, and one example of interest is given by the point-wise state constraint

$$y(x_0) \ge -\delta, \tag{18.20}$$

with $x_0 \in \Omega$ and $\delta > 0$ conveniently fixed.

An important case covered by Eqs. (18.18) to (18.20) is the minimization of the volume (thickness) of the plate such that the deflection y remains above a given tolerance $-\delta$ (in one or in any point in Ω), for a prescribed load $f \in L^2(\Omega)$. This is a natural safety requirement.

In the sequel, we shall denote by $a(e, y, v)$ the functional of Eq. (18.10), and we assume $0 \in \mathcal{K}$, in order to simplify the writing.

PROPOSITION 18.4
Let $e_n \to e$ in $L^\infty(\Omega)$ strongly, and let $y^n,\ y$ denote the corresponding solutions to Eq. (18.11). *Then, $y^n \to y$ strongly in $H_0^2(\Omega)$.*

PROOF By Corollary 18.2 and Eqs. (18.12) and (18.13), we get

$$\int_\Omega e_n^3\left(e_n^{-6} g^2\right) dx \ge \int_\Omega e_n^3\left\{w_n^2 + 2(1-\tau)\left(y^n_{,12}\right)^2 + 2(\tau - 1)y^n_{,11}y^n_{,22}\right\} dx, \tag{18.21}$$

obtained by the admissible choice $\tilde y^n = 0$, $\tilde w_n = -e_n^{-3} g$. Then Eq. (18.19) and Eq. (18.21) show that, for any n:

$$\int_\Omega \left\{w_n^2 + 2(1-\tau)\left(y^n_{12}\right)^2 + 2(\tau - 1)y^n_{11}y^n_{22}\right\} dx \le c.$$

Arguing again as in Eq. (18.16), we see that $\{w_n\}$, $\{y^n\}$ are bounded in $L^2(\Omega)$ and $H_0^2(\Omega)$, respectively. Denoting by $\tilde{y} \in \mathcal{K}$ the weak limit of y^n in $H_0^2(\Omega)$, on a subsequence, we can use the form of Eqs. (18.10) and (18.11) of the variational inequality to see that $\tilde{y} = y$, by the weak lower semicontinuity of quadratic forms. By summing $a(e_n, y^n, y^n - y^m)$ and $a(e_m, y^m, y^m - y^n)$ according to Eq. (18.11) and to the uniform (in e) coercivity of $a(e, y, v)$ on $H_0^2(\Omega)$, we obtain, for some $c > 0$:

$$c|y^n - y^m|^2_{H_0^2(\Omega)} \leq a(e_m, y^m, y^m - y^n) - a(e_n, y^m, y^m - y^n).$$

Using Eq. (18.10), and the uniform convergence of $\{e_n\}$, a short computation gives the strong convergence in $H_0^2(\Omega)$ for $\{y^n\}$, and the proof is finished. ∎

COROLLARY 18.3
The optimization problem of Eqs. (18.18) *to* (18.20) *has at least one optimal solution $e^* \in E_{ad}$ if it has admissible elements.*

This is a consequence of the compact embedding of $W^{1,t}(\Omega)$ in $C(\bar{\Omega})$, $t > 2$, by the Sobolev theorem and of Proposition 18.3.

REMARK Corollary 18.3 is a partial extension of results obtained by Hlavacek et al. [10], Bendsoe [3], and Sprekels and Tiba [14]. If Eq. (18.11) is the obstacle problem, Sokolowski and Rao [13] have studied its sensitivity with respect to variations around e^*.

In the present more general setting, we prove a weaker differentiability-type property. We fix some $b \in L^\infty(\Omega)$, and we denote by y^λ the solution of Eq. (18.11) associated with $e + \lambda b$, $\lambda \in \mathbb{R}$. By Proposition 18.4, $y^\lambda \to y$ strongly in $H_0^2(\Omega)$ as $\lambda \to 0$. Denote by $v^\lambda = \frac{y^\lambda - y}{\lambda} \in H_0^2(\Omega)$.

PROPOSITION 18.5
$\{v^\lambda\}$ is bounded in $H_0^2(\Omega)$. If $\hat{v}$ is a limit point of $\{v^\lambda\}$, then it satisfies:

$$a(e, y, \hat{v}) = \int_\Omega f \hat{v}\, dx, \tag{18.22}$$

$$\begin{aligned} 0 \geq a(e, \hat{v}, \hat{v} - l) + \int_\Omega 3e^2 b[&y_{,11}(\hat{v}_{,11} - l_{,11}) + \tau y_{,11}(\hat{v}_{,22} - l_{,22}) + \tau y_{,22}(\hat{v}_{,11} - l_{,11}) \\ &+ y_{,22}(\hat{v}_{,22} - l_{,22}) + 2(1-\tau) y_{,12}(\hat{v}_{,12} - l_{,12})]\, dx, \quad \forall l \in \hat{Z}, \end{aligned} \tag{18.23}$$

with $\hat{Z} \subset H_0^2(\Omega)$ a closed convex nonvoid set defined in the proof.

PROOF By adding $a(e, y, y - y^\lambda)$ and $a(e + \lambda b,\ y^\lambda, y^\lambda - y)$ and by Eq. (18.11), we get

$$\begin{aligned} 0 \geq\ & a(e, y^\lambda - y, y^\lambda - y) + \lambda \int_\Omega (3e^2 b + 3\lambda e b^2 + \lambda^2 b^3)\big[y^\lambda_{,11}\big(y^\lambda_{,11} - y_{,11}\big) \\ &+ \tau y^\lambda_{,11}\big(y^\lambda_{,22} - y_{,22}\big) + \tau y^\lambda_{,22}\big(y^\lambda_{,11} - y_{,11}\big) + y^\lambda_{,22}\big(y^\lambda_{,22} - y_{,22}\big) \\ &+ 2(1-\tau) y^\lambda_{,12}\big(y^\lambda_{,12} - y_{,12}\big)\big]\, dx\,. \end{aligned} \tag{18.24}$$

Dividing by λ^2 in Eq. (18.24), and using the coercivity of $a(e, \cdot, \cdot)$ and the convergence of y^λ, we find that $\{v^\lambda\}$ is bounded in $H_0^2(\Omega)$. Let $\hat{v}$ be a limit point of $\{v^\lambda\}$, on some subsequence. Passing to the limit $\lambda \searrow 0$ in

$$a(e, y, -v^\lambda) \le -\int_\Omega f v^\lambda \, dx,$$
$$a(e+\lambda b, y^\lambda, v^\lambda) \le \int_\Omega f v^\lambda \, dx,$$

we get Eq. (18.22).

Consider now test functions $l^\lambda \in Z_\lambda = [\frac{1}{\lambda}(\mathcal{K} - y) \cap \frac{1}{\lambda}(y^\lambda - \mathcal{K})] \subset H_0^2(\Omega)$, $\lambda > 0$. Notice that Z_λ is a nonvoid closed convex set and $0 \in Z_\lambda$, $v^\lambda \in Z_\lambda$. If $l^\lambda \in Z_\lambda$, then $y + \lambda l^\lambda \in \mathcal{K}$, $y^\lambda - \lambda l^\lambda \in \mathcal{K}$, $\lambda > 0$. We use these test functions in Eq. (18.11) to obtain:

$$a(e, y, y - y^\lambda + \lambda l^\lambda) \le -\int_\Omega f(y - y^\lambda + \lambda l^\lambda) \, dx,$$
$$a(e+\lambda b, y^\lambda, y^\lambda - y - \lambda l^\lambda) \le \int_\Omega f(y^\lambda - y - \lambda l^\lambda) \, dx.$$

Adding these inequalities and dividing by λ^2, we have

$$\begin{aligned} 0 \ge\ & a(e, v^\lambda, v^\lambda - l^\lambda) + \int_\Omega (3e^2 b + 3\lambda e b^2 + \lambda^2 b)\big[y^\lambda_{,11}\big(v^\lambda_{,11} - l^\lambda_{,11}\big) \\ & + \tau y^\lambda_{,11}\big(v^\lambda_{,22} - l^\lambda_{,22}\big) + \tau y^\lambda_{,22}\big(v^\lambda_{,11} - l^\lambda_{,11}\big) + y^\lambda_{,22}\big(v^\lambda_{,22} - l^\lambda_{,22}\big) \\ & + 2(1-\tau) y^\lambda_{,12}\big(v^\lambda_{,12} - l^\lambda_{,12}\big)\big] dx, \quad \forall l^\lambda \in Z_\lambda. \end{aligned} \tag{18.25}$$

If $\lambda_n \searrow 0$ is chosen such that $v_{\lambda_n} \to \hat{v}$ weakly in $H_0^2(\Omega)$, we denote by $\hat{Z} = \liminf_{\lambda \to 0} Z_{\lambda_n} = \{p \in H_0^2(\Omega);\ \exists p_{\lambda_n} \in Z_{\lambda_n},\ p_{\lambda_n} \to p \text{ in } H_0^2(\Omega)\}$. This is a nonvoid closed convex subset of $H_0^2(\Omega)$. Passing to the limit in Eq. (18.25) gives Eq. (18.23), which ends the proof. □

REMARK The dependence of $\hat{Z}$ and of $\hat{v}$ on the way we choose a convergent subsequence of $\{v^\lambda\}$ shows that they may be not uniquely determined.

18.4 Curved Rods and Shells

For the three-dimensional curved rods, we relax the usual regularity hypotheses on the parametrization, of type $W^{3,\infty}(0, L)$, by avoiding the use of the classical Frenet or Darboux frames (see Cartan [5]). A new local system of axes valid for $C^1[0, L]$ or $W^{2,\infty}(0, L)$ curves was introduced in Ignat et al. [12]. As here we are mainly interested in optimization questions, we perform a direct parametrization of the tangent vector,

$$\bar{t}(\cdot) = (\sin\tau(\cdot)\cos\psi(\cdot), \quad \sin\tau(\cdot)\sin\psi(\cdot), \quad \cos\tau(\cdot)) \,. \tag{18.26}$$

The curve is then parametrized by

$$\bar{\theta}(x_3) = \int_0^{x_3} \bar{t}(s) ds, \; x_3 \in [0, L]. \tag{18.27}$$

Notice that in this way a unit speed curve $\bar{\theta}$ in $\mathbb{R}^3$ with fixed length $L > 0$ is automatically generated. Moreover, the local frame can be obtained by algebraic means,

$$\bar{n} = (\cos\tau\cos\psi, \cos\tau\sin\psi, -\sin\tau), \tag{18.28}$$

$$\bar{b} = (-\sin\psi, \cos\psi, 0). \tag{18.29}$$

The mappings $\tau, \psi \in C^1[0, L]$ give the real parametrization. If $\omega(x_3) \subset I\!R^2$ is a bounded domain, not necessarily simply connected, we define the open set

$$\Omega = \bigcup_{x_3 \in]0,L[} [\omega(x_3) \times \{x_3\}] \subset I\!R^3. \tag{18.30}$$

The curved rod $\tilde{\Omega}$ associated to $\bar{\theta}$ is then obtained by the transformation

$$\begin{aligned} \bar{x} &= (x_1, x_2, x_3) \in \Omega \mapsto F\bar{x} = \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \\ &= \bar{\theta}(x_3) + x_1\bar{n}(x_3) + x_2\bar{b} \in \tilde{\Omega}, \quad \forall \bar{x} \in \Omega. \end{aligned} \tag{18.31}$$

The Jacobian J of the transformation F satisfies $\det J(\bar{x}) \geq c > 0, \forall \bar{x} \in \Omega$, if the sets $\omega(x_3)$ are all contained in a sufficiently small disk in $I\!R^2$. In Ciarlet [8] it is proved that F is one-to-one and that $\tilde{\Omega}$ is well defined.

We make the geometrical assumption that the displacement has the following form for $\tilde{x} \in \tilde{\Omega}$:

$$\bar{y}(\tilde{x}) = \bar{\rho}(x_3) + x_1\bar{N}(x_3) + x_2\bar{B}(x_3), \quad \bar{x} = F^{-1}(\tilde{x}). \tag{18.32}$$

The unknowns are $\bar{\rho}, \bar{N}, \bar{B} \in H_0^1(0, L)^3$, and Eq. (18.32) enters the category of polynomial models. Comparing with the shell model considered later in this section, we may say that Eq. (18.32) gives a generalized Naghdi model for curved rods. By introducing Eq. (18.32) into the elasticity system, we get the following variational equation (here $(h_{ij}) = J^{-1}$ and $\tilde{\lambda}, \ \tilde{\mu}$ are the Lamé coefficients) for $\bar{\rho}, \ \bar{N}, \bar{B}$:

$$\begin{aligned}
&\tilde{\lambda} \int_\Omega \sum_{i,j=1}^{3} [N_i(x_3)h_{1i}(\bar{x}) + B_i(x_3)h_{2i}(\bar{x}) + (\rho_i'(x_3) + x_1N_i'(x_3) \\
&\quad + x_2B_i'(x_3))h_{3i}(\bar{x})][M_j(x_3)h_{1j}(\bar{x}) + D_j(x_3)h_{2j}(\bar{x}) + (\mu_j'(x_3) + x_1M_j'(x_3) \\
&\quad + x_2D_j'(x_3))h_{3j}(\bar{x})] |\det J(\bar{x})| \, d\bar{x} + \tilde{\mu} \int_\Omega \sum_{i<j} \{N_i(x_3)h_{1j}(\bar{x}) + B_i(x_3)h_{2j}(\bar{x}) \\
&\quad + [\rho_i'(x_3) + x_1N_i'(x_3) + x_2B_i'(x_3)]h_{3j}(\bar{x}) + N_j(x_3)h_{1i}(\bar{x}) + B_j(x_3)h_{2i}(\bar{x}) \\
&\quad + [\rho_j'(x_3) + x_1N_j'(x_3) + x_2B_j'(x_3)]h_{3i}(\bar{x})\}\{M_i(x_3)h_{1j}(\bar{x}) + D_i(x_3)h_{2j}(\bar{x}) \\
&\quad + [\mu_i'(x_3) + x_1M_i'(x_3) + x_2D_i'(x_3)]h_{3j}(\bar{x}) + M_j(x_3)h_{1i}(\bar{x}) + D_j(x_3)h_{2i}(\bar{x}) \\
&\quad + [(\mu_j'(x_3) + x_1M_j'(x_3) + x_2D_j'(x_3)]h_{3j}(\bar{x})\} |\det J(\bar{x}| \, d\bar{x} + 2\tilde{\mu} \int_\Omega \sum_{i=1}^{3} \{N_i(x_3)h_{1i}(\bar{x}) \\
&\quad + B_i(x_3)h_{2i}(\bar{x}) + [\rho_i'(x_3) + x_1N_i'(x_3) + x_2B_i'(x_3)]h_{3i}(\bar{x})\} \\
&\quad [M_i(x_3)h_{1i}(\bar{x}) + D_i(x_3)h_{2i}(\bar{x}) + (\mu_i'(x_3) + x_1M_i'(x_3) + x_2D_i'(x_3)h_{3i}(\bar{x})] |\det J(\bar{x})| \, d\bar{x} \\
&= \sum_{l=1}^{3} \int_\Omega f_l(\bar{x})[\mu_l(x_3) + x_1M_l(x_3) + x_2D_l(x_3)] |\det J(\bar{x})| \, d\bar{x} + \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\partial\Omega} g_l(\bar{x}) \\
&\quad [\mu_l(x_3) + x_1M_l(x_3) + x_2D_l(x_3)] |\det J(\bar{x})| \sqrt{\nu_i g^{ij} \nu_j} \, d\tau.
\end{aligned} \tag{18.33}$$

Above, $\bar{\mu}, \bar{M}, \bar{D} \in H_0^1(0, L)^3$ are test functions, (ν_i) is the normal vector to $\partial\Omega$, $(g^{ij}) = J^{-1}(J^T)^{-1}$, and $\bar{f} \in L^2(\Omega)^3$, $\bar{g} \in L^2(\partial\Omega)^3$ are the acting forces.

The coercivity of the bilinear form is established under the assumption that $\omega(x_3) \supset \omega$, $\forall x_3 \in [0, L]$, and

$$0 = \int_\omega x_1\, dx_1\, dx_2 = \int_\omega x_2\, dx_1\, dx_2 = \int_\omega x_1 x_2\, dx_1\, dx_2.$$

The argument in Ignat et al. [12] is a direct one. It is based on the algebraic identity

$$\begin{aligned}&\frac{1}{2}\left[(z_1h_{32} + z_2h_{31})^2 + (z_2h_{33} + z_3h_{32})^2 + (z_1h_{33} + z_3h_{31})^2\right] + \frac{3}{2}\left(z_1^2h_{31}^2 + z_2^2h_{32}^2 + z_3^2h_{33}^2\right)\\ &= \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2\right)\left(h_{31}^2 + h_{32}^2 + h_{33}^2\right) + \frac{1}{2}(z_1h_{31} + z_2h_{32})^2 + \frac{1}{2}(z_1h_{31} + z_3h_{33})^2\\ &\quad + \frac{1}{2}(z_2h_{32} + z_3h_{33})^2.\end{aligned}$$

A general formulation of optimization problems associated with curved rods is (see Eq. [18.32]):

$$\operatorname*{Min}_{\tau,\psi} \{\Pi(\tau, \psi) = j(\bar\theta, \bar y)\}, \tag{18.34}$$

subject to Eq. (18.33) and to constraints $\bar\theta \in \mathcal{K} \subset C^2(0, L)^3$, bounded closed subset. A typical example for Eq. (18.34) is the quadratic case, for instance $j(\bar\theta, \bar y) = \Sigma_{i=1}^3 |\rho_i|^2_{H_0^1(0,L)}$ (minimization of the displacement of the line of centroids). Notice that our construction eliminates degenerate cases like rods of length zero. By imposing the constraint $0 \le \tau(x_3) \le \frac{\pi}{2} - \varepsilon$, $x_3 \in [0, L]$, self-intersecting curves are also eliminated. The partial periodicity constraint

$$\int_0^L t_1\, dx_3 = \int_0^L t_2\, dx_3 = 0$$

can be used for the optimization of spirals, etc.

THEOREM 18.1

If the set of admissible $\{\tau, \psi\}$ *is compact in* $C^1[0, L]^2$*, and if* $j : C^2[0, L]^3 \times H_0^1(0, L)^9 \to I\!R$ *is lower semicontinuous, then the problem of Eqs.* (18.34) *and* (18.33) *admits at least one optimal curved rod* $\bar\theta^*$.

In Arnăutu et al. [2] it is also proved that the mapping $\{\tau, \theta\} \mapsto y$ is Gâteaux differentiable from $C^1[0, L]^2$ to $H_0^1(0, L)^9$, and the directional derivative for the cost of Eq. (18.34) are computed together with the first-order optimality conditions. Many numerical examples may be found in Ignat et al. [12] and in Arnăutu et al. [2]. Some of them have a clear physical meaning, which may be interpreted as a validation of the model.

In the case of shells, we consider an open bounded set $\omega \subset I\!R^2$, not necessarily simply connected and $\varepsilon > 0$, "small." We denote by $\Omega = \omega \times\,]-\varepsilon, \varepsilon[$ and by $p : \omega \to I\!R$ a $C^2(\bar\omega)$ mapping whose graph represents the middle surface of the shell. The shell $\hat\Omega$ is obtained via the transformation $\hat F : \Omega \to \hat\Omega$, $\hat F(x_1, x_2, x_3) = (x_1, x_2, p(x_1, x_2)) + x_3\bar n(x_1, x_2, x_3)$ where $\bar n$ is the normal vector:

$$\bar n = (n_1, n_2, n_3) = \frac{1}{\sqrt{1 + p_1^2 + p_2^2}}(-p_1, -p_2, 1)$$

and where p_1, p_2 are the partial derivatives of p. The shell is assumed to be partially clamped along $\hat\Gamma_0 = \hat F(\Gamma_0)$, with $\Gamma_0 = \gamma_0 \times]-\varepsilon, \varepsilon[$ and $\gamma_0 \subset \partial\omega$ being some open part. The displacement $\hat u \in V(\hat\Omega) = \{\hat v \in H^1(\hat\Omega)^3;\ \hat v|_{\hat\Gamma_0} = 0\}$ is supposed to be of the form

$$\hat u(\hat x) = \bar u(x_1, x_2) + x_3\bar r(x_1, x_2), \quad (x_1, x_2, x_3) = \hat F^{-1}(\hat x).$$

The unknowns $\bar{u}, \bar{r} \in V(\omega) = \{\bar{v} \in H^1(\omega)^3;\ \bar{v}|_{\gamma_0} = 0\}$ represent the displacement of the middle surface of the shell, respectively, the modification of the normal vector. This is allowed to change the length as well (that is, the elastic material can dilate or contract), which is a generalization of the classical Naghdi model studied, for instance, by Blouza [4] under similar regularity conditions. For ε "small," we get $\det J(\bar{x}) \geq c > 0$, $J = \nabla \hat{F}$, which justifies the above construction. If we denote by $[h_{ij}(\bar{x})] = J(\bar{x})^{-1}$, the same approach as for the curved rods, based on the linear elasticity system, generates the following boundary value problem:

$$
\begin{aligned}
&\tilde{\lambda} \int_\Omega \left\{ \sum_{i=1}^{3} \left[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2i} + r_i h_{3i} \right] \right\} \\
&\quad \times \left\{ \sum_{j=1}^{3} \left[\left(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \varrho_j}{\partial x_1} \right) h_{1j} + \left(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \varrho_j}{\partial x_2} \right) h_{2j} + \varrho_j h_{3j} \right] \right\} |\det J(\bar{x})|\, d\bar{x} \\
&\quad + 2\tilde{\mu} \int_\Omega \sum_{i=1}^{3} \left[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2i} + r_i h_{3i} \right] \\
&\quad \times \left[\left(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \varrho_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \varrho_i}{\partial x_2} \right) h_{2i} + \varrho_i h_{3i} \right] |\det J(\bar{x})|\, d\bar{x} + \tilde{\mu} \int_\Omega \sum_{i<j} \\
&\quad \left\{ \left[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1j} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2j} + r_i h_{3j} + \left(\frac{\partial u_j}{\partial x_1} + x_3 \frac{\partial r_j}{\partial x_1} \right) h_{1i} \right.\right. \\
&\quad \left. + \left(\frac{\partial u_j}{\partial x_2} + x_3 \frac{\partial r_j}{\partial x_2} \right) h_{2i} + r_j h_{3i} \right] \left[\left(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \varrho_i}{\partial x_1} \right) h_{1j} + \left(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \varrho_i}{\partial x_2} \right) h_{2j} \right. \\
&\quad \left.\left. + \varrho_i h_{3j} + \left(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \varrho_j}{\partial x_1} \right) h_{1i} + \left(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \varrho_j}{\partial x_2} \right) h_{2i} + \varrho_j h_{3i} \right] \right\} |\det J(\bar{x})|\, d\bar{x} \\
&= \int_\Omega \sum_{l=1}^{3} f_l(\mu_l + x_3 \varrho_l) |\det J(\bar{x})|\, d\bar{x} \\
&\quad + \int_{\partial\Omega - \Gamma_0} \sum_{l=1}^{3} \sum_{i,j=1}^{3} g_l(\mu_l + x_3 \varrho_l) |\det J(\bar{x})| \sqrt{\nu_i(\bar{x}) g^{ij}(\bar{x}) \nu_j(\bar{x})}\, d\tau.
\end{aligned}
\tag{18.35}
$$

Here, the notations are similar to Eq. (18.33). To prove the existence and the uniqueness of the solution $(\bar{u}, \bar{r}) \in V(\omega)^2$ in Eq. (18.35), we have established the coercivity of the bilinear form by applying Korn's inequality (see Sprekels and Tiba [17]). Moreover, in Arnăutu et al. [2], by using an extension technique to $H^1(\mathbb{R}^3)$, it is shown that this coercivity constant is independent of the geometry (of p) in some given classes. We associate with Eq. (18.35) the shape optimization problem

$$
\operatorname*{Min}_{p \in \mathcal{K}} \{\Pi(p) = j(\bar{y}, \bar{p})\}
\tag{18.36}
$$

with $\bar{y} = (\bar{u}, \bar{r}) \in H^1(\omega)^6$ and $\mathcal{K} \subset C^2(\bar{\omega})$ closed. The mapping $j : H^1(\omega)^6 \times C^2(\bar{\omega}) \to \mathbb{R}$ is of general type. Some well-known examples of cost functionals and of constraints $\mathcal{K}$ are:

$$
j(\bar{y}, p) = |u_1|^2_{H^1(\omega)} + |u_2|^2_{H^1(\omega)} + |u_3|^2_{H^1(\omega)}
$$

(minimization of the displacement of the middle surface of the shell), respectively,

$$
\int_\omega \sqrt{1 + p_1^2 + p_2^2}\, dx_1\, dx_2 \leq const
$$

(area limitation for the shell).

THEOREM 18.2
If $\mathcal{K} \subset C^2(\bar{\omega})$ is compact and $j : H^1(\omega)^6 \times C^2(\bar{\omega}) \to I\!R$ is lower semicontinuous, then the shape-optimization problem of Eq. (18.35) *and Eq.* (18.36) *has at least one optimal solution.*

REMARK It is possible to compute directional derivatives of the mapping $p \mapsto \bar{y}$ and to write optimality conditions (see Arnăutu et al. [2]). However, numerical experiments seem very difficult to perform as the coercivity constant is of the order ε^3, which shows the lack of stability properties in the computations.

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Chapter 19

Global Exact Controllability on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of Semilinear Wave Equations with Neumann $L_2(0,T;L_2(\Gamma_1))$-Boundary Control[1]

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Appendix C: Proof of Theorem 19.3.1: Regularity of the ϕ-Problem of Eq. (19.3.9) = Eq. (19.6.3) = Eq. (19.A.4) (Same as the v_2-Problem of Eq. (19.B.17)) with I.C. in $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ 330
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Abstract We provide four proofs that the semilinear wave equation with globally Lipschitz nonlinear term is exactly controllable on the finite energy space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ within the class of $L_2[0, T; L_2(\Gamma_1)]$-boundary controls exercised in the Neumann boundary conditions, whenever the corresponding linear equation satisfies the same property; and on the same time interval. One proof is based on the global inversion approach applied to the original, controlled problem, which was proposed by Lasiecka and Triggiani [29]. In contrast, the other three proofs are based, on a uniform continuous observability inequality of the dual uncontrolled problem. Precise links are exhibited and exploited among these four approaches. All proofs are essentially operator-theoretic with a partial differential equation (PDE) interpretation. A common thread of the four proofs is an analysis of suitable families of collectively compact operators, which then admit uniform inversions in the style of Reference 29. This way, compactness and uniqueness arguments are entirely dispensed with.

19.1 Controlled Dynamics: Wave Equation with Neumann Control

Let Ω be an open-bounded domain of $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$. In Ω we consider the following semilinear problem for the wave equation in the solution $w(t, x)$, with Neumann-boundary control u based on Σ_1:

$$\begin{cases} w_{tt} = \Delta w + f(w) & \text{in } (0, T] \times \Omega \equiv Q; & (19.1.1a)\\ w(0, \cdot) = w_0, w_t(0, \cdot) = w_1 & \text{in } \Omega; & (19.1.1b)\\ w|_{\Sigma_0} \equiv 0; \ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = u & \text{in } (0, T] \times \Gamma_i \equiv \Sigma_i, \quad i = 0, 1. & (19.1.1c)\end{cases}$$

Here $\nu(x)$ is the unit outward normal on $x \in \Gamma$. The standing *assumption* on the nonlinearity f is a global Lipschitz condition [Reference 44, p. 108] as follows:

$$f : \mathbb{R} \to \mathbb{R} \text{ is an absolutely continuous function with first derivative a.e., which is uniformly bounded on } \mathbb{R}: \quad |f'(r)| \leq \text{const.} \quad \text{a.e., } r \in \mathbb{R}. \tag{19.1.2}$$

REMARK 19.1.1 To streamline the treatment, we shall also *assume* throughout the paper that $f(0) = 0$, in order not to be bothered by what are essentially benign terms in the analysis. This assumption of convenience $f(0) = 0$ will be maintained throughout this paper unless otherwise noted.

19.1.1 Exact Controllability

Under some geometric conditions and for all $T >$ some universal $T_0 > 0$, the linear problem Eq. (19.1.1) with $f \equiv 0$ is exactly controllable over $[0, T]$, on the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the class of controls $u \in L_2(0, T; L_2(\Gamma_1))$ [4, 22, 39, 27, 49–51], etc. The most general results are in Reference 4: one such set of specific conditions is given in Eq. (19.3.6) and (19.3.7) below, for sake of concreteness. The result below extends the same (global) exact controllability property to the semilinear problem of Eq. (19.1.1) subject to Eq. (19.1.2), over the same time interval.

THEOREM 19.1.1
Let $T > 0$ be a time for which exact controllability of the linear problem with $f \equiv 0$ holds true in the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ within the class of $L_2(0, T; L_2(\Gamma_1))$-controls u. Let f satisfy the assumption of Eq. (19.1.2). *Then, a similar exact controllability result holds true for the original problem Eq.* (19.1.1) *for the same time T: for any pair $\{w_0, w_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of initial conditions, and any pair $\{\tilde{w}_0, \tilde{w}_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of target states, there exists a suitable control function $u \in L_2(0, T; L_2(\Gamma_1))$ such that the corresponding solution $\{w, w_t\}$ of the problem in Eq.* (19.1.1) *satisfies $\{w(T, \cdot), w_t(T, \cdot)\} = \{\tilde{w}_0, \tilde{w}_1\}$.*

19.1.2 Literature

We confine our discussion to second-order hyperbolic equations, in fact for dim $\Omega \geq 2$. For dim $\Omega = 1$, we refer to Reference 6. Theorem 19.1.1 above is new.

In the case of *Dirichlet* boundary control: $w|_{\Sigma_0} \equiv 0$, $w|_{\Sigma_1} = u$, exact controllability of the semilinear wave equation of Eq. (19.1.1a) under the same assumption of Eq. (19.1.2) on the nonlinearity was given in Reference 29 in the state space $H^1_0(\Omega) \times L_2(\Omega)$ within the class of $H^1_0(0, T; L_2(\Gamma_1))$-controls and in the state space $L_2(\Omega) \times H^{-1}(\Omega)$ within the class of $L_2(0, T; L_2(\Gamma_1))$-controls. Hence, by interpolation, in the state space $H^\gamma_0(\Omega) \times H^{\gamma-1}(\Omega)$, $0 \leq \gamma \leq 1$, $\gamma \neq \frac{1}{2}$ (resp., in the state space $H^{1/2}_{00}(\Omega) \times [H^{1/2}_{00}(\Omega)]'$ for $\gamma = \frac{1}{2}$) within the class of $H^\gamma_0(0, T; L_2(\Gamma_1))$-controls (resp., within the class of $H^{1/2}_{00}(0, T; L_2(\Gamma_1))$-controls, for $\gamma = \frac{1}{2}$). These topological settings are optimal [26, 24, 32, 39, 34], (Reference 34, Section 10.5). The paper in Reference 29 was prompted by Zuazua [61] who used a Schauder fixed-point theorem approach (the approach of using a fixed-point theorem is a well-established strategy that goes back to the mid-1960s [11] in the case of finite-dimensional systems). This way, however, Zuazua [61] obtains the above exact controllability results with Dirichlet boundary control but only in the range $0 < \gamma < 1$ and under an *a priori* assumption of a uniqueness result, for which Ruiz [45] could provide verification only with control applied on the entire boundary. Since then, appropriate uniqueness results have become available also in the case where the control is applied on a suitable portion of the boundary [20, 38, 60]. The limit cases $\gamma = 0$ and $\gamma = 1$—the most interesting ones—are explicitly excluded from Reference 61, as the requirement $0 < \gamma < 1$ is essential to its treatment. The case $\gamma = 0$ is excluded for lack of compactness in the Schauder fixed-point argument. The case $\gamma = 1$ is excluded because not enough information is available in passing to the limit in the compactness-uniqueness argument [see Reference 61, Remark 2.6, p. 375]. The same restriction $0 < \gamma < 1$, this time without the additional uniqueness assumption, is present in the result of Zhang [Reference 60, Theorem 4.2, p. 831] for a more general wave model, as this work invokes on this point Zuazua [61]. In contrast, the approach of Lasiecka and Triggiani [29] is based on a *global inversion theorem*, as carried out for an abstract equation under assumptions that are intended to capture dynamical properties of hyperbolic and hyperbolic-like partial differential equation (PDE) problems (waves, plates, etc.). The aforementioned results for semilinear wave equations with Dirichlet controls are then obtained in Reference 29 by specializing the abstract results. The global inversion theorem continues to be the carrier of the analysis of the present paper with Neumann boundary control as well, at least in its Part I. Moreover, Parts II, III, IV—which deal with the dual uncontrolled linearized problem—either proceed through an approach dual to Part I or else eventually fall into Part I. We also refer to [8].

19.1.3 Orientation

As explained above, the (global) exact controllability of the problem in Eq. (19.1.1) with Neumann-boundary control is studied in Part I of this chapter (Sections 19.2–19.4) by virtue of the *global inversion approach* of Reference 29, here revisited to fit present circumstances. Proposition 19.4.1(a) below provides an operator-theoretic characterization for the solution of the (global) exact controllability problem of the system in Eq. (19.1.1) in terms of the global inversion (homeomorphism) of the map: $u \to [u + \Lambda_T(u)]$. A sufficient condition for this to occur is that the Frechet derivative

$[I + \Lambda'_T(u)]$ of this map has the inverse, which is uniformly bounded in u,

$$\|[I + \Lambda'_T(u)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^\perp)} \leq \text{const.}, \quad \forall u \in [\mathcal{N}(\mathcal{L}_T)]^\perp. \tag{19.1.3}$$

This result is contained in Eq. (19.4.18b) of Proposition 19.4.1(b), indeed, in its version Eq. (19.4.64) of Lemma 19.4.6. On the other hand, Proposition 19.4.3, Eq. (19.4.39a) yields that the operator $[I + \Lambda'_T(u)]$ coincides with the operator $\mathcal{M}_T[y_u]$ modulo, an (algebraic) isomorphism as a factor. Here, $\mathcal{M}_T[y_u]$ is the map $\tilde{u} \to z(T) = \mathcal{M}_T[y_u]\tilde{u}$ in Eqs. (19.3.15) and (19.4.33), with u as a parameter, where z is the solution of the corresponding linearized problem with I.C. $z_0 = 0$; see Eq. (19.2.18), whose PDE version is given by Eq. (19.2.37). Thus, the uniform bounded inversion $[I + \Lambda'_T(u)]^{-1}$ is, in turn, equivalent to the condition that the adjoint operator $\mathcal{M}^*_T[y_u]$ *be bounded below uniformly in u*, as in Eq. (19.5.5), see Proposition 19.5.1. The latter condition is then ultimately verified as a *continuous observability inequality of the linearized problem, uniformly with respect to the potential $q(t, x) \in L_\infty(Q)$ in a given ball*, where, in the relevant case, $q = f'[w(t, x)]$. This inequality of Eq. (19.5.10) for the dual uncontrolled ϕ-problem in Eq. (19.3.9) = Eq. (19.6.3) actually proved as inequality of Eq. (19.A.21) of Theorem 19.A.3 in Appendix 19.A. Two different strategies are then pursued here to establish (global) exact controllability of Eq. (19.1.1), with the second strategy receiving three different implementations.

In Part I, Sections 19.2 to 19.4, following Reference 29, extract some key properties of the linearization system. A first group of properties consists of structural properties of the operators defining the linearization system: they are labeled here (P.1) = Eq. (19.2.14), (P.2) = Eq. (19.2.27), and (P.3) = Eq. (19.2.34). A second group of properties consists of control-theoretic properties labeled in Section 19.3 as (C.1), exact controllability of the linear problem, and (C.2), an approximate controllability property of the linearized system. Following the strategy of Reference 29, it is here shown in Part I that, cumulatively, properties (P.1), (P.2), (P.3), (C.1), and (C.2) imply the desired uniform bound Eq. (19.1.3) = Eq. (19.4.64), whereby then exact controllability of the semilinear system Eq. (19.1.1) is established via a global inversion theorem, as stated in Theorem 19.1.1. Part I deals entirely with the *original controlled* problem.

A second strategy centers on establishing the *uniform continuous observability inequality* (COI) of Eq. (19.5.5) of the linearized problem, involving $\mathcal{M}^*_T[q]$, uniformly in the potential $q \in L_\infty(Q)$ in a fixed finite ball. To this end, three different implementations may be pursued (see Section 19.5).

The first such implementation makes up Part II (Sections 19.6 and 19.7). Unlike Part I, it deals entirely this time with the *dual uncontrolled* problem in Eq. (19.3.9) = Eq. (19.6.3) and seeks to prove *directly* the uniform COI of Eq. (19.5.11) of the linearized problem by the operator-theoretic properties of Lemmas 19.4.4 and 19.4.5 established in Part I.

A second implementation—pursued in Part III (Section 19.8)—consists again in showing the uniform COI of Eq. (19.5.5) or Eq. (19.5.10), however, this time by seeking to prove the *equivalent* inequality of Eq. (19.8.4) = Eq. (19.8.7) on traces or their operator representations. But, again, it does so *not* directly but by proving, in turn, an *equivalent* inequality of Eq. (19.8.8) (*equivalence via an isomorphism*), which is precisely the operator-theoretic inequality of Eq. (19.8.9), in turn, established via Lemmas 19.4.4 and 19.4.5 of Part I.

A third implementation of the second strategy makes up Part IV (Section 19.9); this also deals entirely with the *dual uncontrolled* problem and seeks likewise to establish the uniform COI of Eq. (19.5.5) or Eq. (19.5.10). However, unlike Part III, it achieves this goal by proving the trace inequality (or its operator reformulation) of Eq. (19.8.4) = Eq. (19.8.7) this time *directly*, not its *equivalent* (via an isomorphism) version Eq. (19.8.8). Indeed, there are some advantages in pursuing this route. One still falls in the "collectively compact family of operators" mode that characterizes this entire paper (following Reference 29) and consequent application of its basic theory [1]. However, now only the counterpart of Lemma 19.4.4 is needed. The counterpart of Lemma 19.4.5 is entirely dispensed with on this Part IV. In short, Part IV requires only one uniform inversion (as in Lemma 19.4.4), not two uniform inversions (as in Lemma 19.4.5). This way, it provides an alternative proof of Theorem 19.1.1. Figure 19.5.1 in Section 19.5 illustrates graphically the four different paths of the four proofs.

The chapter closes by presenting, in Section 19.10, some comparison comments, also in light of a generalization of the Dirichlet-boundary control case with $L_2(\Sigma_1)$-control (case $\gamma = 0$) for a far more general model than in past literature, based on the extension in the Riemannian setting [59], of prior sharp results in References 20, 38, and 60 in the Euclidean setting.

Some essential but more standard background material is collected, sometimes for completeness, in Appendices A to C.

Part I: Proof of Theorem 19.1.1 by the Abstract Global Inversion Approach of Reference 29 via the Controlled Problem

19.2 Abstract Formulation of Equation (19.1.1) and Corresponding Structural Properties

Abstract Model

Equation (19.1.1) can be cast into the following abstract equation with $y(t) = [w(t), w_t(t)]$:

$$\dot{y} = Ay + F(y) + Bu \in [\mathcal{D}(A^*)]', \quad y(0) = y_0 \in Y \tag{19.2.1}$$

to be interpreted as specified below, where the relevant spaces and operators are identified.

1. First, let $\mathcal{A} : L_2(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L_2(\Omega)$ be the positive, self-adjoint operator defined by $\mathcal{A}h = -\Delta h$, $\mathcal{D}(\mathcal{A}) \equiv \{h \in H^2(\Omega), h|_{\Gamma_0} \equiv \frac{\partial h}{\partial \nu}|_{\Gamma_1} \equiv 0\}$. Then, $-\mathcal{A}$ generates a strongly continuous (s.c.) cosine operator $C(t)$ on $L_2(\Omega)$ with 'sine'-operator $S(t) = \int_0^t C(\tau)\, d\tau$. With $y(t) = [w(t), w_t(t)]$, the operator A in the model Eq. (19.2.1) is given by

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}, \mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big), \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) = H^1_{\Gamma_0}(\Omega) = \{h \in H^1(\Omega), h|_{\Gamma_0} = 0\}. \tag{19.2.2}$$

Then, A generates the unitary s.c. group e^{At} given by

$$e^{At} = \begin{bmatrix} C(t) & S(t) \\ -\mathcal{A}S(t) & C(t) \end{bmatrix}, \|C(t)\|_{L[L_2(\Omega)]} + \left\|\mathcal{A}^{\frac{1}{2}} S(t)\right\|_{L[L_2(\Omega)]} \le \text{const.} \tag{19.2.3}$$

on either of the spaces

$$H \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \ \text{ or } \ Y \equiv L_2(\Omega) \times \left[\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\right]', \tag{19.2.4}$$

which are topologically equivalent to $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ and $L_2(\Omega) \times [H^1_{\Gamma_0}(\Omega)]'$, respectively, where $[\quad]'$ denotes duality with respect to $L_2(\Omega)$ as a pivot space.

2. Next, let N be the Neumann map (harmonic extension of boundary data) defined by

$$Ng = h \iff \left\{ \Delta h = 0 \text{ in } \Omega;\ h|_{\Gamma_0} = 0,\ \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_1} = g \right\}; \tag{19.2.5a}$$

$$N : \text{ continuous } L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) \equiv \mathcal{D}\big(\mathcal{A}^{\frac{3}{4}-\epsilon}\big), \quad \forall\, \epsilon > 0 \tag{19.2.5b}$$

[40, 34]. Then, with $U = L_2(\Gamma_1)$, the operator B in the model in Eq. (19.2.1) is

$$Bu = \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix}, \quad A^{-1}Bu = \begin{bmatrix} -Nu \\ 0 \end{bmatrix}, \quad A^{-1}B \in L(U; H), \tag{19.2.6}$$

where $\mathcal{A}$ in $\mathcal{A}Nu$ is actually the isomorphic extension, say, $L_2(\Omega) \to [\mathcal{D}(\mathcal{A})]'$ of the original operator $\mathcal{A}$ defined above in Eq. (19.2.2).

3. Finally, the nonlinear operator F in the model in Eq. (19.2.1) and its Frechet derivative $F'[\eta] \in L(Y)$ at the point $\eta \in Y$ are given by

$$F(y) = \begin{Bmatrix} 0 \\ f(y_1(\cdot)) \end{Bmatrix}, \, F'[\eta](y) = \begin{Bmatrix} 0 \\ f'(\eta_1(\cdot))y_1(\cdot) \end{Bmatrix}, \; F(0) = 0, \tag{19.2.7}$$

$y = [y_1, y_2] \in Y$, $\eta = [\eta_1, \eta_2] \in Y$, with f the function in Eq. (19.1.1a). Because of assumption in Eq. (19.1.2) on f, we have that F is a continuous operator on H or on Y, and, moreover, that

$$\|F'[\eta]\|_{L(H)} + \|F'[\eta]\|_{L(Y)} \leq \text{const., uniformly in } \eta \in Y, \tag{19.2.8}$$

whereas $F(0) = 0$ is a consequence of $f(0) = 0$, Remark 19.1.1.

Additional properties Instead of the differential version of Eq. (19.2.1) on, say, $[\mathcal{D}(A^*)]'$, duality with respect to H as a pivot space, we consider its variation of parameter version

$$y(t) = e^{At}y_0 + (\mathcal{L}u)(t) + (\mathcal{R}Fy)(t), \; y(t) = [w(t), w_t(t)]; \tag{19.2.9}$$

$$y(T) = e^{AT}y_0 + \mathcal{L}_T u + \mathcal{R}_T Fy. \tag{19.2.10}$$

Regarding the operators $\mathcal{L}$ and $\mathcal{L}_T$ The following critical regularity property of the problem in Eq. (19.1.1) with $f \equiv 0$ and $\{w_0, w_1\} = 0$ is known

$$\begin{bmatrix} w_{f\equiv 0}(t) \\ (w_t)_{f\equiv 0}(t) \end{bmatrix} = (\mathcal{L}u)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau = \begin{bmatrix} \mathcal{A}\displaystyle\int_0^t S(t-\tau)Nu(\tau)\,d\tau \\ \mathcal{A}\displaystyle\int_0^t C(t-\tau)Nu(\tau)\,d\tau \end{bmatrix} \tag{19.2.11a}$$

$$: \text{ continuous } L_2(0, T; L_2(\Gamma_1)) \to C([0, T]; H^{\alpha}_{\Gamma_0}(\Omega) \times H^{\alpha-1}(\Omega)) \tag{19.2.11b}$$

$$: \text{closed } L_2(0, T; L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}) \to L_2(0, T; H), \tag{19.2.11c}$$

$\alpha = \frac{3}{4}$ for Ω a parallelepiped, $\alpha = \frac{2}{3}$ for a general smooth bounded domain [28, 32, 53]. The corresponding result with $\alpha > \frac{3}{4}$ is false for dim $\Omega \geq 2$ [28]. For the formulas in Eq. (19.2.11a), see References 57, 26, 32, and 34. Moreover, we have

$$H^{\alpha}_{\Gamma_0}(\Omega) \equiv \mathcal{D}\big(\mathcal{A}^{\frac{\alpha}{2}}\big), \quad H^{\alpha-1}(\Omega) = \big[H_0^{1-\alpha}(\Omega)\big]' \equiv [H^{1-\alpha}(\Omega)]' = \big[\mathcal{D}\big(\mathcal{A}^{(1-\alpha)/2}\big)\big]', \tag{19.2.12}$$

because $H^{\rho}(\Omega) \equiv H_0^{\rho}(\Omega)$, $0 \leq \rho \leq \frac{1}{2}$ [40]. Thus

$$\begin{bmatrix} w_{f\equiv 0}(T) \\ (w_t)_{f\equiv 0}(T) \end{bmatrix} = \mathcal{L}_T u \equiv (\mathcal{L}u)(T): \text{ continuous } L_2(0, T; L_2(\Gamma_1)) \to H^{\alpha}_{\Gamma_0}(\Omega) \times H^{\alpha-1}_{\Gamma_0}(\Omega), \tag{19.2.13a}$$

$$\text{closed } L_2(0, T; L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \to H. \tag{19.2.13b}$$

When viewing $\mathcal{L}_T$ as a closed operator as in Eq. (19.2.13b), we shall consider $\mathcal{D}(\mathcal{L}_T)$ as a Hilbert space endowed with the graph norm.

For the analysis of the semilinear exact controllability problem to be carried out in Section 19.4, the following property—identified as (P.1)—plays a critical role:

(P.1) (key property of $\mathcal{L}$)

$$\mathcal{L}: \text{ compact } L_2(0,T;L_2(\Gamma_1)) \to \mathcal{E}_T \equiv L_2(0,T;Y);\ Y \equiv L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'. \tag{19.2.14}$$

The property in Eq. (19.2.14) is obtained by application of Aubin's lemma [2], which requires (the weaker version, L_2 in time of) Eq. (19.2.11b), as well as

$$\left(\frac{d}{dt}\mathcal{L}u\right)(t) = \left\{\begin{array}{c} \mathcal{A}\displaystyle\int_0^t C(t-\tau)Nu(\tau)\,d\tau \\ \mathcal{A}\left[Nu(t) - \mathcal{A}\displaystyle\int_0^t S(t-\tau)Nu(\tau)\,d\tau\right] \end{array}\right\} \tag{19.2.15a}$$

$$: \text{ continuous } L_2(0,T;L_2(\Gamma_1)) \to L_2(0,T;H^{\alpha-1}(\Omega) \times (\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))'), \tag{19.2.15b}$$

along with compactness in space, from $H^{\alpha}(\Omega) \times H^{\alpha-1}(\Omega) \to Y$.

Regarding the operators $\mathcal{R}$, $\mathcal{R}_T$ We recall from Eq. (19.2.4) that $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ and $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. Then

$$(\mathcal{R}g)(t) = \int_0^t e^{A(t-\tau)}g(\tau)\,d\tau : \text{ continuous } L_1(0,T;Y) \to C([0,T];Y); \tag{19.2.16a}$$

$$\text{continuous } L_1(0,T;H) \to C([0,T];H). \tag{19.2.16b}$$

$$\mathcal{R}_T g = (\mathcal{R}g)(T) : \text{ continuous } L_1(0,T;Y) \to Y; \tag{19.2.17a}$$

$$\text{continuous } L_1(0,T;H) \to H. \tag{19.2.17b}$$

The linearized problem The linearized problem corresponding to Eq. (19.2.1) is given by

$$\dot{z} = Az + F'[\eta]z + Bu, \quad z(0) = z_0 \in H, \eta \in Y; \tag{19.2.18}$$

$$z(t) = e^{At}z_0 + (\mathcal{L}u)(t) + (\mathcal{K}[\eta]z)(t); \tag{19.2.19}$$

$$z(T) = e^{AT}z_0 + \mathcal{L}_T u + \mathcal{K}_T[\eta]z \tag{19.2.20}$$

(its PDE version will be given in Eq. [19.2.37] below), with $\mathcal{L}$, $\mathcal{L}_T$ defined in Eq. (19.2.11) and (19.2.13). Moreover, $\mathcal{K}[\eta]g$ and $\mathcal{K}_T[\eta]g$, are defined as follows for $\eta = [\eta_1, \eta_2]$, $g = \{g_1, g_2\} \in \mathcal{E}_T \equiv L_2\{0,T;L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'\}$, via Eqs. (19.2.7), (19.2.3), and (19.2.16):

$$\mathcal{K}[\eta] = \mathcal{R}F'[\eta]; \quad (\mathcal{K}[\eta]g)(t) = \int_0^t e^{A(t-\tau)}F'[\eta(\tau)]g(\tau)\,d\tau \tag{19.2.21}$$

$$= \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix} d\tau = \begin{bmatrix} \displaystyle\int_0^t S(t-\tau)f'(\eta_1(\tau))g_1(\tau)\,d\tau \\ \displaystyle\int_0^t C(t-\tau)f'(\eta_1(\tau))g_1(\tau)\,d\tau \end{bmatrix} \tag{19.2.22a}$$

$$: \text{continuous } \mathcal{E}_T \equiv L_2(0,T;Y) \to C([0,T];H) \subset \mathcal{E}_T. \tag{19.2.22b}$$

Recalling $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, we obtain via also (19.1.2) used in (19.2.22),

$$\|\mathcal{K}[\eta]g\|_{\mathcal{E}_T} \leq \sqrt{T}\|\mathcal{K}[\eta]g\|_{C([0,T];H)} \leq C\|g_1\|_{L_2(0,T;L_2(\Omega))} \leq C\|g\|_{\mathcal{E}_T},$$

$$\text{uniformly in } \eta \in \mathcal{E}_T \equiv L_2\left(0,T;L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'\right); \tag{19.2.23}$$

$$\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta]: \text{ continuous } \mathcal{E}_T \to H, \text{ uniformly in } \eta \in \mathcal{E}_T. \tag{19.2.24a}$$

$$\mathcal{K}_T[\eta]g = \int_0^T e^{A(T-t)} F'[\eta(t)]g(t)\,dt = \int_0^T e^{A(T-t)} \begin{bmatrix} 0 \\ f'(\eta_1(t))g_1(t) \end{bmatrix} dt. \tag{19.2.24b}$$

Similarly, from Eq. (19.2.22),

$$\left(\frac{d}{dt}\mathcal{K}[\eta]g\right)(t) = \begin{bmatrix} \displaystyle\int_0^t C(t-\tau) f'(\eta_1(\tau))g_1(\tau)\,d\tau \\ f'(\eta_1(t))g_1(t) - \mathcal{A}\displaystyle\int_0^t S(t-\tau) f'(\eta_1(\tau))g_1(\tau)\,2d\tau \end{bmatrix}; \tag{19.2.25}$$

$$\left\| \frac{d}{dt}\mathcal{K}[\eta]g \right\|_{L_2(0,T;L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')} \leq C\|g_1\|_{L_2(0,T;L_2(\Omega))} \leq C\|g\|_{\mathcal{E}_T},$$
$$\text{uniformly in } \eta \in \mathcal{E}_T \equiv L_2\big(0,T;L_2(\Omega)\times\big[\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\big]'\big). \tag{19.2.26}$$

Again via Aubin's Lemma [2], which uses Eq. (19.2.23) and (19.2.26), we obtain the following property of $\mathcal{K}[\eta] \equiv \mathcal{R}F'[\eta]$, which plays a critical role in the subsequent analysis of the semilinear exact controllability problem in Section 19.4.

(P.2) (key property of $\mathcal{K}[\eta] \equiv \mathcal{R}F'[\eta]$)

With $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, not only do we have that

$$\mathcal{K}[\eta] = \mathcal{R}F'[\eta]: \text{ compact } \mathcal{E}_T \to \mathcal{E}_T, \quad \text{for each } \eta \text{ fixed } \in \mathcal{E}_T \equiv L_2(0,T;Y), \tag{19.2.27}$$

but, in addition, we have that

(a)

$$\begin{cases} \text{the family of operators } \{\mathcal{K}[\eta]\}_{\eta\in\mathcal{E}_T} \text{ is collectively compact on the space} \\ \mathcal{E}_T \equiv L_2(0,T;Y)\text{: that is Reference 1, p. 4, the set union } \bigcup_{\eta\in\mathcal{E}_T} \mathcal{K}[\eta] \\ \text{(unit ball of } \mathcal{E}_T) \text{ is a precompact set in } \mathcal{E}_T, \end{cases} \tag{19.2.28}$$

where the union of the image of the unit ball of $\mathcal{E}_T$ under the operator $\mathcal{K}[\eta]$ is taken over all $\eta \in \mathcal{E}_T$. The validity of the property in Eq. (19.2.28) is, of course, because estimates of Eq. (19.2.23) and (19.2.26) are *uniform* in $\eta \in \mathcal{E}_T$.

Moreover, $\mathcal{K}[\eta]$ satisfies the following additional property.

(b) For any sequence $\eta_n \in \mathcal{E}_T \equiv L_2(0,T;Y)$, $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, we can extract a subsequence η_{n_k} such that

$$\mathcal{K}[\eta_{n_k}] \equiv \mathcal{R}F'[\eta_{n_k}] \to \mathcal{K}^0 \equiv \mathcal{R}F_0 \text{ strongly in } \mathcal{E}_T \equiv L_2(0,T;Y), \tag{19.2.29a}$$

$$(K^0 g)(t) = (\mathcal{R}F_0 g)(t) = \int_0^t e^{A(t-\tau)} F_0(\tau)g(\tau)\,d\tau \tag{19.2.29b}$$

$$= \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ f_0(\tau)g_1(\tau) \end{bmatrix} d\tau, \tag{19.2.29c}$$

for a suitable operator $F_0 \in L(Y)$, possibly depending on the subsequence. The proof of the property in Eq. (19.2.29) follows as in Reference 29, Proposition 3.3(b), p. 129. In short, given $\eta_n \in \mathcal{E}_T$, the assumption of Eq. (19.1.2) yields $|f'(\eta_{n,1})| \leq$ const., uniformly in n, so that by Alaoglou's Theorem one can extract a subsequence $\eta_{n_k,1} \in L_2(0,T;L_2(\Omega))$ such that $f'(\eta_{n_k,1}) \to$ some f_0 in $L^\infty(\mathbb{R})$-weak star. One then defines

$$F_0 y \equiv \begin{bmatrix} 0 \\ f_0(\,\cdot\,)y_1 \end{bmatrix}, \ \|F_0 y\|_Y = \|f_0(\,\cdot\,)y_1\|_{[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \le c\|f_0(\,\cdot\,)y_1\|_{L_2(\Omega)} \le c\|y_1\|_{L_2(\Omega)} \le c\|y\|_Y. \tag{19.2.30}$$

Then, one obtains that

$$\begin{bmatrix} \mathcal{A}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{A}^{\frac{1}{2}} \end{bmatrix} \mathcal{K}[\eta_{n_k}] \to \begin{bmatrix} \mathcal{A}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{A}^{\frac{1}{2}} \end{bmatrix} \mathcal{R}F_0 \text{ weakly in } \mathcal{E}_T \equiv L_2(0,T;Y), \tag{19.2.31}$$

as well as

$$\frac{d\mathcal{K}[\eta_{n_k}]}{dt} \to \frac{d\mathcal{R}F_0}{dt} \text{ weakly in } \mathcal{E}_T \equiv L_2(0,T;Y). \tag{19.2.32}$$

Then, as a consequence of the weak converge of Eq. (19.2.31) (that takes care of the "space"-coordinates) and of Eq. (19.2.32) (that takes care of the "time"-coordinate), as well as of the compactness of $\mathcal{A}^{-\frac{1}{2}}$ on $L_2(\Omega)$, we deduce that the strong convergence of Eq. (19.2.29) holds true, as desired, and property (P.2)(b) is established.

Consequence of (P.2)(a) and (b) As a consequence of the collectively compact property of Eq. (19.2.28) and of the strong convergence of Eq. (19.2.29), it then follows [A1, p. 5] that

$$\mathcal{K}^0 \text{ is compact on } \mathcal{E}_T; \ \|\mathcal{K}[\eta]\|_{L(\mathcal{E}_T)} \le \text{const., uniformly in } \eta \in \mathcal{E}_T \equiv L_2(0,T;Y). \tag{19.2.33}$$

There is one more property this time for the operator $\mathcal{K}_T[\eta] = \mathcal{R}_T F'[\eta]$ that will be needed in the analysis of the exact controllability property of the semilinear model of Eqs. (19.2.1) or (19.1.1):

(P.3) (key property of $\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta]$) For each $\eta \in \mathcal{E}_T \equiv L_2(0,T;Y)$ fixed, where $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, we have

$$\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta] : \text{ continuous } \mathcal{E}_T \equiv L_2(0,T;Y) \to H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega) \tag{19.2.34}$$

as already noted in Eqs. (19.2.23) and (19.2.24) and, moreover, given any sequence $\eta_n \in \mathcal{E}_T$, we can extract a subsequence η_{n_k}, such that the following weak convergence takes place:

$$\begin{cases} \mathcal{K}_T[\eta_{n_k}] \equiv \mathcal{R}_T F'[\eta_{n_k}] \to \text{ some } \mathcal{K}_T^0 \equiv \mathcal{R}_T F_0, \\ \text{weakly from } \mathcal{E}_T \equiv L_2(0,T;Y) \to H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega), \end{cases} \tag{19.2.35a}$$

$$K_T^0 g = \mathcal{R}_T F_0 g = \int_0^T e^{A(T-t)} F_0(t) g(t)\, dt = \int_0^T e^{A(T-t)} \begin{bmatrix} 0 \\ f_0(t) g_1(t) \end{bmatrix} dt, \tag{19.2.35b}$$

for some operator $F_0 \in L(Y)$, possibly depending on the subsequence, which can be taken to be the same that arises in Eq. (19.2.29).

The validity of the property of Eq. (19.2.35) follows through an argument similar to that yielding property (P.2)(b) = Eq. (19.2.29); see details in Refercence 29, Proposition 3.4, p. 131.

Consequence of (P.3) As a consequence of the properties of Eqs. (19.2.35) and (19.2.8), the latter being a consequence of the standing assumption of Eq. (19.1.2) on f, we obtain

$$\|\mathcal{K}_T[\eta]\|_{L(\mathcal{E}_T;H)} \le \text{const., uniformly in } \eta \in \mathcal{E}_T. \tag{19.2.36}$$

In PDE terms, the linearized z-problem of Eq. (19.2.18) is given by

$$\begin{cases} \zeta_{tt} = \Delta\zeta + f'[\eta_1(\cdot)]\zeta & \text{in } (0,T]\times\Omega \equiv Q; & (19.2.37a)\\ \zeta(0,\cdot) = \zeta_0, \zeta_t(0,\cdot) = \zeta_1 & \text{in } \Omega; & (19.2.37b)\\ \zeta|_{\Sigma_0} \equiv 0; \left.\dfrac{\partial\zeta}{\partial\nu}\right|_{\Sigma_1} = u & \text{in } (0,T]\times\Gamma_i \equiv \Sigma_i,\ i=0,1, & (19.2.37c)\end{cases}$$

$z_0 = \{\zeta_0, \zeta_1\} \in H$, where $\eta_1 \in L_2(0,T; L_2(\Omega))$. Its dual version is precisely the ϕ-problem in Eqs. (19.3.9a–c) below with potential $q(t,x) = f'(\eta_1(\cdot))$, eventually $q(t,x) = f'(w(t,x))$, therefore satisfying the assumption of only Eq. (19.3.10) below: $q \in L_\infty(Q)$.

19.3 Control-Theoretic Properties

In this subsection we collect control-theoretic properties enjoyed by the system in Eq. (19.1.1) (or Eq. (19.2.1)), to be used in the subsequent analysis of the semilinear exact controllability property in Section 19.4. (1) exact controllability on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of the linear problem of Eqs. (19.1.1) or (19.2.1), with $f \equiv 0$; and (2) an approximate controllability property of the corresponding linearized abstract problem of Eq. (19.2.18), rewritten in PDE terms in Eq. (19.2.37).

Preliminaries

However, before doing so, we need to address the issue that, in the present Neumann-boundary control case, a pathological feature arises (that has no counterpart in the corresponding Dirichlet-boundary control case), namely, the space $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ of finite energy where exact controllability is sought is much smoother (for dim $\Omega \geq 2$) than the space $H^\alpha(\Omega) \times H^{\alpha-1}(\Omega)$, $\alpha = \frac{2}{3}$ or $\frac{3}{4}$, of optimal regularity of the boundary control $\to$ solution operator $\mathcal{L}$ (see Eq. (19.2.11b)) or corresponding operator $\mathcal{L}_T$ (see Eq. (19.2.13a)). In other words, $\mathcal{L}_T$ is *not* continuous from $L_2(0,T; L_2(\Gamma_1))$ to $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, where $H^1_{\Gamma_0}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$. However, $\mathcal{L}$ is *closed* as an operator (see Eq. (19.2.11a)):

$$(\mathcal{L}u)(t) = A\int_0^t e^{A(t-\tau)}A^{-1}Bu(\tau)\,d\tau,\ A^{-1}B \in L(U;H) \tag{19.3.1a}$$

$$: L_2(0,T;U) \supset \mathcal{D}(\mathcal{L}) \to C([0,T]; H \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega)) \text{ closed.} \tag{19.3.1b}$$

This is so, because A is closed on H, $A^{-1} \in L(H)$, and the integral term in Eq. (19.3.1a) is bounded from $L_2(0,T;U)$ to $C([0,T];H)$; Reference 19, p. 164. In addition, we have with reference to Eq. (19.2.11a)

$$\begin{bmatrix} w_{f\equiv 0}(t) \\ (w_t)_{f\equiv 0}(t)\end{bmatrix} = (\mathcal{L}u)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau = \begin{bmatrix} \mathcal{A}\displaystyle\int_0^t S(t-\tau)Nu(\tau)\,d\tau \\ \mathcal{A}\displaystyle\int_0^t C(t-\tau)Nu(\tau)\,d\tau\end{bmatrix} \tag{19.3.2a}$$

$$: \text{ continuous } H^1(0,T;L_2(\Gamma_1)) \to C([0,T];H); \tag{19.3.2b}$$

$H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$. To see Eq. (19.3.2b), one integrates by parts in t each component in Eq. (19.3.2a) and invokes Eq. (19.2.11b) or else invokes Reference 30. Similarly, we have that (see Eq. (19.2.13b))

$$\mathcal{L}_T u \equiv (\mathcal{L}u)(T) \quad : L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \to H = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega) \text{ is closed;} \tag{19.3.3}$$

$$: H^1(0,T;L_2(\Gamma_1)) \to H \text{ is continuous.} \tag{19.3.4}$$

(C.1) Exact Controllability of the Linear Problem in Eq. (19.1.1) ($f \equiv 0$), on the Space $H = H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the Class of $L_2(0, T; L_2(\Gamma_1))$-Neumann Boundary Controls for $T > T_0$

This is the property for the closed operator $\mathcal{L}_T$ that

$$\mathcal{L}_T : L_2(0, T; L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \text{ is surjective } \stackrel{\text{onto}}{\longrightarrow} H \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega). \tag{19.3.5a}$$

Equivalently, (see Reference 56, p. 235) its corresponding (Hilbert) adjoint is bounded below

$$\|\mathcal{L}_T^* h\|_{L_2(0,T;L_2(\Gamma_1))} \geq C_T \|h\|_H, \quad h \in H. \tag{19.3.5b}$$

It is well known that such property for the problem in Eq. (19.1.1) with $f \equiv 0$ holds true under some geometric conditions and for all T sufficiently large [4, 27, 39, 49–51].

One Setting of Geometric Conditions For the sake of concreteness we shall make reference to one such setting even though it is not the most general one, which is in Reference 4. Let $d : \Omega \Rightarrow \mathbb{R}^+$ be a C^3-strictly convex function and let $h(x) = \nabla d(x)$ be the corresponding conservative vector field, so that, if $\mathcal{H}_d(x)$ and $J_h(x)$ denote, respectively, the Hessian matrix of d and Jacobian matrix of h, then it holds that

$$\mathcal{H}_d(x) \equiv J_h(x) \geq 2\rho > 0, \quad \forall\, x \in \Omega, \tag{19.3.6}$$

for some constant $\rho > 0$. Assume further that

$$h(x) \cdot \nu(x) \leq 0 \ \text{ on } \Gamma_0, \quad \nu(x) = \text{outward unit normal}. \tag{19.3.7}$$

Define

$$T_0 = 2\left[\frac{\max_{x\in\overline{\Omega}} d(x)}{\rho}\right]^{\frac{1}{2}}, \ \rho \text{ as in Eq. (19.3.6)}. \tag{19.3.8}$$

Then, exact controllability of the linear problem of Eq. (19.1.1) with $f \equiv 0$ is *guaranteed to hold true* on $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ *within the class* of $L_2[0, T; L_2(\Gamma_1)]$ *controls, under the geometric conditions* of Eqs. (19.3.6) and (19.3.7), *for all* $T > T_0$ [33, 36, 37].

(C.2) Approximate Controllability (Unique Continuation) of the Linearized System of Eq. (19.2.18) or Its PDE-Version Eq. (19.2.37) and Its Limit Version Involving the Limits in Eqs. (19.2.29) and (19.2.35)

We consider the following homogeneous backward problem:

$$\begin{cases} \phi_{tt} = \Delta\phi + q(t,x)\phi & \text{in } (0, T] \times \Omega \equiv Q; \quad (19.3.9a) \\ \phi(T, \cdot) = \phi_0, \phi_t(T, \cdot) = \phi_1 & \text{in } \Omega; \quad (19.3.9b) \\ \phi|_{\Sigma_0} \equiv \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } (0, T] \times \Gamma_i \equiv \Sigma_i, \quad (19.3.9c) \end{cases}$$

under the standing assumption that

$$\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times \big[\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\big]'; \quad q(t,x) \in L_\infty(Q). \tag{19.3.10}$$

We begin by recalling a relevant regularity result.

THEOREM 19.3.1

With reference to the ϕ-problem of Eq. (19.3.9)*, with q as in Eq.* (19.3.10)*, we have*

$$\{\phi_0, \phi_1\} \in L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]' \Rightarrow \{\phi, \phi_t\} \in C\left([0, T]; L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'\right) \quad (19.3.11a)$$

continuously. More precisely,

$$\|\{\phi, \phi_t\}\|^2_{C([0,T];L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')} \leq C_T(1 + e^{M_q T}\|q\|_{L_\infty(Q)})\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \quad (19.3.11b)$$

One may provide several standard proofs of Eq. (19.3.11a): by fixed point (contraction) by the evolution operator $U(t, s)$ as in Appendix 19.B below, etc. In Appendix 19.C, Theorem 19.C.1, we provide, for completeness, a proof that has the advantage of showing the dependence upon q of the constant of continuity, as in Eq. (19.3.11b).

Unique Continuation Property We now pass to the relevant unique continuation result, or (by duality) approximate controllability.

THEOREM 19.3.2

Consider the problem in Eqs. (19.3.9a–b) *with $\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ and B.C.*

$$\phi|_\Sigma \equiv 0, \quad \left.\frac{\partial \phi}{\partial \nu}\right|_{\Sigma_1} \equiv 0, \quad (19.3.12)$$

in place of Eq. (19.3.9c)*. Assume the geometric conditions of Eqs.* (19.3.6) *and* (19.3.7)*, and let $T > T_0$ as defined in Eq.* (19.3.8)*. Then, in fact, $\phi_0 = \phi_1 = 0$.*

PROOF OF THEOREM 19.3.2 [see Reference 29; p. 133–134].

Step 1. First, under the weaker B.C. Eq. (19.3.9c), Theorem 19.3.1 provides the *a-priori* regularity of Eq. (19.3.11a) for $\{\phi, \phi_t\} \in C([0, T]; L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$. Using this information, we next boost the regularity using the full strength of the over determined B.C. of Eq. (19.3.12) and obtain that, for $T > T_0$, we have

$$\infty > \|\phi\|^2_{L_\infty(0,T;L_2(\Omega))} \geq C_T(T - T_0)\|\{\phi, \phi_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)}. \quad (19.3.13)$$

In view of Eq. (19.3.13), we actually have that the corresponding solution satisfies

$$\{\phi_0, \phi_1\} \in \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega), \quad \text{hence } \{\phi, \phi_t\} \in C\left([0, T]; \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega)\right), \quad (19.3.14)$$

as it follows by multiplying Eq. (19.3.9) by ϕ_t and using the Gronwall inequality (as in Eq. (19.C.11) below). The inequality of Eq. (19.3.13) for $T > T_0$ (defined in Eq. (19.3.8)) is already in the literature. In fact, one invokes Reference 33, Theorem 2.1.2, Eq. (2.1.10a), p. 222, or Reference 36, Theorem 3.4, Eq. (3.15), p. 30 (with forcing term $\equiv 0$), combined with the inequality of Reference 33, (2.4.3), p. 237 or Reference 36, (5.2.1), p. 46, yielding the boundary terms $\overline{BT}|_\Sigma = \frac{1}{2}\int_{\Sigma_0} e^{\tau\varphi}\left(\frac{\partial\phi}{\partial\nu}\right)^2 h \cdot \nu \, d\Sigma_0 \leq 0$ under the B.C. of Eq. (19.3.12), with φ as in this Reference.

Step 2. With the regularity boosted to the $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$-level as stated in Eq. (19.3.14), we can now invoke available uniqueness theorem [38, 60, 59]. Indeed, this uniqueness result follows as a very special case of inequality of Eq. (19.10.12) below in a Riemannian setting, taken from Reference 59, and previously available in the Euclidean setting [20, 38, 60]. However, References 45, 16, and 17 do not apply, as they require $\Gamma_0 = \emptyset$. In this connection, we quote also [3, 12–15].

Approximate Controllability of a ζ-Problem The above treatment casts property Eq. (19.C.2) as a unique continuation property of the ϕ-problem Eq. (19.3.9) with I.C. $\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. We now justify why this property Eq. (19.C.2) is also, equivalently, an approximate controllability property.

Indeed, we note that the ϕ-problem of Eq. (19.3.9) is dual to the ζ-problem of Eq. (19.2.37) with $f'(\eta_1(t,x)) \equiv q(t,x)$, see Appendix A or Appendix B. [Ultimately, $\eta_1(t,x) = w(t,x)$]. Then, the uniqueness property in Theorem 19.3.2 with $q(t,x) = f'(\eta_1(t,x))$ means, by duality, that

$$\text{the map } u \in L_2(0,T;U) \to \mathcal{M}_T[q]u \equiv \{\zeta(T), \zeta_t(T)\} \text{ has dense range in } H, \tag{19.3.15}$$

where $U = L_2(\Gamma_1)$, $H = H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, which is precisely an approximate controllability property of the corresponding controlled ζ-problem.

When referring particularly to the z-problem Eq. (19.2.18) or its PDE version, the ζ-problem Eq. (19.2.37), both depending on the parameter $\eta = [\eta_1, \eta_2] \in \mathcal{E}_T$, we shall write alternatively $\mathcal{M}_T[\eta]$ in place of $\mathcal{M}_T[q]$ for the map in Eq. (19.3.15) where, of course, $q(t,x) = f'[\eta_1(t,x)]$. Below, in the analysis of Section 19.4, we shall obtain an explicit formula for the map $\mathcal{M}_T[\eta]$. It is given by (see Eq. [19.4.33])

$$\mathcal{M}_T[\eta] \equiv \mathcal{L}_T + \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L} : L_2(0,T;U) \supset \mathcal{D}(\mathcal{M}_T[\eta]) = \mathcal{D}(\mathcal{L}_T) \to H. \tag{19.3.16}$$

Similarly, for the limit version given by Eqs. (19.2.29a) and (19.2.35a), we have likewise

$$\mathcal{M}^0_T \equiv \mathcal{L}_T + \mathcal{K}^0_T(I - \mathcal{K}^0)^{-1}\mathcal{L} : \mathcal{D}(\mathcal{M}_T) \to H. \tag{19.3.17}$$

This corresponds to the ζ-problem of Eq. (19.2.37) with potential $q = f_0(\cdot)$ as obtained in Eq. (19.2.29b) or (19.2.35b).

Thus, the denseness property of Eq. (19.3.15) encompasses, in particular, the following statement: for each fixed $\eta \in \mathcal{E}_T$, the map $\mathcal{M}_T[\eta]$ in Eq. (19.3.16), as well as the map $\mathcal{M}^0_T$ in Eq. (19.3.17), each have *range dense* in H.

19.4 Exact Controllability of the Semilinear Problem of Eqs. (19.1.1) and (19.1.2) under Properties (P.1), (P.2), (P.3), (C.1), (C.2). Proof of Theorem 19.1.1 by the Global Inversion Approach [29]

So far, our treatment of the semilinear problem of Eq. (19.1.1), under the assumption of Eq. (19.1.2) on the nonlinearity, has recast Eq. (19.1.1) as the abstract semilinear Eq. (19.2.1) subject to the property of Eq. (19.2.8) and has extracted several additional key properties of Eq. (19.1.1), which have all been expressed in abstract form: the structural properties (P.1), (P.2), (P.3), as well as the control-theoretic properties (C.1) and (C.2). We shall now use these properties to establish exact controllability of Eq. (19.1.1) through the abstract approach of Reference 29, based on a global inversion theorem [7, 46]. We thus make reference to Eq. (19.2.1) where $y(t) = \{w(t), w_t(t)\}$ in regard to Eq. (19.1.1).

The present section corresponds to Reference 29, Section 2.1 revisited, with the key operator $\mathcal{L}_T$, however, unbounded (but closed) $L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T \text{ onto } H)$.

Step 1. Let $y_0 \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ and $u \in \mathcal{D}(\mathcal{L})$, where $H^1(0,T;L_2(\Gamma_1)) \subset \mathcal{D}(\mathcal{L}) \subset L_2[0,T;L_2(\Gamma_1)]$, see Eqs. (19.3.1b) and (19.3.2b). By a fixed point argument, we then obtain that $y(t) \equiv \{w(t), w_t(t)\} \in C([0,T];H)$. In this setting, we return to the variation of parameter formula of Eq. (19.2.9), rewritten as $[I - \mathcal{R}F]y_u = e^{A\cdot}y_0 + \mathcal{L}u \in C([0,T];H)$ (we note the dependence of y on u) and obtain the solution

$$y_u(t) = \{[I - \mathcal{R}F]^{-1}[e^{A\cdot}y_0 + \mathcal{L}u]\}(t) \in C([0,T];H), \ u \in \mathcal{D}(\mathcal{L}), \tag{19.4.1}$$

directly in terms of the data, in particular for $u \in H^1(0, T; L_2(\Gamma_1))$. In view of Eq. (19.1.2), we have

$$[I - \mathcal{R}F]^{-1} : \text{ bounded and continuous } C([0, T]; H) \to C([0, T]; H). \tag{19.4.2}$$

The validity of Eq. (19.4.2) follows from solving uniquely on successive intervals of suitably small but equal size $[0, T_1]$, $[T_1, 2T_1]$, $T_1 > 0$, etc., the corresponding equation

$$[I - \mathcal{R}F]\xi = \mathcal{X} \in C([0, T]; H), \quad \text{or} \quad \xi(t) - \int_0^t e^{A(t-\tau)} F[\xi(\tau)]\, d\tau = \mathcal{X}(t), \tag{19.4.3}$$

using the property, from Eq. (19.1.2), that F in Eq. (19.2.7) is globally Lipschitz on $H : \|F(\xi)\|_H \leq \text{const}\|\xi\|_H$.

We return to the variation of the parameter formula of Eq. (19.2.10) for $t = T$:

$$y_u(T) = e^{AT} y_0 + \mathcal{L}_T u + \mathcal{R}_T F y_u \in H, \quad y_0 \in H, \quad u \in \mathcal{D}(\mathcal{L}_T), \tag{19.4.4}$$

in particular, for $u \in H^1[0, T; L_2(\Gamma_1)]$. Without loss of generality for our present exact controllability problem, we *can restrict to controls* $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$ throughout this section, where

$$\mathcal{N}(\mathcal{L}_T) = \{u \in \mathcal{D}(\mathcal{L}_T) : \mathcal{L}_T u = 0\}; \tag{19.4.5}$$

a closed subspace of $L_2(0, T; U)$ by the closedness of $\mathcal{L}_T$ in Eqs. (19.3.3), and $[\quad]^\perp$ denotes the orthogonal complement in $L_2[0, T; L_2(\Gamma_1)]$ to the closed subspace $[\quad]$,

$$L_2(0, T; L_2(\Gamma_1)) = [\mathcal{N}(\mathcal{L}_T)] + [\mathcal{N}(\mathcal{L}_T)]^\perp. \tag{19.4.6}$$

The closed complementary subspaces $\mathcal{N}(\mathcal{L}_T)$ and $[\mathcal{N}(\mathcal{L}_T)]^\perp$ will be *always topologized* by *the inherited* $L_2(0, T; U)$-norm, $U = L_2(\Gamma_1)$. Thus, the *exact controllability problem* under investigation *can be reformulated as follows*: Given $T > 0$, $y_0 \in H$, $y_T \in H$, seek, if possible, a control $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0, T; L_2(\Gamma_1))$ such that

$$y_T = y_u(T) = e^{AT} y_0 + \mathcal{L}_T u + \mathcal{R}_T F y_u, \quad u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \tag{19.4.7}$$

Henceforth, $[\mathcal{N}(\mathcal{L}_T)]^\perp$ will be *our control space* and $\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}$, by which we mean the restriction of $\mathcal{L}_T$ on $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$, is injective.

Step 2. We now invoke the exact controllability property (19.C.1) = Eq. (19.3.5) for the linear system, whereby

$$\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp} : \text{closed, injective, surjective from } [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \text{ onto } H. \tag{19.4.8}$$

By the open mapping theorem, we then obtain that

$$\mathcal{L}_T^{\#} \equiv \left(\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}\right)^{-1}: \text{ bounded, injective, surjective} H \overset{\text{onto}}{\longrightarrow} [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0, T; U), \tag{19.4.9}$$

where $\mathcal{L}_T^{\#}$ is the pseudo-inverse of $\mathcal{L}_T$. Thus, $\mathcal{L}_T^{\#}$ is an *isomorphism* between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$, in the sense of Reference 56, p. 15: a one-to-one correspondence between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$. $\mathcal{L}_T^{\#}$ is bounded in Eq. (19.4.9), but $(\mathcal{L}_T^{\#})^{-1} = \mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}$ is not bounded from $L_2(0, T; U)$ to H. It is bounded only if the subspace $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ of $L_2(0, T; L_2(\Gamma_1))$ is topologized by the inherited norm

$$\|u\|_{[\mathcal{N}(\mathcal{L}_T)]^\perp} \equiv \|(\mathcal{L}|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})u\|_H,$$

which is equivalent to the graph norm, in which case $\mathcal{L}_T^{\#}$ becomes a *topological isomorphism* ($\mathcal{L}_T^{\#}$ and $(\mathcal{L}_T^{\#})^{-1}$ both continuous with such topologies) or *linear homeomorphism*. We shall not use this property. An explicit formula for $\mathcal{L}_T^{\#}$ is known, under the present property (19.C.1) = Eq. (19.3.5). Indeed, by a well-known result in Functional Analysis (see Reference 56, p. 235), the surjectivity property of Eq. (19.3.5a) for the closed operator $\mathcal{L}_T$ is *equivalent* to the condition that the Hilbert space adjoint: $\mathcal{L}_T^* : H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0, T; L_2(\Gamma_1))$ is bounded below as in Eq. (19.3.5b)

$$\|\mathcal{L}_T^* x\|_{L_2(0,T;L_2(\Gamma_1))} \geq C_T \|x\|_H, \quad \forall\, x \in \mathcal{D}(\mathcal{L}_T^*) \tag{19.4.10}$$

(sometimes the Banach-space adjoint is more convenient to take).

REMARK 19.4.1 Throughout this paper, $*$ will denote adjoint with respect to the space H of exact controllability. Other adjoints, with respect to other spaces, will be denoted by a different symbol, as documented in Remark 19.4.3.

The inequality of Eq. (19.4.10) is the corresponding COI for the linear problem of Eq. (19.1.1) with $f \equiv 0$. Based on Eq. (19.4.10), which is under the exact controllability property (C.1) = Eq. (19.3.5), the minimization problem of steering the origin "O" (rest) to the target $y_T \in H$ while minimizing the $L_2(0, T; U)$-norm leads via a Lagrange multiplier argument [Reference 58, Appendix] to the following explicit expression for $\mathcal{L}_T^{\#}$:

$$\mathcal{L}_T^{\#} = \mathcal{L}_T^*[\mathcal{L}_T\mathcal{L}_T^*]^{-1} : \text{ bounded, injective } H \to [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(0, T; L_2(\Gamma_1)) \tag{19.4.11}$$

($\mathcal{L}_T\mathcal{L}_T^*$ defines an isomorphism from H onto its dual H' by the Lax-Milgram lemma applied to Eq. (19.4.10)). The pseudo-inverse $\mathcal{L}_T^{\#}$ enjoys the usual properties

$$\mathcal{L}_T\mathcal{L}_T^{\#} = \text{identity on } H; \quad \mathcal{L}_T^{\#}\mathcal{L}_T = \text{identity on } [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \tag{19.4.12}$$

Next, applying $\mathcal{L}_T^{\#}$ on Eq. (19.4.7) and using Eq. (19.4.12) (right side), we see that the *exact controllability problem* under investigation *can be reformulated as follows*: Given $T > 0$, $y_0 \in H$, $y_T \in H$, seek, if possible, a control $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0, T; L_2(\Gamma_1))$, such that

$$u + \mathcal{L}_T^{\#}\mathcal{R}_T F(y_u) = \mathcal{L}_T^{\#}\left(y_T - e^{AT} y_0\right), \tag{19.4.13}$$

or, after substituting Eq. (19.4.1) for y_u into Eq. (19.4.13), such that

$$u + \mathcal{L}_T^{\#}\mathcal{R}_T F[I - \mathcal{R}F]^{-1}[e^{A\cdot} y_0 + \mathcal{L}u] = \mathcal{L}_T^{\#}\left[y_T - e^{AT} y_0\right]. \tag{19.4.14}$$

On the other hand, the exact controllability property of Eq. (19.C.1) = Eq. (19.3.5) for the linear system ensures that, given y_T and y_0 in H, there exists a control $v_T^0 \in \mathcal{D}(\mathcal{L}_T)$—which without loss of generality we may also take in $[\mathcal{N}(\mathcal{L}_T)]^\perp$—such that (recall Eq. (19.4.12) (right side)):

$$\begin{aligned} &\mathcal{L}_T v_T^0 = y_T - e^{AT} y_0, \\ &\text{hence} \quad \mathcal{L}_T^{\#}\mathcal{L}_T v_T^0 = v_T^0 = \mathcal{L}_T^{\#}\left[y_T - e^{AT} y_0\right], \quad v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \end{aligned} \tag{19.4.15}$$

Thus, using Eq. (19.4.15) on the right-hand side of Eq. (19.4.14), we conclude the following: Given $T > 0$, in order to solve the semilinear exact controllability problem steering y_0 at $t = 0$ to y_T at $t = T$ along Eq. (19.1.1) or Eq. (19.2.1), we seek a control $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$, such that

$$u + \Lambda_T(u) = v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \equiv \text{range}(\mathcal{L}_T^{\#}), \tag{19.4.16}$$

where we have set, via Eqs. (19.4.13) and (19.4.14),

$$\begin{aligned} \Lambda_T(u) \equiv \mathcal{L}_T^{\#}\mathcal{R}_T F(y_u) &= \mathcal{L}_T^{\#}\mathcal{R}_T F[I - \mathcal{R}F]^{-1}[e^{A\cdot} y_0 + \mathcal{L}u] \subset \text{range}(\mathcal{L}_T^{\#}) \\ &= [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \end{aligned} \tag{19.4.17}$$

and where $v_T^0 = \mathcal{L}_T^{\#}[y_T - e^{AT}y_0] \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ is, by Eq. (19.4.15) (left), the control steering y_0 at $t = 0$ to y_T at time $t = T$, along the corresponding *linear* dynamics of Eq. (19.1.1) with $f \equiv 0$.

Notice that given $v_T^0 \in \text{range}(\mathcal{L}_T^{\#})$, then if Eq. (19.4.16) is solvable for $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$ (the control space), then by necessity, $u \in \text{range}(\mathcal{L}_T^{\#})$ as well, because $\Lambda_T(u) \in \text{range}(\mathcal{L}_T^{\#})$.

The converse is also true. Let $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ be a control steering $y_0 \in H$ at $t = 0$ to $y_T \in H$ at $t = T$, along the solution of the semilinear problem of Eq. (19.2.1). Then, u satisfies Eq. (19.4.7) and, by Eq. (19.4.12), u satisfies Eqs. (19.4.13) and (19.4.14) as well. Then the control $v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ defined by Eq. (19.4.16) has the property that $\mathcal{L}_T v_T^0 = y_T - e^{AT}y_0$, see Eqs. (19.4.15) and (19.4.12), and hence steers y_0 at $t = 0$ to the state y_T at $t = T$ along the linear dynamics of Eq. (19.1.1) with $f \equiv 0$. Thus, (C.1) = Eq. (19.3.5) is satisfied. We have obtained the following result.

PROPOSITION 19.4.1 *Assume (1.2).*

(a) Given $T > 0$, $y_0 = \{w_0, w_1\} \in H$, $y_T = \{\tilde{w}_0, \tilde{w}_1\} \in H$, $H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, there exists a control $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T) \subset L_2(0, T; L_2(\Gamma_1))$ steering y_0 at $t = 0$ to y_T at $t = T$ along the semilinear system of Eq. (19.1.1) *subject to Eq.* (19.1.2) *[or the system of Eq.* (19.2.1) *subject to Eq.* (19.2.8) *if and only if* (19.C.1) = *of Eq.* (19.3.5) *holds true and such steering control u satisfies identity of Eq.* (19.4.16)*, where $v_T^0 = \mathcal{L}_T^{\#}[y_T - e^{AT}y_0] \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ is then the control steering likewise y_0 at $t = 0$ to the target y_T at time $t = T$, along the corresponding linear dynamics of Eq.* (19.1.1) *with $f \equiv 0$ [(2.1) with $F \equiv 0$].*

(b) A sufficient condition for part (a) to hold true—that is, the exact controllability of the semilinear problem of Eq. (19.1.1) *or Eq.* (19.2.1)*—is the global inversion of the C^1 map $u \to u + \Lambda_T(u)$: $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \to$ itself, in Eq.* (19.4.16)*. More precisely, such a map $u \to u + \Lambda_T(u)$ is a homeomorphism (bijective, continuous, with continuous inverse), provided that (Reference 7, p. 153):*

(b1) its Frechet derivative $I + \Lambda_T'(u)$ has a bounded inverse: $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \to [\mathcal{N}(\mathcal{L}_T)]^{\perp}$;

(b2) the norm of such inverse operator grows at most linearly in $\|u\|_{L_2(0,T;L_2(\Gamma_1))}$:

$$\|[I + \Lambda_T'(u)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^{\perp})} \leq C\left[1 + \|u\|_{L_2(0,T;L_2(\Gamma_1))}\right], \quad \forall\, u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \tag{19.4.18a}$$

$[[\mathcal{N}(\mathcal{L}_T)]^{\perp}$ is topologized by $L_2(0, T; L_2(\Gamma_1))]$ a condition a-fortiori *satisfied provided that*

$$\|[I + \Lambda_T'(u)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^{\perp})} \leq C, \quad \forall\, u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}. \tag{19.4.18b}$$

An explicit expression of $\Lambda_T'(u)$, $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$, is:

$$\begin{aligned}\Lambda_T'(u) = \Lambda_T'(y_u) = \mathcal{L}_T^{\#}\mathcal{K}_T[y_u]y_u' &\equiv \mathcal{L}_T^{\#}\mathcal{K}_T[y_u][I - \mathcal{K}[y_u]]^{-1}\mathcal{L} \\ &: \text{compact } L_2(0, T; U) \supset [\mathcal{N}(\mathcal{L}_T)]^{\perp} \to [\mathcal{N}(\mathcal{L}_T)^{\perp};\end{aligned} \tag{19.4.19a}$$

$$\begin{aligned}\{I - \mathcal{K}[y_u]\}^{-1} &\text{ bounded on } C([0, T]; H);\ \text{or on } L_2(0, T; H); \\ &\text{or on } C([0, T]; Y);\ \text{or on } L_2(0, T; Y).\end{aligned} \tag{19.4.19b}$$

PROOF Part (a) was established in Step 2 above.

Part (b) is nothing but a well-known criterion for global inversion in nonlinear analysis—see Reference 7, Theorem 15.4, p. 153 for Eq. (19.4.18a), and References 5 and 46, p. 16—for Eq. (19.4.18b). We now verify Eq. (19.4.19). The operator $\Lambda_T(u) = \mathcal{L}_T^{\#}\mathcal{R}_T F(y_u)$ in Eq. (19.4.17) is Frechet differentiable on, say, $H^1(0, T; L_2(\Gamma_1))$ and its Frechet derivative is

$$\Lambda_T'(u) = \mathcal{L}_T^{\#}\mathcal{R}_T F'(y_u)y_u' = \mathcal{L}_T^{\#}\mathcal{K}_T[y_u]y_u', \tag{19.4.20}$$

recalling Eq. (19.2.24). On the other hand, from Eq. (19.2.9), we obtain by differentiation in $u \in H^1(0, T; U)$,

$$y_u - \mathcal{R}F(y_u) = e^{A\cdot}y_0 + \mathcal{L}u \in C([0, T]; H) \to y_u' - \mathcal{R}F'(y_u)y_u' = \mathcal{L}, \tag{19.4.21}$$

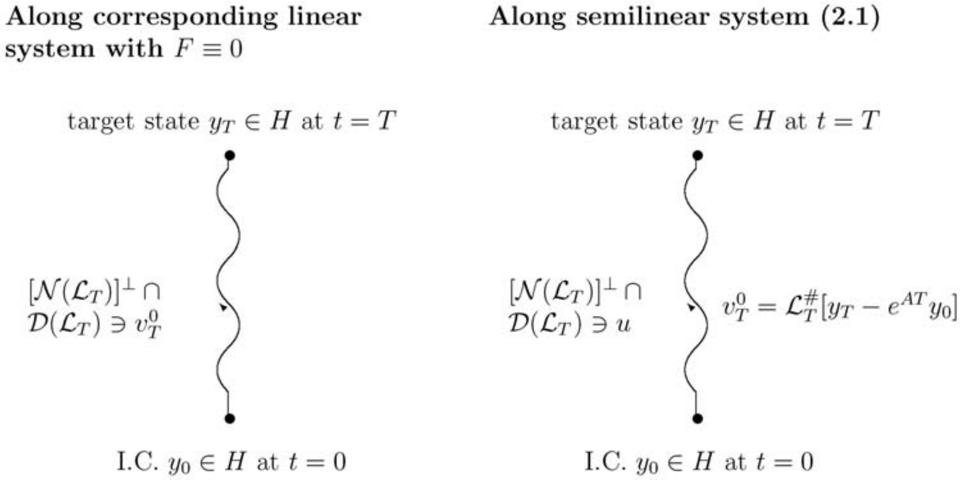

FIGURE 19.4.1: Proposition 19.4.1(a): Given T, $y_0 \in H$, $y_T \in H$, there exists a steering control $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ for the semilinear system of Eq. (19.2.1) if and only if (19.C.1) = Eq. (19.3.5) holds true and u satisfies $u + \Lambda_T(u) = v_T^0 = \mathcal{L}_T^{\#}[y_T - e^{AT}y_0]$, where $v_T^0 \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ is the steering control for the corresponding linear system.

or, recalling $\mathcal{R}F'(y_u) = \mathcal{K}[y_u]$ from Eq. (19.2.21) and $U = L_2(\Gamma_1)$,

$$\begin{aligned} \{I - \mathcal{K}[y_u]\}y_u' &= \mathcal{L} : H^1(0, T; U) \to C([0, T]; H) \ \text{ hence} \\ y_u' &= \{I - \mathcal{K}[y_u]\}^{-1}\mathcal{L} : H^1(0, T; U) \to C([0, T]; H). \end{aligned} \tag{19.4.22}$$

The inversion in Eq. (19.4.22) satisfying Eq. (19.4.19b) is justified due to the assumption of Eq. (19.1.2) or Eq. (19.2.8) that f is globally Lipschitz, in the same way as the inversion $[I - \mathcal{R}F]^{-1}$ was justified in Eqs. (19.4.2) and (19.4.3). Notice the following regularity properties: $\mathcal{L}$: compact $L_2(0, T; U) \to L_2(0, T; Y)$ by (P.1) = Eq. (19.2.14); $\{I - \mathcal{K}[y_u]\}^{-1}$ continuous on $L_2(0, T; Y)$, by Eq. (19.4.19b); $\mathcal{K}_T[y_u]$: continuous $L_2(0, T; Y) \to H$, *a fortiori* from Eq. (19.2.24) (this is property [P.3] in Eq. [19.2.34]); and finally, $\mathcal{L}_T^{\#}$: continuous H into $\{\mathcal{N}(\mathcal{L}_T)\}^{\perp}$, see (19.4.11). Then, these properties combined yield that $\Lambda_T'(u)$ is compact as an operator $L_2(0, T; U) \supset [\mathcal{N}[\mathcal{L}_T)]^{\perp} \to [\mathcal{N}(\mathcal{L}_T)]^{\perp}$ (with $L_2(0, T; U)$-topology), as claimed in Eq. (19.4.19a). □

Step 3. Here we analyze the boundedness condition for $[I + \Lambda_T'(u)]^{-1}$ in Eq. (19.4.18). Thus, let $v \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T) \equiv \mathcal{L}_T^{\#}H$, $\mathcal{L}_T^{\#}$ injective, see Eq. (19.4.9), so that there exists a unique $h \in H$ such that $v = \mathcal{L}_T^{\#}h$. Let $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$ be a fixed parameter. We seek to solve the problem

$$\tilde{u} + \Lambda_T'(u)\tilde{u} = v = \mathcal{L}_T^{\#}h \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T), \tag{19.4.23}$$

for a unique $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$, hence necessarily $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$, so as to obtain the bound

$$\|\tilde{u}\|_{L_2(0,T;U)} = \|[I + \Lambda_T'(u)]^{-1}v\|_{L_2(0,T;U)} \le C[1 + \|u\|_{L_2(0,T;U)}]\|v\|_{L_2(0,T;U)}, \tag{19.4.24}$$

$U = L_2(\Gamma_1)$. To this end, we return to the linearized z-problem in Eq. (19.2.18), with parameter $\eta = y_u$, u the parameter in Eq. (19.4.23), and rewrite Eqs. (19.2.18) and (19.2.19) accordingly with control $\tilde{u}$ as:

$$\dot{z} = Az + F'[y_u]z + B\tilde{u}, \ z(0) = z_0, \quad [I - \mathcal{K}[y_u]]z = e^{A\cdot}z_0 + \mathcal{L}\tilde{u} \Rightarrow \tag{19.4.25}$$

$$z = [I - \mathcal{K}[y_u]]^{-1}[e^{A\cdot}z_0 + \mathcal{L}\tilde{u}], \tag{19.4.26}$$

where the inverse in Eq. (19.4.26) is justified, see Eq. (19.4.19b), under the assumption of Eq. (19.1.2) by the property in Eq. (19.2.8) as in the case of obtaining Eq. (19.4.2) via Eq. (19.4.3). Evaluating Eq. (19.4.25) at $t = T$ yields, see Eq. (19.2.20):

$$z(T) = \mathcal{K}_T[y_u]z + e^{AT}z_0 + \mathcal{L}_T\tilde{u}, \quad \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \tag{19.4.27}$$

The following result shows the importance of extracting the vector $h \in H$ in solving Eq. (19.4.23): it provides an *equivalence* between solving Eq. (19.4.23) for $\tilde{u}$ and exact controllability from the origin of the linearized system via $\tilde{u}$ with target h at $t = T$.

PROPOSITION 19.4.2

Assume Eq. (19.1.2) *and let* $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$ *be a fixed parameter. Let* $v \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ *or, equivalently, let* $h \in H$ *be given, with* $v = \mathcal{L}_T^\# h$*. Then, the following equivalence holds true. Assume that there exists a control* $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ *which steers the origin* $z_0 = 0$ *at* $t = 0$ *to the target* $z_T = h \in H$ *at* $t = T$ *along the linearized z-equation of Eq.* (19.4.25)*, that is, via Eq.* (19.4.27)*. Then, that* $\tilde{u}$ *is a solution of Eq.* (19.4.23)*.*

Conversely, let $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ *be a solution of Eq.* (19.4.23)*. Then, that* $\tilde{u}$ *is a control which steers the origin* $z_0 = 0$ *at* $t = 0$ *to the target* $z_T = h \in H$ *at* $t = T$*, along the linearized z-equation of Eq.* (19.4.25)*, that is, via Eq.* (19.4.27)*.*

In short, solving Eq. (19.4.23) *for* $\tilde{u}$ *as the state h runs over the space H is equivalent to exact controllability of the linearized z-system of Eq.* (19.4.25)*, that is, Eq.* (19.4.27) *on the space H over the interval* $[0, T]$*.*

PROOF In one direction, let $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ be a control which steers the corresponding solution of the z-system of Eq. (19.4.25) from the origin $z_0 = 0$ at $t = 0$ to the target $z_T = h \in H$ at $t = T$. Then, via Eq. (19.4.27), we have

$$z_T = h = z(T) = \mathcal{K}_T[y_u]z + \mathcal{L}_T\tilde{u}, \quad \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \tag{19.4.28}$$

Apply $\mathcal{L}_T^\#$ across Eq. (19.4.28), use $\mathcal{L}_T^\#\mathcal{L}_T =$ identity on $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ by Eq. (19.4.12) (right) and obtain

$$\tilde{u} + \mathcal{L}_T^\#\mathcal{K}_T[y_u]z = \mathcal{L}_T^\# h = v \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T). \tag{19.4.29}$$

After substituting z in Eq. (19.4.26) with $z_0 = 0$ in Eq. (19.4.29) yields

$$\tilde{u} + \mathcal{L}_T^\#\mathcal{K}_T[y_u][I - \mathcal{K}[y_u]]^{-1}\mathcal{L}\tilde{u} = v = \mathcal{L}_T^\# h, \tag{19.4.30}$$

that is, recalling Eq. (19.4.19a), we conclude that

$$\tilde{u} + \Lambda_T'(u)\tilde{u} = v = \mathcal{L}_T^\# h \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T), \quad \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T), \tag{19.4.31}$$

and $\tilde{u}$ satisfies Eq. (19.4.23), as desired.

The above argument is reversible. Conversely, let $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ be a solution of Eq. (19.4.23) = Eq. (19.4.31). Then, recalling Eq. (19.4.19a), such $\tilde{u}$ satisfies Eq. (19.4.30), and hence Eq. (19.4.29) via Eq. (19.4.26). Apply now $\mathcal{L}_T$ across Eq. (19.4.29), use twice $\mathcal{L}_T\mathcal{L}_T^\# =$ identity on H by Eq. (19.4.12) (left) and obtain

$$\mathcal{L}_T\tilde{u} + \mathcal{K}_T[y_u]z = h. \tag{19.4.32}$$

Then, in view of Eq. (19.4.27) with $z_0 = 0$, we see that Eq. (19.4.32) says that $z(T) = h$, and then this $\tilde{u}$ steers $z_0 = 0$ at $t = 0$ to $z(T) = h$ at $t = T$ along the z-system, as desired. ∎

Further properties of the operator $I + \Lambda_T'(u)$ follow next. To the end, we return to Eq. (19.2.20) and Eq. (19.4.26) and recall the operator introduced in Eq. (19.3.15):

$$\mathcal{M}_T[\eta] : u \to \mathcal{M}_T[\eta]u \equiv z(T) = \mathcal{L}_T u + \mathcal{K}_T[\eta]z = \{\mathcal{L}_T + \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L}\}u; \tag{19.4.33a}$$

$$L_2(0, T; U) \supset \mathcal{D}(\mathcal{M}_T[\eta]) \equiv [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \to H,$$
$$\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L} : \text{ compact } L_2(0, T; U) \to H, \tag{19.4.33b}$$

see Remark 19.4.2, where $z(T)$ is the solution of Eq. (19.2.20) for $z_0 = 0$ and $\eta \in Y$, with z given by Eq. (19.4.26) with $z_0 = 0$.

REMARK 19.4.2 The operator $\mathcal{M}_T[\eta]$ in Eq. (19.4.33) is given in perturbation form over $\mathcal{L}_T$. The formula of Eq. (19.4.33) for $\mathcal{M}_T[\eta]$ has to be interpreted as follows, with $U \equiv L_2(\Gamma_1)$, $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$:

1. $\mathcal{L}$: continuous (compact) $L_2(0, T; U) \to \mathcal{E}_T \equiv L_2(0, T; Y)$ as in Eq. (19.2.14);
2. $\mathcal{K}[\eta]$: continuous (compact) $\mathcal{E}_T \to \mathcal{E}_T$, as in Eq. (19.2.27), hence $(I - \mathcal{K}[\eta])^{-1}$: continuous $\mathcal{E}_T \to \mathcal{E}_T$;
3. $\mathcal{K}_T[\eta]$: continuous $\mathcal{E}_T \to H$, as in Eq. (19.2.24);
4. $\mathcal{L}_T$: closed operator: $L_2(0, T; U) \supset [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \to H$, as in Eq. (19.3.3).

Properties (1), (2), and (3) imply the regularity in Eq. (19.4.33b).

Because $z(T) = \{\zeta(T), \zeta_t(T)\}$, where ζ is the solution of the problem in Eq. (19.2.37) with I.C. $\{\zeta_0, \zeta_1\} = z_0 = 0$, for $\eta_1 = w(t, x)$, then the last part of Section 19.3, including Eqs. (19.3.15) to (19.3.17) says that the maps:

$$\mathcal{M}_T[\eta] \equiv \mathcal{L}_T + \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L}, \quad \text{fixed } \eta \in \mathcal{E}_T \tag{19.4.34a}$$
$$\mathcal{M}_T^0 \equiv \mathcal{L}_T + \mathcal{K}_T^0(I - \mathcal{K}^0)^{-1}\mathcal{L} \tag{19.4.34b}$$
$$: L_2(0, T; U) \supset [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \to H \tag{19.4.34c}$$

have range dense in H, for each $\eta \in Y$. Equivalently, the Hilbert adjoint operators of Eqs. (19.4.34a–b):

$$\mathcal{M}_T^*[\eta] \equiv \mathcal{L}_T^* + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] : H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0, T; U); \tag{19.4.35a}$$

$$\left(\mathcal{M}_T^0\right)^* = \mathcal{L}_T^* + \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}\left(\mathcal{K}_T^0\right)^* : H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0, T; U) \tag{19.4.35b}$$

where $\mathcal{L}_T^* = (\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})^*$ in our present setting, and where by Eq. (19.4.33b), or Remark 19.4.3 below

$$\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta], \quad \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*$$
$$: \text{ compact } H \to L_2(0, T; U), \tag{19.4.35c}$$

are *injective*; that is, they have a trivial null space on H:

$$\mathcal{N}\{\mathcal{M}_T^*[\eta]\} = \mathcal{N}\{\mathcal{L}_T^* + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]\} = \{0\}, \text{ fixed } \eta \in \mathcal{E}_T; \tag{19.4.36a}$$

$$\mathcal{N}\left\{\left(\mathcal{M}_T^0\right)^*\right\} = \mathcal{N}\left\{\mathcal{L}_T^* + \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}\left(\mathcal{K}_T^0\right)^*\right\} = \{0\}. \tag{19.4.36b}$$

REMARK 19.4.3 Consistently with Remarks 19.4.1 and 19.4.2, we have chosen the following notation for the adjoints in Eqs. (19.4.35) and (19.4.36):

1. $\mathcal{L}_T^*$: $H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0, T; U)$ as in Remark 19.4.1;

2. $\mathcal{L}^{\circledast}$: compact $\mathcal{E}_T \to L_2(0, T; U)$—see also Eq. (19.6.12) below for explicit definition;

3. $\mathcal{K}^{\circledast}[\eta]$; $(\mathcal{K}^0)^{\circledast}$: compact $\mathcal{E}_T \to \mathcal{E}_T$; hence, $(I - \mathcal{K}^{\circledast}[\eta])^{-1}$, $[I - (K^0)^{\circledast}]^{-1}$: continuous $\mathcal{E}_T \to \mathcal{E}_T$;

4. $\mathcal{K}^*_T[\eta]$; $(\mathcal{K}^0_T)^*$: continuous $H \to \mathcal{E}_T$.

With this premise, we now note a quantitative version of Proposition 19.4.2. To this end, we set for convenience, recalling Eq. (19.4.19a),

$$\Lambda'_T[\eta] \equiv C_T[\eta] \equiv \mathcal{L}^\#_T \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L}: \text{ compact } [\mathcal{N}(\mathcal{L}_T)]^\perp \to \text{itself}, \quad (19.4.37)$$

$$C^0_T[\eta] \equiv \mathcal{L}^\#_T \mathcal{K}^0_T (I - \mathcal{K}^0)^{-1}\mathcal{L}: \text{ compact } [\mathcal{N}(\mathcal{L}_T)]^\perp \to \text{itself}, \quad (19.4.38)$$

where $\mathcal{K}^0_T$, $\mathcal{K}^0$ are the limits in Eqs. (19.2.29a) and (19.2.35a). $[\mathcal{N}(\mathcal{L}_T)]^\perp$ could be replaced by $L_2(0, T; L_2(\Gamma_1))$.

PROPOSITION 19.4.3

With reference to $\Lambda'_T(u)$ in Eq. (19.4.37) *and to Eq.* (19.4.34) *with $\eta = y_u$, we have*

a.

$$I + \Lambda'_T(u) = \mathcal{L}^\#_T \mathcal{M}_T[y_u] = I + \mathcal{L}^\#_T \mathcal{K}_T[y_u](I - \mathcal{K}[y_u])^{-1}\mathcal{L}: \textit{continuous}$$
$$L_2(0, T; U) \supset [\mathcal{N}(\mathcal{L}_T)]^\perp \to [\mathcal{N}(\mathcal{L}_T)]^\perp, \quad (19.4.39a)$$

with $\mathcal{L}^\#_T$ isomorphism between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$, see below Eq. (19.4.9)*; hence*

$$I + [\Lambda'_T(u)]^* = \mathcal{M}^*_T[y_u](\mathcal{L}^\#_T)^*: \textit{ continuous } [\mathcal{N}(\mathcal{L}_T)]^\perp \to [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(0, T; U); \quad (19.4.39b)$$

$$(\Lambda'_T[u])^* = C^*_T[u] = \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[y_u])^{-1}\mathcal{K}^*_T[y_u](\mathcal{L}^\#_T)^* \quad (19.4.40a)$$
$$\textit{compact } [\mathcal{N}(\mathcal{L}_T)]^\perp \to \textit{itself}; \quad (19.4.40b)$$

b. the bounded operator $I + (\Lambda'_T[u])^* = I + C^*_T[u]$ is injective on $[\mathcal{N}(\mathcal{L}_T)]^\perp$:

$$\tilde{u} + [\Lambda'_T(u)]^*\tilde{u} = \tilde{u} + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[y_u])^{-1}\mathcal{K}^*_T[y_u]\left(\mathcal{L}^\#_T\right)^*\tilde{u} = 0; \quad \tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^\perp$$
$$\Rightarrow \tilde{u} = 0 \textit{ (for each fixed value of } u \in [\mathcal{N}(\mathcal{L}_T)]^\perp), \quad (19.4.41)$$

so that the original operator $I + \Lambda'_T(u)$ has dense range as an operator from $[\mathcal{N}(\mathcal{L}_T)]^\perp$ into itself. Similarly, the operator $I + (C^0_T)^*$ is injective on $[\mathcal{N}(\mathcal{L}_T)]^\perp$, where

$$\left(C^0_T\right)^* = \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}\left(\mathcal{K}^0_T\right)^*\left(\mathcal{L}^\#_T\right)^*: \textit{compact } [\mathcal{N}(\mathcal{L}_T)]^\perp \to \textit{itself}; \quad (19.4.42)$$

c. In fact, more is true: both operators

$$[I + \Lambda'_T(u)]^{-1} \textit{ and } \{I + (\Lambda'_T(u)\}^*]^{-1} \textit{ exist as bounded operators } \in L([\mathcal{N}(\mathcal{L}_T)]^\perp), \quad (19.4.43)$$

for each $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$. Similarly, both operators

$$\left[I + C^0_T\right]^{-1} \textit{ and } \left[I + \left(C^0_T\right)^*\right]^{-1} \textit{ exist as bounded operators in } L([\mathcal{N}(\mathcal{L}_T)]^\perp). \quad (19.4.44)$$

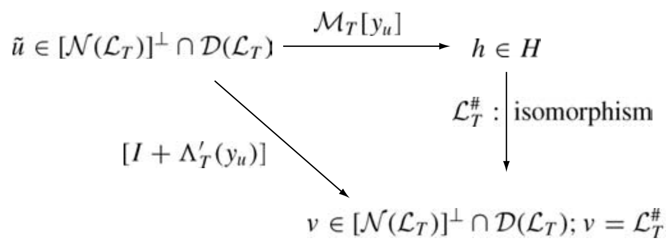

FIGURE 19.4.2: Propositions 19.4.2 and 19.4.3 given $v \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$, equivalently, given $h \in H$, with $v = \mathcal{L}_T^{\#} h$, there exists $\tilde{u} \in [\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ such that $\tilde{u} + \Lambda_T'(y_u)\tilde{u} = v$ if and only if such $\tilde{u}$ also satisfies $\mathcal{M}_T[y_u]\tilde{u} = z(T) = h$ along the linearized z-equation (19.4.25) with $z(0) = 0$ (exact controllability of the z-problem). In short, $I + \Lambda_T'(y_u) = \mathcal{L}_T^{\#}\mathcal{M}_T[y_u]$, and $[I + \Lambda_T'(y_u)]$ is surjective from $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ onto itself if and only if $\mathcal{M}_T[y_u]$ is surjective from $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ onto H.

PROOF

1. We apply $\mathcal{L}_T^{\#}$ to the identity in Eq. (19.4.33) giving $\mathcal{M}_T[\eta]$ on $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ and obtain for $\eta = y_u$:

$$\mathcal{L}_T^{\#}\mathcal{M}_T[y_u] = \mathcal{L}_T^{\#}\mathcal{L}_T + \mathcal{L}_T^{\#}\mathcal{K}_T[y_u](I - \mathcal{K}[y_u])^{-1}\mathcal{L} \tag{19.4.45}$$

$$= I + \Lambda_T'(u), \tag{19.4.46}$$

recalling that $\mathcal{L}_T^{\#}\mathcal{L}_T =$ Identity on $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$, via Eq. (19.4.12), as well as the definition in Eq. (19.4.19a) = Eq. (19.4.37) for $\Lambda_T'(u)$. Then Eq. (19.4.46), extended to $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$, proves Eq. (19.4.39a) from which Eq. (19.4.39b) follows by duality.

2. In Eq. (19.4.39b), we have that both $(\mathcal{L}_T^{\#})^* = ([\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^{\perp}}]^{-1})^*$ is (plainly) injective from $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$ to H and $\mathcal{M}_T^*[y_u]$ is injective on H as noted in Eq. (19.4.36a). So $I + [\Lambda_T'(u)]^*$ is injective, as claimed in Eq. (19.4.41). Similarly, $[I + (C_T^0)^*]$ is injective, by invoking now that $(\mathcal{M}_T^0)^*$ is injective, as noted in Eq. (19.4.36b).

3. But $(\Lambda_T'(u))^*$ is compact, because so is $C_T[u] = \Lambda_T'(u)$ by Eq. (19.4.19a) = Eq. (19.4.37). Therefore, injectivity of $[I + \{\Lambda_T'(u)\}^*]$ is equivalent to bounded invertibility of $[I + \{\Lambda_T'(u)\}^*]$, hence to bounded invertibility of $[I + \Lambda_T'(u)]$, and Eq. (19.4.43) follows.

The conclusion of Eq. (19.4.44) follows in a similar way, now using that C_T^0 is compact by Eq. (19.4.38). ∎

Step 4. What remains to be done to complete the proof of Theorem 19.1.1 is to show that the family of inverse operators

$$[I + \Lambda_T'(y_u)]^{-1} = \left\{I + \mathcal{L}_T^{\#}\mathcal{K}_T[y_u][I - \mathcal{K}[y_u]]^{-1}\mathcal{L}\right\}^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^{\perp}), \quad \forall\, u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}, \tag{19.4.47}$$

each member of which is in $L([\mathcal{N}(\mathcal{L}_T)]^{\perp})$ by Proposition 19.4.3(c), Eq. (19.4.43), possesses, in fact, a bound that is uniform with respect to $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$; that is,

$$\|(I + \Lambda_T'[y_u])^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^{\perp})} \le \text{const., uniformly in } u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}. \tag{19.4.48}$$

Two inversions are involved in Eq. (19.4.47), and both have to be shown to be uniformly bounded in u. It is at this level that properties (P.2) and (P.3) in Section 19.2 play a critical role, as described below. The first uniform inversion is accomplished in Lemma 19.4.4 below. The second and conclusive uniform inversion is accomplished in Lemma 19.4.6 below. To this end, we notice that in order to exploit that $\mathcal{L}$ is compact: $L_2(0, T; U) \to \mathcal{E}_T \equiv L_2(0, T; Y), U = L_2(\Gamma_1), Y = L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, see Eq. (19.2.14), we shall then seek a uniform bound for $[I - \mathcal{K}[\eta]]^{-1}$ in the uniform norm of $L(Y)$, with $\eta \in \mathcal{E}_T$, *a-fortiori* for $\eta = y_u \in C([0, T]; H)$, see Step 1.

LEMMA 19.4.4

Not only do we have $(I - \mathcal{K}[\eta])^{-1} \in L(\mathcal{E}_T)$, for each $\eta \in \mathcal{E}_T \equiv L_2(0, T; Y)$, $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, as was shown in Eq. (19.4.19b), *but moreover:*

1.

$$\|(I - \mathcal{K}[\eta])^{-1}\|_{L(\mathcal{E}_T)} \leq \textit{const., uniformly in } \eta \in \mathcal{E}_T. \tag{19.4.49}$$

2. *Let η_{n_k} be a subsequence of a given arbitrary sequence $\eta_n \in \mathcal{E}_T$, such that*

$$\mathcal{K}[\eta_{n_k}] \to \textit{ some } \mathcal{K}^0 = \mathcal{R}F_0 \textit{ strongly in } \mathcal{E}_T, \tag{19.4.50}$$

for some $F_0 \in L(Y)$, as guaranteed by property (P.2)(b), Eq. (19.2.29), *whereby then $\mathcal{K}^0$ is compact on $\mathcal{E}_T$, as noted in Eq.* (19.2.33). *Then*

$$[I - \mathcal{K}^0]^{-1} \in L(\mathcal{E}_T), \quad \textit{and} \quad (I - \mathcal{K}[\eta_{n_k}])^{-1} \to (I - \mathcal{K}^0)^{-1} \textit{ strongly in } \mathcal{E}_T. \tag{19.4.51}$$

PROOF (a2) The proof relies entirely on property (P.2), which then allows one to invoke a standard result (Reference 1, Theorem 1.6, p. 6) on a family of collectively compact operators. As already noted in Eq. (19.2.33), the operator $\mathcal{K}^0$ is of the form $\mathcal{K}^0 \equiv \mathcal{R}F_0$. This then, as in Eq. (19.4.3) allows one to obtain that $[I - \mathcal{K}^0]^{-1} \in L(\mathcal{E}_T)$, as desired. One could also use the present property that $\mathcal{K}^0$ is compact, whereby then it suffices to establish that $[I - \mathcal{K}^0]$ is injective: $[I - \mathcal{R}F_0]f = 0 \Rightarrow f = 0$, which is true because this leads by differentiation, via Eq. (19.2.16), to $\dot{f} = Af + F_0 f$, $f(0) = 0 \Rightarrow f \equiv 0$. Moreover, the strong convergence in Eq. (19.4.50) for the collectively compact family $\mathcal{K}[\eta]$, as noted in (P.2)(b) = Eq. (19.2.29), combined with the existence of $[I - \mathcal{K}^0]^{-1} \in L(\mathcal{E}_T)$ just established implies via Reference 1, Theorem 1.6, p. 6 that $[I - \mathcal{K}[\eta_{n_k}]]^{-1} \in L(\mathcal{E}_T)$ for all k sufficiently large, and, moreover, that the strong convergence in Eq. (19.4.51) takes place. Part (a2) is proved and this establishes part (a1) because the sequence $\{\eta_n\}$ was arbitrary. ∎

Step 5. Let us define the family of operators

$$W_T[\eta] \equiv \mathcal{L}_T^{\#}\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1} \tag{19.4.52a}$$

$$: \text{ continuous } \mathcal{E}_T \equiv L_2(0, T; Y) \to [\mathcal{N}(\mathcal{L}_T)]^{\perp} \text{ for each } \eta \in \mathcal{E}_T, \tag{19.4.52b}$$

so that

$$W_T[\eta]\mathcal{L} \equiv \mathcal{L}_T^{\#}\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}\mathcal{L} \equiv \Lambda_T'[\eta] \equiv C_T[\eta], \tag{19.4.53}$$

by Eq. (19.4.19a) or Eq. (19.4.37a), where the indicated regularity property in Eq. (19.4.52b) is obtained by combining Eq. (19.4.49), but Eq. (19.4.19b) would suffice, with the property Eq. (19.2.34) for $\mathcal{K}_T[\eta]$ and the property of Eq. (19.4.9) for $\mathcal{L}_T^{\#}$. Moreover, we have that: given any sequence $\eta_n \in \mathcal{E}_T$, we can extract a subsequence η_{n_k} such that

$$W_T[\eta_{n_k}] \to W^0 \equiv \mathcal{L}_T^{\#}\mathcal{K}_T^0(I - \mathcal{K}^0)^{-1} \text{ weakly from } \mathcal{E}_T \text{ to } [\mathcal{N}(\mathcal{L}_T)]^{\perp}. \tag{19.4.54}$$

This follows by combining property (P.2)(b), Eq. (19.2.29a), with property (P.3), Eq. (19.2.35a) as well as Eq. (19.4.51) (right) of Lemma 19.4.4 and the regularity in Eq. (19.4.9) for $\mathcal{L}_T^{\#}$.

The desired conclusion then follows as a specialization of the following result.

LEMMA 19.4.5

Let Z_i be two Banach spaces, $i = 1, 2$. Let Q be a compact operator $Z_1 \to Z_2$, and let $W(p)$ be a family of bounded operators $Z_2 \to Z_1$ depending on the parameter $p \in \mathcal{P}$, such that

$$W(p_n) \to W^0 \text{ weakly} \tag{19.4.55}$$

for any sequence p_n, with $W^0 : Z_2 \to Z_1$ depending on the sequence. Assume further that the operators $[I + W(p)Q]$ and $[I + W^0Q]$ are all injective on Z_1 and hence are boundedly invertible on Z_1, as $W(p)Q$, W^0Q are compact. Then, in fact,

$$\|[I + W(p)Q]^{-1}\|_{L(Z_1)} \leq \text{const., uniformly in } p \in \mathcal{P}, \tag{19.4.56}$$

and the weak convergence

$$[I + W(p_n)Q]^{-1} \to [I + W^0Q]^{-1} \text{ weakly in } Z_1 \tag{19.4.57}$$

holds true. Moreover, denoting by $*$ the obvious adjoints,

$$Q^*W^*(p) \text{ is a collectively compact family in } Z_1 \text{ in the parameter } p \in \mathcal{P}; \tag{19.4.58}$$

$$Q^*W^*(p_n) \to Q^*(W^0)^* \text{ strongly in } Z_1; \tag{19.4.59}$$

$$[I + Q^*W^*(p_n)]^{-1} \to [I + Q^*(W^0)^*]^{-1} \text{ strongly in } Z_1; \tag{19.4.60}$$

$$\|[I + Q^*W^*(p_n)]^{-1}\|_{L(Z_2)} \leq \text{const., uniformly in } n. \tag{19.4.61}$$

PROOF Two proofs—one direct on these operators, one on their adjoints—are given in Reference 29, Lemma 19.2.2, p. 123. ∎

Specialization to Our Present Problem and Completion of Proof of Theorem 19.1.1 We apply Lemma 19.4.5 with:

$$\begin{aligned} Z_1 &\equiv [\mathcal{N}(\mathcal{L}_T)]^{\perp} \subset L_2(0, T; L_2(\Gamma_1)); \quad Z_2 \equiv \mathcal{E}_T \equiv L_2(0, T; Y); \\ Y &\equiv L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'; \end{aligned} \tag{19.4.62}$$

$$Q \equiv \mathcal{L} \text{ defined in Eq. (19.2.14)}; \quad W(p) \equiv W_T[\eta] \equiv \mathcal{L}_T^{\#}\mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1}, \tag{19.4.63}$$

as defined in Eq. (19.4.52). We now verify the required assumptions of Lemma 19.4.5. First, the required weak convergence of Eq. (19.4.55) is assured by the established property of Eq. (19.4.54). Here the parameter $p \in \mathcal{P}$ specializes to the parameter $\eta \in \mathcal{E}_T$. Next, via Eq. (19.4.53), injectivity of $[I + W(p)Q] = I + \Lambda_T'[\eta] = I + C_T[\eta]$ and of $I + W^0Q = I + C_T^0$ is assured by Eqs. (19.4.43) and (19.4.44), respectively. Then, the conclusion of Eq. (19.4.56) of Lemma 19.4.5 specializes, under the above setting, to the following result.

LEMMA 19.4.6

Under the assumption of Eq. (19.1.2)*, we have*

$$\begin{aligned} \|[I + \Lambda_T'[\eta]]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^{\perp})} &\equiv \left\|\left\{I + \mathcal{L}_T^{\#}\mathcal{K}_T[\eta][I - \mathcal{K}[\eta]]^{-1}\mathcal{L}\right\}^{-1}\right\|_{L([\mathcal{N}(\mathcal{L}_T)]^{\perp})} \\ &\leq \text{const., uniformly in } \eta \in \mathcal{E}_T \equiv L_2(0, T; Y), \end{aligned} \tag{19.4.64}$$

$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$; in particular, uniformly in $\eta = y_u \in C([0,T]; H)$, $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, see Step 1.

Theorem 19.1.1 is now proved via Lemma 19.4.6 via Proposition 19.4.1(b), followed by Proposition 19.4.1(a).

19.5 Two Main Strategies for Exact Controllability of the Semilinear System of Eq. (19.1.1) and Their Equivalence. Four Implementations

Orientation

Proposition 19.4.1(a)—depicted in Figure 19.4.1—states an *equivalence* between the following two facts: on the one hand, the sought-after exact controllability of the semilinear system of Eq. (19.2.1), subject to Eq. (19.2.8), or Eq. (19.1.1), subject to Eq. (19.1.2) from the initial point $y_0 \in H$ at $t = 0$ to the target state $y_T \in H$ at $t = T$, by means of the steering control $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$; and, on the other hand, exact controllability of the corresponding *linear* system of Eq. (19.2.1) with $F \equiv 0$, Eq. (19.1.1) with $f \equiv 0$, from the same initial point y_0 at $t = 0$ to the same target state y_T at $t = T$ but this time by virtue of the control $v_T^0 = u + \Lambda_T(u) \in [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$.

Next, Proposition 19.4.1(b) provides a *sufficient* condition for Proposition 19.4.1(a) to actually hold true—that is, for the solvability in u of the equation $u + \Lambda_T(u) = v_T^0$—in terms of the uniform invertibility condition of Eq. (19.4.18) on the inverse $[I + \Lambda'_T(u)]^{-1}$: more precisely, the condition of Eq. (19.4.18) implies that the C^1-map $u \to u + \Lambda_T(u)$ is a homeomorphism (bijective, continuous, with continuous inverse) from $[\mathcal{N}(\mathcal{L}_T)]^\perp$ onto itself.

In turn, Proposition 19.4.2, quantified further in Proposition 19.4.3 and depicted in Figure 19.4.2, states in particular that for any parameter $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$ *fixed*, consequently with solution $y_u \in C([0,T]; H)$ fixed (see Section 19.4, Step 1), we have the following equivalence:

$$\begin{cases} \text{the linear map } \tilde{u} \to \tilde{u} + \Lambda'_T(y_u)\tilde{u} \text{ in Eq. (19.4.23) is surjective from} \\ [\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \text{ onto itself,} \end{cases} \tag{19.5.1}$$

if and only if the z-problem of Eq. (19.2.18) with $\eta_1 = y_{u,1} = w$ and $z_0 = 0$ is exactly controllable on H in $[0, T]$: that is,

$$\begin{cases} \text{the closed linear map } \tilde{u} \to \mathcal{M}_T[y_u]\tilde{u} = z(T) \text{ in Eq. (19.4.33) is surjective} \\ \text{from } L_2(0,T;U) \supset [\mathcal{N}[\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T) \text{ onto } H, \end{cases} \tag{19.5.2}$$

because $I + \Lambda'_T[y_u] = \mathcal{L}_T^\# \mathcal{M}_T[y_u]$ by Eq. (19.4.39a), where $\mathcal{L}_T^\#$ in Eq. (19.4.39a) is an isomorphism between H and $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ as noted in Eq. (19.4.9); in turn, by duality [Reference 56, p. 235], if and only if

$$\|u\|_{L_2(\Sigma_1)} \le C'_{Ty_u} \|[I + \Lambda'_T(y_u)]^* u\|_{L_2(\Sigma_1)}, \quad u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(\Sigma_1) \tag{19.5.3}$$

(duality on Eq. (19.5.1)), $L_2(\Sigma_1) = L_2(0,T; L_2(\Gamma_1))$, and, in turn, if and only if

$$\|h\|_H \le C_{Ty_u} \|\mathcal{M}_T^*(y_u)h\|_{L_2(\Sigma_1)}, \quad h \in \mathcal{D}(\mathcal{M}_T^*(y_u)) = \mathcal{D}(\mathcal{L}_T^*) \subset H \tag{19.5.4}$$

(duality on Eq. [19.5.2]). The positive constants C'_{Ty_u} and C_{Ty_u} are independent of u and h, respectively. Next, we recall from Eq. (19.4.19a) that we are currently writing $\Lambda'_T(u)$ to really mean $\Lambda'_T(y_u)$, that is, the dependence of Λ'_T is directly in y_u, rather than u. In fact, more precisely, referring to the operators $\mathcal{K}$ in Eqs. (19.2.21) and (19.2.22) and, similarly, $\mathcal{K}_T$—which define Λ'_T—we see that the actual dependence is on $f'(y_{1,u}(t,x)) = f'(w_u(t,x))$, with reference to Eq. (19.2.1) or Eq. (19.1.1), respectively. Because of the assumption of Eq. (19.1.2), we can therefore consider Λ'_T as, ultimately, depending on a parameter $q \in L_\infty(Q)$, where, at the end, $q(t,x) = f'[w_u(t,x)]$, where, moreover, q is confined in a fixed sphere $B(0,r)$ centered at the origin, of radius $r > 0$, in $L_\infty(Q)$. We shall accordingly write $\Lambda'_T(q)$, $q \in B(0,r) \subset L_\infty(Q)$. We next let q run over $B(0,r)$ and show that the corresponding *equivalent* inequalities of Eqs. (19.5.3) and (19.5.4)—except, this time, with constants uniform in q—are then *equivalent* to the uniform invertibility condition of Eq. (19.4.18b) on the inverse $[I + \Lambda'_T(q)]^{-1}$, which, in turn, by Proposition 19.4.1(b), guarantes the identity of Eq. (19.4.16), and hence by Proposition 19.4.1(a), implies the exact controllability of the semilinear problem of Eq. (19.1.1) or Eq. (19.2.1) over the interval $[0,T]$ on the state space H.

PROPOSITION 19.5.1

Let $q \in B(0,r) \subset L_\infty(Q)$, $r > 0$.

1. *Then, the following inequalities are equivalent:*
   (i) (uniform COI)

$$\|h\|_H \le C_{Tr} \|\mathcal{M}_T^*(q)h\|_{L_2(\Sigma_1)}, \quad h \in \mathcal{D}(\mathcal{M}_T^*(q)) = \mathcal{D}(\mathcal{L}_T^*) \subset H; \tag{19.5.5}$$

   (ii) (surjectivity of the family of operators $[I + \Lambda'_T(q)]$ from $L_2[0,T; L_2(\Gamma_1)] = L_2(\Sigma_1) \supset [\mathcal{N}(\mathcal{L}_T)]^\perp$ onto itself, "uniformly in $q \in B(0,r)$," see below Eq. (19.5.6))

$$\|u\|_{L_2(\Sigma_1)} \le C'_{Tr} \|[I + \Lambda'_T(q)]^* u\|_{L_2(\Sigma_1)}, \quad u \in [\mathcal{N}(\mathcal{L}_T)]^\perp. \tag{19.5.6}$$

   By the proof of Reference 29, Theorem 19.9.4, in particular, the paragraph below Eq. (19.9.1), p. 236, this means that:

$$\overline{[I + \Lambda'_T(q)](S_1)} \supset C_{C'_{Tr}}, \quad \forall\, q \in B(0,r),$$

   where $S_m = \{\mu \in [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(\Sigma_1) : \|u\|_{L_2(\Sigma_1)} \le m\}$;
   (iii) (the condition of Eq. (19.4.18b))

$$\|[I + \Lambda'_T(q)]^{-1}\|_{L([\mathcal{N}(\mathcal{L}_T)]^\perp)} \le \; const_{Tr}. \tag{19.5.7}$$

   All of the positive constants in Eqs. (19.5.5), (19.5.6), and (19.5.7) depend only on T and the radius $r > 0$ of $B(0,r)$, not on h or u or q.

2. *Accordingly, by Proposition 19.4.1(b), each of the above conditions in Eqs. (19.5.5), (19.5.6), or (19.5.7) guarantees that the map $u \to u + \Lambda_T(u)$ is surjective from $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ onto itself, that is the condition of Eq. (19.4.16). Hence, each of the above conditions of Eqs. (19.5.5), (19.5.6), and (19.5.7) implies, by Proposition 19.4.1(a), the exact controllability of the semilinear problem of Eqs. (19.1.1) or Eq. (19.2.1) on the state space H over the interval $[0,T]$.*

PROOF The equivalence Eq. (19.5.5) $\Longleftrightarrow$ (19.5.6) stems, again, from the identity $[I + \Lambda'_T(q)]^* = \mathcal{M}_T^*(q)(\mathcal{L}_T^\#)^*$ in Eq. (19.4.39b). Assume Eq. (19.5.6). For $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$, set $h = (\mathcal{L}_T^\#)^* u \in H$, where $(\mathcal{L}_T^\#)^*$ is a bounded operator from $[\mathcal{N}(\mathcal{L}_T)]^\perp$ into H. In fact, Range$(\mathcal{L}_T^\#)^* = (\mathcal{L}_T^\#)^*[\mathcal{N}(\mathcal{L}_T)]^\perp = \mathcal{D}(\mathcal{L}_T^*)$, because

$$\left(\mathcal{L}_T^\#\right)^* = \left[(\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})^{-1}\right]^* = [(\mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp})^*]^{-1} = (\mathcal{L}_T^*)^{-1},$$

in our present setting and notation: $\mathcal{L}_T = \mathcal{L}_T|_{[\mathcal{N}(\mathcal{L}_T)]^\perp}$. Moreover, $(\mathcal{L}_T^\#)^*$ is injective on $[\mathcal{N}(\mathcal{L}_T)]^\perp$, because $\mathcal{L}_T^\#$: H onto $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ by (19.4.9), hence with range dense in $[\mathcal{N}(\mathcal{L}_T)]^\perp$. In short,

$(\mathcal{L}_T^{\#})^*$ *is an isomorphism from* $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$ onto $\mathcal{D}(\mathcal{L}_T^*)$. Hence, $u = ((\mathcal{L}_T^{\#})^*)^{-1}h$ and then, by the assumed inequality of Eqs. (19.5.6) and (19.4.39b) recalled above, we obtain

$$\frac{1}{\|(\mathcal{L}_T^{\#})^*\|}\|h\|_H \leq \left\|\left(\left(\mathcal{L}_T^{\#}\right)^*\right)^{-1}h\right\|_{L_2(\Sigma_1)} \leq C'_{Tr}\|\mathcal{M}_T^*(q)h\|_{L_2(\Sigma_1)}. \tag{19.5.8}$$

All $h \in \mathcal{D}(\mathcal{L}_T^*) = \mathcal{D}[\mathcal{M}_T^*(q)]$, see Eq. (19.4.35a) arise in this way. In conclusion, inequality of Eq. (19.5.8) has been shown for all $h \in \mathcal{D}[\mathcal{M}_T^*(q)] = \mathcal{D}(\mathcal{L}_T^*)$. Thus, we have obtained Eq. (19.5.5).

Conversely, assume Eq. (19.5.5). For $h \in \mathcal{D}[\mathcal{M}_T^*(q)] = \mathcal{D}(\mathcal{L}_T^*)$, we set, recalling the above isomorphism on $(\mathcal{L}_T^{\#})^*$, $u = [(\mathcal{L}_T^{\#})^*]^{-1}h$, or $(\mathcal{L}_T^{\#})^*u = h$, so that, $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$. Then, Eq. (19.5.5) implies via Eq. (19.4.39b)

$$\begin{aligned} c\|u\|_{L_2(\Sigma_1)} \leq \left\|\left(\mathcal{L}_T^{\#}\right)^* u\right\|_H &\leq C_{Tr}\left\|\mathcal{M}_T^*(q)\left(\mathcal{L}_T^{\#}\right)^* u\right\|_{L_2(\Sigma_1)} \\ &= C_{Tr}\|[I + \Lambda'_T(q)]^* u\|_{L_2(\Sigma_1)}, \end{aligned} \tag{19.5.9}$$

for all $u \in [\mathcal{N}(\mathcal{L}_T)]^{\perp}$, where the inequality at the extreme left is obtained via Reference 56, p. 235, because $\mathcal{L}_T^{\#}$: H onto $[\mathcal{N}(\mathcal{L}_T)]^{\perp} \cap \mathcal{D}(\mathcal{L}_T)$. Thus, we have proved Eq. (19.5.6).

Next, the implication of Eq. (19.5.7) $\Rightarrow$ Eq. (19.5.6) is obvious, whereas the reverse implication of Eq. (19.5.6) $\Rightarrow$ Eq. (19.5.7) follows via the open mapping theorem, once we know that the surjective operator $[I + \Lambda'_T(q)]$ is also *injective* for each $q \in L_\infty(Q)$, a fact that is *a-fortiori* true by Eq. (19.4.43) of Proposition 19.4.3. ∎

We shall see in Appendix 19.A, Eq. (19.A.17), that a reformulation of the dual condition of Eq. (19.5.5) may be written *equivalently* in terms of the solution $\phi(t; \Phi_0)$, $\Phi_0 = [\phi_0, \phi_1]$ of the problem of Eq. (19.3.9), same as Eq. (19.6.3) below.

PROPOSITION 19.5.2

An equivalent reformulation of the sufficient condition of Eq. (19.5.5) *for exact controllability of the semilinear system of Eq.* (19.2.1) *subject to Eq.* (19.2.8)*, that is, Eq.* (19.1.1) *subject to Eq.* (19.1.2)*, in the state space H over the interval* $[0, T]$ *is that there exists a constant $C_T > 0$, independent of the I.C. and of the potential $q \in B(0, r)$ (a ball of radius $r > 0$ in $L_\infty(Q)$) such that the following inequality holds true for the solutions of the problem of Eq.* (19.3.9)

$$\|\{\phi_0, \phi_1\}\|_Y^2 \leq C_{Tr}\int_0^T\int_{\Gamma_1}\phi^2 d\Sigma_1, \tag{19.5.10}$$

where $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$.

REMARK 19.5.1 Abstractly, the inequality of Eq. (19.5.10) can be written in the following form: Eq. (19.5.10) is the explicit version of

$$\|V_0\|_H^2 \leq C_{Tr}\int_0^T \|B^*V(t; V_0)\|_U^2 dt \tag{19.5.11}$$

where $V(t; V_0)$, $V_0 \in H$, solves the equation below; see Eqs. (19.B.14) to (19.B.16) in Appendix 19.B:

$$V_t = -\mathbb{A}^*(t)V = -A^*V - (F'[y_u])^*V, \quad V(T) = V_0 \in H, \tag{19.5.12}$$

$F'[y_u] = [0, q(t, x)]^{tr}$, $q(t, x) = f'[w(t, x)]$, according to Eq. (19.2.7), with $\eta = y_u$, where the adjoint B^* refers to H, see Eq. (19.B.8) of Appendix 19.B.

PROOF OF PROPOSITION 19.5.2 AND REMARK 19.5.1. This result is contained in the subsequent (independent) Appendix 19.B. Indeed, Eq. (19.B.13) gives $\mathcal{M}_T^* h = B^* V(\cdot\,; V_0)$, where $V_0 = h$ by Eq. (19.B.12). This way, Eq. (19.5.5) becomes Eq. (19.5.10). Moreover, $V(t; V_0)$ satisfies Eq. (19.B.14): $V_t(t; V_0) \equiv -\mathbb{A}^*(t)V(t; V_0)$, where $\mathbb{A}^*(t) \equiv A^* + (F'[q])^*$ by Eq. (19.B.15a). Proposition 19.5.2 and Remark 19.5.1 are thus recovered.

REMARK 19.5.2 The problem of Eq. (19.5.12) is dual to the original linearized controlled z-problem of Eq. (19.2.18) (with $z_0 = 0$), which defines $\mathcal{M}_T[y_u]$ via Eq. (19.4.33). The PDE reformulation of Eq. (19.5.12) is given by the ϕ-problem of Eq. (19.3.9) = Eq. (19.6.3) or the v_2-problem of Eq. (19.B.17) in Appendix 19.B, where $V(t; V_0) = [v_1(t; V_0), v_2(t; V_0)]^{tr}$ by Eq. (19.B.12), $V_0 = [h_0, h_1] \in H$, $\phi_0 = h_1 \in L_2(\Omega)$, $\phi_1 = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$.

Two strategies. Four implementations

Thus, on the basis of Proposition 19.5.1, two strategies arise in order to show exact controllability of the semilinear system of Eq. (19.1.1), subject to Eq. (19.1.2):

Strategy 1 The first strategy consists of establishing the uniform invertibility condition Eq. (19.5.7) = Eq. (19.4.18b), thus satisfying Proposition 19.4.1(b) and Proposition 19.5.1. This strategy was pursued in Part I in the proof of Theorem 19.1.1 given in Section 19.4 and based on the *original controlled problem* of Eq. (19.1.1) or Eq. (19.2.1). Critical ingredients are Lemmas 19.4.4 and 19.4.5. The final step was achieved in Lemma 19.4.6, Eq. (19.4.64). Then, Proposition 19.4.1(a) (or Proposition 19.5.1) completes the proof of Theorem 19.1.1.

Strategy 2 A second strategy consists of establishing the condition of Eq. (19.5.5) of Proposition 19.5.1 (uniform continuous observability inequality) of the linearized problem uniformly with respect to $L_\infty(Q)$-potentials $q(t, x)$ in a fixed ball of radius r, where then Eq. (19.5.5) is attained for $f'(w(t, x)) = q(t, x)$, and then $F'[y_u] = [0, q(t, x)]^{tr}$ according to Eq. (19.2.7) with $\eta = y_u$. In turn, the uniform COI of Eq. (19.5.5) is given explicitly by the inequality of Eq. (19.5.10) of Proposition 19.5.2. Thus, Strategy 2 centers on the *dual uncontrolled* problem of Eq. (19.3.9) = Eq. (19.6.3) in PDE-form Eq. [(19.5.12) in abstract form], rather than the original controlled problem of Eq. (19.1.1) or Eq. (19.2.1).

To pursue Strategy 2 centered on the uniform COI of Eq. (19.5.5), equivalently Eq. (19.5.10), three different implementations are possible:

Strategy 2, Implementation (i) Here one seeks to show the uniform COI of Eq. (19.5.5), equivalently Eq. (19.5.10), of the linearized problem by a dual analysis of the operator-theoretic approach of Section 19.4, centered this time on the dual uncontrolled problem of Eq. (19.3.9) = Eq. (19.6.3). This implementation is pursued in Part II (Sections 19.6–19.7). It shows the uniform COI of Eq. (19.5.5) = Eq. (19.7.4) *directly*, by virtue of the critical Lemmas 19.4.4 and 19.4.5, this time in dual versions, however.

Strategy 2, Implementation (ii) This is pursued in Part III (Section 19.8). It is still focused, as in implementation (b_1), on showing the uniform COI of Eq. (19.5.5) of the *dual, uncontrolled* problem. However, unlike implementation (b_1), the key point now is that Part III aims at proving the explicit COI of Eq. (19.5.10) by seeking to establish the *equivalent* comparison trace inequality of Eq. (19.8.4) or its operator-theoretic version of Eq. (19.8.7). However, Part III does *not* prove Eq. (19.8.4) = Eq. (19.8.7) *directly*. Rather, Part III seeks to prove the *equivalent* inequality of Eq. (19.8.8) (where the equivalence is described by an isomorphism). But the inequality of Eq. (19.8.8) is precisely the same as the inequality of Eq. (19.8.9), expressed in terms of quantities of Part I. Thus, the inequality of Eq. (19.8.9) (= Eq. [19.8.8]) is ultimately established by falling into Part I and invoking Lemmas 19.4.4 and 19.4.5 of Part I.

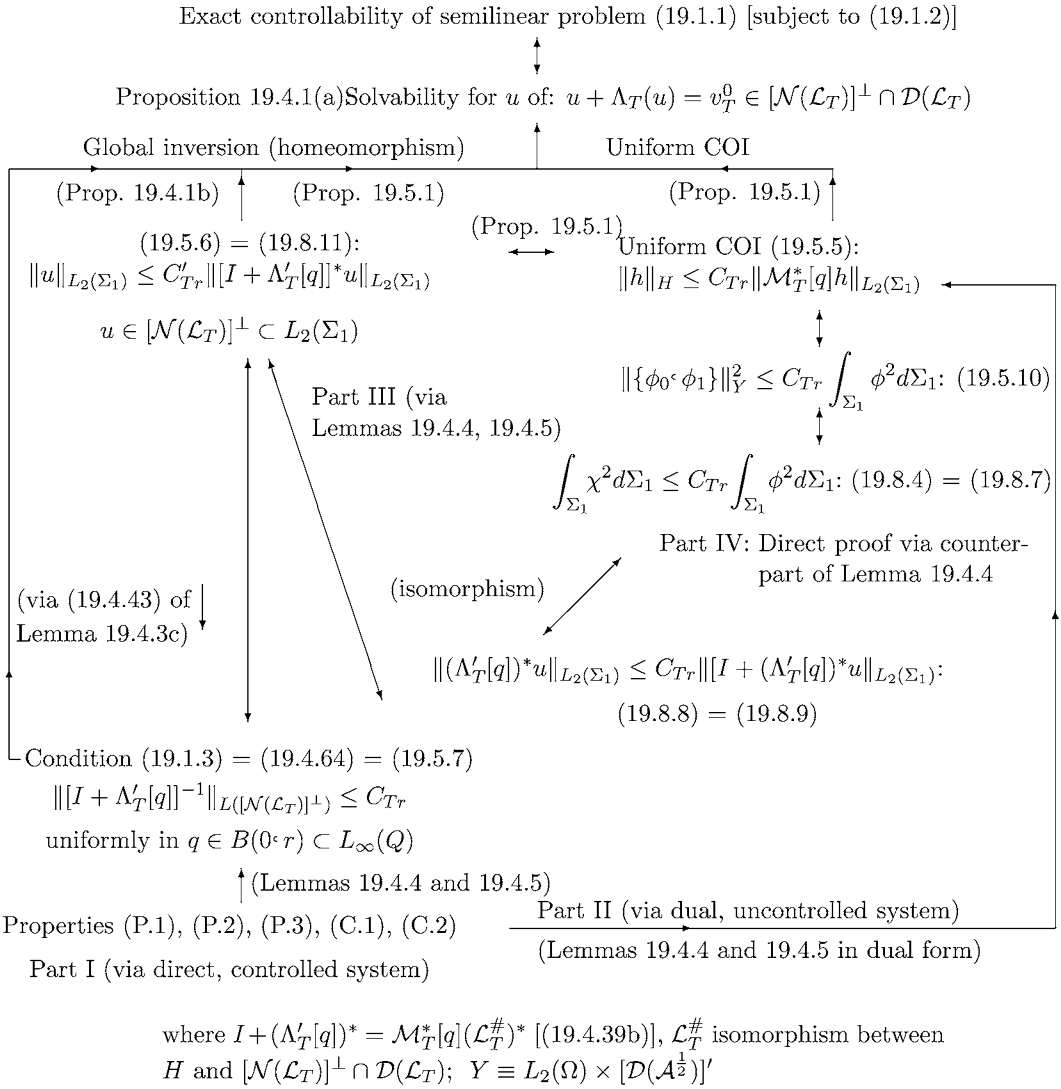

FIGURE 19.5.1: Part I establishes the left column through an operator-theoretic approach on the controlled problem Eq. (19.1.1) or Eq. (19.2.1). Part II establishes the uniform COI of Eq. (19.5.5) through a dual operator-theoretic approach on the dual uncontrolled problem Eq. (19.3.9) = Eq. (19.6.3). Part III establishes Eq. (19.8.4) = Eq. (19.8.7) not directly but via the equivalent of Eq. (19.8.8) = Eq. (19.8.9) by invoking Eq. (19.5.6) = Eq. (19.8.11) of Part I. Part IV establishes the uniform COI of Eq. (19.5.10) by proving Eq. (19.8.4) = Eq. (19.8.7) directly still through a dual operator-theoretic approach on the dual uncontrolled problem of Eq. (19.3.9) = Eq. (19.6.3).

Strategy 2, Implementation (iii) This Implementation constitutes Part IV (Section 19.9). Here one likewise seeks to establish the uniform COI of Eq. (19.5.5), equivalently Eq. (19.5.10), via an analysis of the *dual uncontrolled* problem. However, unlike Implementation (ii), now one seeks to prove the trace inequality (or its operator-theoretic reformulation) of Eq. (19.8.4) = Eq. (19.8.7) *directly*, not its *equivalent* (via an isomorphism) version of Eq. (19.8.8) as in Strategy 2, Implementation. This will result in some advantages described in the "Orientation" of Section 19.9.

The above strategies and implementations are pictorially condensed in Figure 19.5.1.

REMARK 19.5.3 As a matter of fact, Strategy 2 admits a fourth purely PDE implementation at the level of showing the trace inequality of Eq. (19.8.4) by a double compactness-uniqueness argument. This is pursued in another paper [35].

Part II: Proof of Theorem 19.1.1 through the Uniform COI of Eq. (19.5.5): Duality over the Abstract Operator Approach of Reference 29, via the Dual Uncontrolled Problem of Eq. (19.3.9) = Eq. (19.6.3)

19.6 A Direct Derivation of the Formula of Eq. (19.4.35a) for the Operator $\mathcal{M}_T^*[\eta]$via the Dual Problem. PDE Interpretation

The purpose of this section is twofold. First, we provide a *direct* proof of the formula of Eq. (19.4.35a) for the operator $\mathcal{M}_T^*[\eta]$, which involves *only the dual linearized* problem (with no control), rather than the original linearized controlled z-(abstract) system in Eq. (19.2.18) or Eq. (19.2.37) in PDE form. Next, we provide in Proposition 19.6.7 a PDE interpretation of the perturbation formula of Eq. (19.4.35a). A systematic study of the dual linearized problem will be given in Appendices 19.A, 19.B, and 19.C. The dual linearized problem in question is the ϕ-problem Eq. (19.3.9) = Eq. (19.6.3) below = Eq. (19.A.4) in Appendix 19.A below, with I.C. $\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, see Eq. (19.3.10)—same as the v_2-problem in Eq. (19.B.17) in Appendix 19.B below—and potential $q(t,x) = f'(\eta_1(t,x)) \in L_\infty(Q)$. We shall, accordingly, invoke some results from (the independent) Appendices. By contrast, in Section 19.4, we first established the formula in Eq. (19.4.33) for the input (control)-solution map $z(T) = \mathcal{M}_T[\eta]u$ for the controlled linearized z-problem (19.2.18) with $z_0 = 0$, and next we obtained the formula in Eq. (19.4.35a) for $\mathcal{M}_T^*[\eta]$ by duality. In short, we seek to avoid the operation of duality from $\mathcal{M}_T[\eta]$ to $\mathcal{M}_T^*[\eta]$ and, instead, derive $\mathcal{M}_T^*[\eta]$ directly.

Derivation of the Formula of Eq. (19.4.35a). Proposition 19.6.1

As in Eq. (19.3.15) *or Eq.* (19.4.33), *let*

$$z(T) = \mathcal{M}_T[\eta]u; \quad \mathcal{M}_T[\eta] : L_2(0,T;U) \supset \mathcal{D}(\mathcal{M}_T[\eta]) \equiv \mathcal{D}(\mathcal{L}_T) \to H, \tag{19.6.1}$$

where $\eta \in \mathcal{E}_T \equiv L_2(0,T;Y)$, $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$; $U = L_2(\Gamma_1)$, $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, *and where z is the solution of the linearized problem in Eq.* (19.2.18), *given by Eq.* (19.2.19) *for I.C.* $z_0 = 0$. *Then, its (Hilbert space) adjoint* $\mathcal{M}_T^*[\eta]$ *is given by*

$$\mathcal{M}_T^*[\eta] = \mathcal{L}_T^* + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] \tag{19.6.2a}$$

$$: H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0,T;U), \tag{19.6.2b}$$

in agreement with the formula in Eq. (19.4.35a), *where the operators on the right-hand-side of Eq.* (19.6.2a) *are specified in Remark* 19.4.3.

PROOF Here we provide a *direct* proof using the *dual* (uncontrolled) problem. ∎

Step 1. Our starting point is such dual (uncontrolled) problem, which is the ϕ-problem in Eq. (19.3.9), same as Eq. (19.6.3) or Eq. (19.A.4) in Appendix 19.A below, with I.C. $\{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, or—which is the same—the v_2-problem in Eq. (19.B.17) in Appendix 19.B with

I.C. $\{v_2(T), v_{2t}(T)\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, where the potential $q(t,x)$ is throughout specialized to $q(t,x) = f'(\eta_1(t,x)) \in L_\infty(Q)$, $\eta = [\eta_1, \eta_2] \in \mathcal{E}_T$. When convenient, we use the notation of the subsequent Appendix 19.B. Thus, the dual uncontrolled problem is, in the present context, given by

$$\begin{cases} \phi_{tt} = \Delta\phi + f'(\eta_1(t,x))\phi & \text{on } Q; \quad (19.6.3a) \\ \phi(T) = \phi_0 = h_1 \in L_2(\Omega);\ \phi_t(T) = \phi_1 = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' & \text{in } \Omega; \quad (19.6.3b) \\ \phi|_{\Sigma_0} \equiv 0, \quad \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{on } \Sigma_i \quad (19.6.3c) \end{cases}$$

(see Eq. (19.B.17) in Appendix 19.B) with $[h_0, h_1] \in H = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$.

In (the independent) Appendices 19.A and 19.B, it is proved in Eq. (19.A.12), or Eqs. (19.B.13) to (19.B.17) that, then, the operator $\mathcal{M}_T^*[\eta]$ is given by

$$\{\mathcal{M}_T^*[\eta]V_0\}(t) = B^*V(t;V_0), \quad V_0 = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \in H, \quad V(t;V_0) = \begin{bmatrix} v_1(t;V_0) \\ v_2(t,V_0) \end{bmatrix} \quad (19.6.4a)$$

$$v_2(t;V_0) = \phi(t;\Phi_0); \quad v_1(t;V_0) = -\mathcal{A}^{-1}\phi_t(t;\Phi_0); \quad \Phi_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \text{ in } Y, \quad (19.6.4b)$$

where the second component $v_2(t;V_0)$ of $V(t;V_0)$ is precisely the solution of the problem in Eq. (19.6.3). Here B^* is defined in Remark 19.B.1 (the symbol $*$ refers to duality with respect to H, as stipulated in Remark 19.4.1), whereas $V(t;V_0)$ is the solution of Eq. (19.B.14) {with $q(t,x) = f'[\eta_1(t,x)]$}, rewritten here as

$$V_t(t;V_0) = -\mathbb{A}^*(t)V(t;V_0) = AV(t;V_0) + \begin{bmatrix} 0 & -\mathcal{A}^{-1}[f'(\eta_1(\cdot))\,\cdot\,] \\ 0 & 0 \end{bmatrix} V(t;V_0); \quad (19.6.5a)$$

$$V(t;V_0) = \begin{bmatrix} v_1(t;V_0) \\ v_2(t;V_0) \end{bmatrix} = \begin{bmatrix} -\mathcal{A}^{-1}\phi_t(t;\Phi_0) \\ \phi(t;\Phi_0) \end{bmatrix}; \quad V(T;V_0) = V_0 = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \in H;$$

$$\Phi_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} h_1 \\ -\mathcal{A}h_0 \end{bmatrix} \in Y, \quad (19.6.5b)$$

recalling Eq. (19.B.15a), $-A^* = A$, and Eq. (19.B.12). More explicitly, we have that the solution $V(t;V_0)$ of Eq. (19.6.5a) is given by

$$V(t;V_0) = e^{A(t-T)}V_0 + \int_T^t e^{A(t-\tau)} \begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau \quad (19.6.6)$$

$$= e^{A^*(T-t)}V_0 + \int_t^T e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau, \quad (19.6.7)$$

because $A^* = -A$ in H (again, $*$ denotes adjoint in H). Applying B^* to Eq. (19.6.7) yields, after recalling Eq. (19.6.4):

$$(\mathcal{M}_T^*[\eta]V_0)(t) = B^*V(t;V_0) = B^*e^{A^*(T-t)}V_0 \quad (19.6.8)$$

$$+ \int_t^T B^*e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix} d\tau, \quad (19.6.9)$$

or recalling the usual dualities [34],

$$\mathcal{M}_T^*[\eta]V_0 = \mathcal{L}_T^* V_0 + \mathcal{L}^* \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\cdot))v_2(\cdot\,; V_0)] \\ 0 \end{bmatrix}, \quad V_0 \in H, \tag{19.6.10}$$

where $\mathcal{L}_T^* : H \supset \mathcal{D}(\mathcal{L}_T^*) \to L_2(0, T; U)$ as defined in Remark 19.4.3, whereas $\mathcal{L}^*$ is defined by

$$(\mathcal{L}u, h)_{L_2(0,T;H)} = (u, \mathcal{L}^* h)_{L_2(0,T;U)}, \quad \forall\, u \in \mathcal{D}(\mathcal{L}),\ h \in \mathcal{D}(\mathcal{L}^*); \tag{19.6.11a}$$

$$\mathcal{L}^* : L_2(0, T; H) \supset \mathcal{D}(\mathcal{L}^*) \to L_2(0, T; U), \tag{19.6.11b}$$

consistently with the agreed-upon notation in Remark 19.4.1, where $*$ denotes adjoint from H.

Step 2. In contrast to Eq. (19.6.11), we now consider the operator $\mathcal{L}$ as in Eq. (19.2.14): continuous (compact) $L_2(0, T; U) \to \mathcal{E}_T \equiv L_2(0, T; Y)$ and, as in Remark 19.4.3, let $\mathcal{L}^{\circledast}$, be the adjoint defined by

$$(\mathcal{L}u, y)_{L_2(0,T;Y)} = (u, \mathcal{L}^{\circledast} y)_{L_2(0,T;U)}, \quad \forall\, u \in L_2(0, T; U), \quad y \in L_2(0, T; Y); \tag{19.6.12a}$$

$$\mathcal{L}^{\circledast} : \text{continuous (compact)} : L_2(0, T; Y) \to L_2(0, T; U). \tag{19.6.12b}$$

LEMMA 19.6.2

With reference to Eqs. (19.6.11) *and* (19.6.12), *we have*

$$\mathcal{L}^{\circledast} y = \mathcal{L}^{\circledast} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathcal{L}^* \begin{bmatrix} \mathcal{A}^{-1} y_1 \\ \mathcal{A}^{-1} y_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in Y \equiv L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'. \tag{19.6.13}$$

PROOF We have that

$$\begin{bmatrix} \mathcal{A}^{-\frac{1}{2}} & 0 \\ 0 & \mathcal{A}^{-\frac{1}{2}} \end{bmatrix} : \text{ isomorphism } Y \equiv L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]' \overset{\text{onto}}{\to} H \equiv \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega), \tag{19.6.14}$$

hence, via Eqs. (19.6.14) and (19.6.11a)

$$(\mathcal{L}u, y)_{L_2(0,T;Y)} = \left(\mathcal{L}u, \begin{bmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \mathcal{A}^{-1} \end{bmatrix} y\right)_{L_2(0,T;H)} = \left(u, \mathcal{L}^* \begin{bmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \mathcal{A}^{-1} \end{bmatrix} y\right)_{L_2(0,T;U)}, \tag{19.6.15}$$

and Eq. (19.6.13) follows by comparing Eq. (19.6.12a) with Eq. (19.6.15). ∎

Applying the identity in Eq. (19.6.13) of Lemma 19.6.2 to Eq. (19.6.10), we obtain

$$\mathcal{M}_T^*[\eta]V_0 = \mathcal{L}_T^* V_0 + \mathcal{L}^{\circledast} \begin{Bmatrix} f'(\eta_1(\cdot))v_2(\cdot\,; V_0) \\ 0 \end{Bmatrix}, \quad V_0 = [h_0, h_1] \in H. \tag{19.6.16}$$

Step 3. Lemma 19.6.3 With reference to the operator $\mathcal{K}_T[\eta] \equiv \mathcal{R}_T F'[\eta] \in L(\mathcal{E}_T; H)$ in Eq. (19.2.24), we have $\mathcal{K}_T^*[\eta] \in L(H; \mathcal{E}_T)$ where

$$(\mathcal{K}_T^*[\eta]h)(t) = \begin{bmatrix} f'(\eta_1(t)) \left\{e^{A^*(T-t)}h\right\}_2 \\ 0 \end{bmatrix}, \quad h \in H, \tag{19.6.17a}$$

where $e^{A^*(T-t)} = e^{A(t-t)}$ ($*$ adjoint in H, see Remark 19.4.1) and recalling Eq. (19.2.3)

$$\left\{e^{A^*(T-t)}h\right\}_2 = \left\{e^{A(t-T)}h\right\}_2 = -\mathcal{A}S(t-T)h_1 + C(t-T)h_2 \tag{19.6.17b}$$

is the second component, on the space $L_2(\Omega)$, of $e^{A^*(T-t)}h$ on H.

PROOF Let $g \in \mathcal{E}_T$, $h \in H$, $\eta \in \mathcal{E}_T$. With $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, we return to Eq. (19.2.24b) and compute

$$\begin{aligned}(\mathcal{K}_T[\eta]g, h)_H &= (\mathcal{R}_T F'[\eta]g, h)_H = (g, \mathcal{K}_T^*[\eta]h)_{\mathcal{E}_T} &(19.6.18)\\
&= \left(\int_0^T e^{A(T-t)}\begin{bmatrix}0\\ f'(\eta_1(t))g_1(t)\end{bmatrix}dt, h\right)_H\\
&= \int_0^T\left(\begin{bmatrix}0\\ f'(\eta_1(t))g_1(t)\end{bmatrix}, e^{A^*(T-t)}h\right)_H dt\\
&= \int_0^T \left[g_1(t), f'(\eta_1(t))\left\{e^{A^*(T-t)}h\right\}_2\right]_{L_2(\Omega)} dt\\
&= \int_0^T\left(\begin{bmatrix}g_1(t)\\ g_2(t)\end{bmatrix}, \begin{Bmatrix}f'(\eta_1(t))\left\{e^{A^*(T-t)}h\right\}_2\\ 0\end{Bmatrix}\right)_Y dt; &(19.6.19)\end{aligned}$$

$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. Thus, comparing the right-hand side of Eq. (19.6.18) with Eq. (19.6.19), we obtain Eq. (19.6.17a). □

Step 4. We return to the operator $\mathcal{K}[\eta] = \mathcal{R}F'[\eta]$: continuous (compact) $\mathcal{E}_T \to \mathcal{E}_T$ as noted in Eq. (19.2.27), and let $\mathcal{K}^{\circledast}[\eta] \in L(\mathcal{E}_T)$ be its adjoint, defined in Remark 19.4.3(iii).

LEMMA 19.6.4
For the adjoint $\mathcal{K}^{\circledast}[\eta]$, we have for $e = [e_1, e_2] \in \mathcal{E}_T$:

$$\left(\mathcal{K}^{\circledast}[\eta]\begin{bmatrix}e_1\\ e_2\end{bmatrix}\right)(t) = \begin{bmatrix}f'(\eta_1(t))\int_t^T \mathcal{A}^{-1}\left\{e^{A(t-\sigma)}e(\sigma)\right\}_2 d\sigma\\ 0\end{bmatrix} \tag{19.6.20a}$$

$$= \begin{bmatrix}f'(\eta_1(t))\int_t^T [S(\sigma-t)e_1(\sigma) + \mathcal{C}(t-\sigma)\mathcal{A}^{-1}e_2(\sigma)]\, d\sigma\\ 0\end{bmatrix}. \tag{19.6.20b}$$

PROOF For $g = [g_1, g_2] \in \mathcal{E}_T$, $e = [e_1, e_2] \in \mathcal{E}_T$, we compute with $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, recalling Eq. (19.2.22):

$$\begin{aligned}(\mathcal{K}[\eta]g, e)_{L_2(0,T;Y)} &\equiv (g, \mathcal{K}^{\circledast}[\eta]e)_{L_2(0,T;Y)} &(19.6.21)\\
&= \int_0^T\left(\int_0^t e^{A(t-\tau)}\begin{bmatrix}0\\ f'(\eta_1(\tau))g_1(\tau)\end{bmatrix}d\tau, \begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}\right)_Y dt &(19.6.22)\\
&= \int_0^T\int_0^t\left(\begin{bmatrix}0\\ f'(\eta_1(\tau))g_1(\tau)\end{bmatrix}, e^{A^{\circledast}(t-\tau)}e(t)\right)_Y d\tau dt &(19.6.23)\end{aligned}$$

(changing the order of integration and using skew-adjointness $A^{\circledast} = -A$ on Y)

$$= \int_0^T \int_\tau^T \left(\begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix}, e^{A(\tau-t)}e(t) \right)_Y dt d\tau \tag{19.6.24}$$

$$= \int_0^T \left(\begin{bmatrix} 0 \\ f'(\eta_1(\tau))g_1(\tau) \end{bmatrix}, \int_\tau^T e^{A(\tau-t)}e(t)\, dt \right)_Y d\tau \tag{19.6.25}$$

$$= \int_0^T \left(f'(\eta_1(\tau))g_1(\tau), \int_\tau^T \mathcal{A}^{-1} \left\{ e^{A(\tau-t)}e(t) \right\}_2 dt \right)_{L_2(\Omega)} d\tau \tag{19.6.26}$$

$$= \int_0^T \left(\begin{bmatrix} g_1(\tau) \\ g_2(\tau) \end{bmatrix}, \begin{bmatrix} f'[\eta_1(\tau)] \int_\tau^T \mathcal{A}^{-1} \left\{ e^{A(\tau-t)}e(t) \right\}_2 dt \\ 0 \end{bmatrix} \right)_Y d\tau. \tag{19.6.27}$$

Thus, comparing the right-hand side of Eq. (19.6.21) with Eq. (19.6.27) yields Eq. (19.6.20a), from which Eq. (19.6.20b) readily follows via Eq. (19.2.3). ∎

Step 5.

LEMMA 19.6.5

Let $V_0 \in H$, $\eta \in \mathcal{E}_T$. With reference to $\mathcal{K}_T^*[\eta]$ in Eq. (19.6.17a) *and $\mathcal{K}^{\circledast}[\eta]$ in Eq.* (19.6.20a)*, we have:*

1.
$$[I - \mathcal{K}^{\circledast}[\eta]] \begin{bmatrix} f'(\eta_1(\cdot))v_2(\cdot\,; V_0) \\ 0 \end{bmatrix} = \mathcal{K}_T^*[\eta]V_0; \tag{19.6.28}$$

2.
$$\begin{bmatrix} f'(\eta_1(\cdot))v_2(\cdot\,; V_0) \\ 0 \end{bmatrix} = [I - \mathcal{K}^{\circledast}[\eta]]^{-1}\mathcal{K}_T^*[\eta]V_0. \tag{19.6.29}$$

PROOF

1. We return to the identity of Eq. (19.6.7), in H, extract the second component in $L_2(\Omega)$, multiply this by $f'[\eta_1(t)]$ and obtain since $V = [v_1, v_2]^{tr}$, $v_2 = \phi$; see Eqs.(19.6.4a and b):

$$\begin{aligned} f'(\eta_1(t))v_2(t; V_0) &= f'(\eta_1(t)) \left\{ e^{A^*(T-t)}V_0 \right\}_2 \\ &\quad + f'(\eta_1(t)) \int_t^T \left\{ e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau; V_0)] \\ 0 \end{bmatrix} \right\}_2 d\tau. \end{aligned} \tag{19.6.30}$$

But, recalling Eq. (19.2.3), we have with $e^{A^*(\tau-t)} = e^{A(t-\tau)}$, because $S(\cdot)$ is odd:

$$\begin{aligned} \left\{ e^{A^*(\tau-t)} \begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau; V_0)] \\ 0 \end{bmatrix} \right\}_2 &= -\mathcal{A}S(t-\tau)\mathcal{A}^{-1}f'(\eta_1(\tau))v_2(\tau; V_0) \\ &= S(\tau - t)f'(\eta_1(\tau))v_2(\tau; V_0). \end{aligned} \tag{19.6.31}$$

Then, Eq. (19.6.30), augmented by a null second component, becomes, by recalling Eqs. (19.6.17a), (19.6.31), and (19.6.20b):

$$\begin{bmatrix} f'(\eta_1(t))v_2(t;V_0) \\ 0 \end{bmatrix} = \begin{bmatrix} f'(\eta_1(t))\left\{e^{A^*(T-t)}V_0\right\}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} f'(\eta_1(t))\displaystyle\int_t^T S(\tau-t)f'(\eta_1(\tau))v_2(\tau;V_0)\,d\tau \\ 0 \end{bmatrix} \tag{19.6.32}$$

$$= (\mathcal{K}_T^*[\eta]V_0)(t) + \left(\mathcal{K}^{\circledast}[\eta]\begin{bmatrix} f'(\eta_1(\cdot))v_2(\cdot;V_0) \\ 0 \end{bmatrix}\right)(t). \tag{19.6.33}$$

Then, Eq. (19.6.33) establishes Eq. (19.6.28) and part (1) is proved.

2. Part (2) follows from Eq. (19.6.28) as in obtaining Eq. (19.4.2) from Eq. (19.4.3).

∎

Step 6. Substituting Eq. (19.6.29) into the right-hand side of Eq. (19.6.16) yields the identity of Eq. (19.6.2a) = Eq. (19.4.35a), as desired.

The following result is contained in the proof of Proposition 19.6.1.

COROLLARY 19.6.6

The following identities hold true:

$$\int_T^t B^*e^{A(t-\tau)}\begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix}d\tau$$

$$= \int_t^T B^*e^{A^*(\tau-t)}\begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau;V_0)] \\ 0 \end{bmatrix}d\tau \tag{19.6.34}$$

$$= \mathcal{L}^*\begin{bmatrix} \mathcal{A}^{-1}[f'(\eta_1(\cdot))v_2(\cdot;V_0)] \\ 0 \end{bmatrix} = \mathcal{L}^{\circledast}\begin{bmatrix} [f'(\eta_1(\cdot))v_2(\cdot;V_0)] \\ 0 \end{bmatrix}$$

$$= \mathcal{L}^{\circledast}[I-\mathcal{K}^{\circledast}(\eta)]^{-1}\mathcal{K}_T^*[\eta]V_0. \tag{19.6.35}$$

PROOF We refer to, and trace, Eqs. (19.6.6), (19.6.9), (19.6.10), (19.6.13), (19.6.16), and (19.6.29). ∎

PDE Interpretation of the Formula in Eq. (19.6.2a) = Eq. (19.4.35)

The following result provides a PDE interpretation of the perturbation formula Eq. (19.6.2a) = Eq. (19.4.35) for $\mathcal{M}_T^*[\eta]$. To this end, we decompose the ϕ-problem of Eq. (19.6.3), $\phi(t;\Phi_0) = v_2(t;V_0)$, as

$$\phi = \psi + \chi, \tag{19.6.36}$$

where ψ and χ are solutions of the following problems

$$\begin{cases} \psi_{tt} = \Delta\psi \\ \psi(T,\cdot) = \phi_0;\ \psi_t(T,\cdot) = \phi_1 \\ \psi|_{\Sigma_0} \equiv 0;\ \left.\frac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 \end{cases} \quad \begin{cases} \chi_{tt} = \Delta\chi + f'[\eta_1(t,x)]\phi & \text{in } Q; \\ \chi(T,\cdot) = 0;\ \chi_t(T,\cdot) = 0 & \text{in } \Omega; \\ \chi|_{\Sigma_0} \equiv 0;\ \left.\frac{\partial\chi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{in } \Sigma_i, \end{cases} \quad \begin{matrix}(19.6.37a)\\(19.6.37b)\\(19.6.37c)\end{matrix}$$

with $\phi_0 = h_1 \in L_2(\Omega)$, $\phi_1 = -\mathcal{A}h_0 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$; $V_0 \equiv \{h_0, h_1\} \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, hence $\{\psi(T,\cdot), \psi_t(T,\cdot)\} = \{\phi_0, \phi_1\} \in L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$. Explicitly, ψ and χ are given by:

$$\psi(t) \equiv C(t-T)h_1 + S(t-T)(-\mathcal{A}h_0) = \left\{e^{A^*(T-t)}V_0\right\}_2, \quad V_0 = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \in H, \quad (19.6.38)$$

recalling, in the last step, Eq. (19.6.17b), or Eq. (19.2.3), and via the variation of parameter formula,

$$\chi(t) = \int_T^t S(t-\tau) f'(\eta_1(\tau)) v_2(\tau; V_0)\, d\tau \in C\left\{[0,T]; \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right\}$$
$$v_2(t; V_0) \equiv \phi(t; \Phi_0) \in C([0,T]; L_2(\Omega)). \quad (19.6.39)$$

We notice that via Eq. (19.6.38) and Eq. (19.6.39), the *decomposition Eq.* (19.6.36) *already appeared in the top line of the identity of* Eq. (19.6.32), stripped by the common factor $f'(\eta_1(t))$ across.

PROPOSITION 19.6.7
With reference to the problem in Eq. (19.6.37) *and the identity of Eq.* (19.6.2a) = *Eq.* (19.4.35a)*:* $\mathcal{M}_T^*[\eta] = \mathcal{L}_T^* + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]$*, we have:*

1. $$\{\mathcal{L}_T^* V_0\}(t) = B^* e^{A^*(T-t)} V_0 = \left[\left\{e^{A^*(T-t)}V_0\right\}_2\right]_{\Sigma_1} \equiv \psi(t)|_{\Sigma_1}; \quad V_0 \in \mathcal{D}(\mathcal{L}_T^*); \quad (19.6.40)$$

2. For $V_0 \in H$*:*

$$\{\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]V_0\}(t)$$
$$= \int_T^t B^* e^{A(t-\tau)} \begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau; V_0)] \\ 0 \end{bmatrix} d\tau \quad (19.6.41)$$
$$= \int_T^t \left[\left\{ e^{A(t-\tau)} \begin{bmatrix} -\mathcal{A}^{-1}[f'(\eta_1(\tau))v_2(\tau; V_0)] \\ 0 \end{bmatrix}\right\}_2\right]_{\Sigma_1} \quad (19.6.42)$$
$$= \int_T^t [S(t-\tau) f'(\eta_1(\tau)) v_2(\tau; V_0)]_{\Sigma_1} d\tau = \chi(t)_{\Sigma_1} \in C([0,T]; H^{\frac{1}{2}}(\Gamma)); \quad (19.6.43)$$

3. For $V_0 \in \mathcal{D}(\mathcal{L}_T^*)$*:*

$$\{\mathcal{M}_T^*[\eta]V_0\}(t) = \{\mathcal{L}_T^* V_0\}(t) + \{\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]V_0\}(t)$$
$$\equiv \psi(t; \Phi_0)|_{\Sigma_1} + \chi(t)|_{\Sigma_1} = v_2(t; V_0)|_{\Sigma_1} = \phi(t; \Phi_0)|_{\Sigma_1}. \quad (19.6.44)$$

PROOF The first identity in Eq. (19.6.40) is the usual duality formula, see Eqs. (19.6.8) and (19.6.9). Next, we recall—see Eq. (19.9.8b)—that (#): $B^*[{}^a_b] = b|_{\Sigma_1}$. This formula and Eq. (19.6.38) then justify the last two identities in Eq. (19.6.40).

The identity in Eq. (19.6.41) was already noted in Corollary 19.6.6, Eqs. (19.6.34) and (19.6.35). Next, we use the above formula (#) for B^* to get Eq. (19.6.42). Next, we recall Eq. (19.6.31) as well as $A^* = -A$ to obtain the left-hand side of Eq. (19.6.43). Finally, we invoke Eq. (19.6.39) to obtain the right-hand side of Eq. (19.6.43). ∎

19.7 Direct Proof of the Uniform COI of Eq. (19.5.5) via Lemmas 19.4.4 and 19.4.5 (in Dual Versions) (Thus Using Properties (P.1), (P.2), (P.3), (C.1), and (C.2))

Orientation

In Section 19.5 we have extracted two basic strategies for proving Theorem 19.1.1, and, moreover, we have shown their equivalence in Proposition 19.5.1. Strategy 1 consists of showing the uniform invertibility condition of Eq. (19.4.18b) = Eq. (19.5.7) for the operator $[I + \Lambda'_T(q)]^{-1}$. Strategy 2 consists of showing the uniform COI of Eq. (19.5.5), that is, the uniform lower bound for the operator $\mathcal{M}^*_T[q]$. The equivalence between these two strategies was proved in Proposition 19.5.1 and rests on two facts: (a) the basic relationship of Eq. (19.4.39b): $\mathcal{M}^*_T[q](\mathcal{L}^\#_T)^* = [I + \Lambda'_T(q)]^*$, between the relevant operators, and (b) the property that $(\mathcal{L}^\#_T)$: the state target space H onto the control space $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$. So far, however, it was the uniform invertibility condition of Eq. (19.5.7) involving $[I + \Lambda'_T(q)]^{-1}$—not the uniform COI of Eq. (19.5.10) involving $\mathcal{M}^*_T(q)$—that has been established in Section 19.4, Lemma 19.4.6, by virtue of Lemma 19.4.4 and Lemma 19.4.5, on the basis of the structural properties (P.1), (P.2), and (P.3) and the control properties (C.1) and (C.2).

In contrast, the *goal* of the present section is to prove the uniform COI of Eq. (19.5.5) *directly*, on the basis of Lemmas 19.4.4 and 19.4.5 in dual versions.

We begin the present proof by returning to Eq. (19.4.35a) and its limit case of Eq. (19.4.35b). We rewrite these formulas for convenience:

$$\mathcal{M}^*_T[\eta] \equiv \mathcal{L}^*_T + \mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}^*_T[\eta] : H \supset \mathcal{D}(\mathcal{L}^*_T) \to L_2(0, T; U); \quad (19.7.1a)$$

$$\left(\mathcal{M}^0_T\right)^* = \mathcal{L}^*_T + \mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}\left(\mathcal{K}^0_T\right)^* : H \supset \mathcal{D}(\mathcal{L}^*_T) \to L_2(0, T; U); \quad (19.7.1b)$$

We recall from Eq. (19.4.8a) that we are throughout considering $\mathcal{L}_T$ as an injective closed surjective operator from $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ onto H [via (C.1) = (19.3.5a)]. Hence, the range $\mathcal{L}^*_T H$ is dense in $[\mathcal{N}(\mathcal{L}_T)]^\perp$ in the $L_2(0, T; U)$-topology. But, by (C.1) in its equivalent version of Eq. (19.3.5b), $(\mathcal{L}^*_T)^{-1}$ is bounded on $\mathcal{L}^*_T H$. Hence, $(\mathcal{L}^*_T)^{-1}$ is bounded on $[\mathcal{N}(\mathcal{L}_T)]^\perp$: $(\mathcal{L}^*_T)^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^\perp; H)$. With reference to Eq. (19.7.1), because the range of $\mathcal{L}^{\circledast}$ need not be contained in $[\mathcal{N}(\mathcal{L}_T)]^\perp$, we now *extend* $(\mathcal{L}^*_T)^{-1}$ *by zero on* $\mathcal{N}(\mathcal{L}_T)$:

$$(\mathcal{L}^*_T)^{-1} \equiv 0 \text{ on } [\mathcal{N}(\mathcal{L}_T)], \text{ so that } (\mathcal{L}^*_T)^{-1} \in L(L_2(0, T; U); H). \quad (19.7.2)$$

A similar extension will be used in Eq. (19.8.12). Next, we apply such (extended) $(\mathcal{L}^*_T)^{-1}$ across Eqs. (19.7.1a) and (19.7.1b), thus obtaining

$$\{I + (\mathcal{L}^*_T)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}^*_T[\eta]\}h = (\mathcal{L}^*_T)^{-1}\mathcal{M}^*_T[\eta]h, \quad h \in \mathcal{D}(\mathcal{L}^*_T) \subset H; \quad (19.7.3a)$$

$$\{I + (\mathcal{L}^*_T)^{-1}\mathcal{L}^{\circledast}(I - (\mathcal{K}^0)^{\circledast}]^{-1}\left(\mathcal{K}^0_T\right)^*\}h = (\mathcal{L}^*_T)^{-1}\left(\mathcal{M}^0_T\right)^* h, \quad h \in \mathcal{D}(\mathcal{L}^*_T) \subset H. \quad (19.7.3b)$$

Our present *goal* is to show the uniform COI of Eq. (19.5.5), rewritten here in terms of the parameter $\eta \in \mathcal{E}_T$ rather than the parameter (potential) $q \in L_\infty(Q)$ {eventually, $q(t, x) = f'[\eta_1(t, x)]$} as

$$\|h\|_H \le C_T \|\mathcal{M}^*_T[\eta]h\|_{L_2(0,T;U)}, \quad \forall\, h \in \mathcal{D}(\mathcal{L}^*_T) \text{ uniformly in } \eta \in \mathcal{E}_T, \quad (19.7.4)$$

for some $C_T > 0$ independent of h and η. Thus, starting from Eq. (19.7.3a), in order to obtain the desired uniform estimate of Eq. (19.7.4), *we need to justify* the following two steps in progression:

(i) first, inversion for each $\eta \in \mathcal{E}_T$,

$$h = \{I + (\mathcal{L}^*_T)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}^*_T[\eta]\}^{-1}(\mathcal{L}^*_T)^{-1}\mathcal{M}^*_T[\eta]h, \quad h \in \mathcal{D}(\mathcal{L}^*_T), \quad (19.7.5a)$$

as well as (we shall see below) its limit version

$$h = \{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^*\}^{-1}(\mathcal{L}_T^*)^{-1}(\mathcal{M}_T^0)^* h, \quad h \in \mathcal{D}(\mathcal{L}_T^*); \tag{19.7.5b}$$

(ii) second, uniform bound in $\eta \in \mathcal{E}_T$ of the inverse of Eq. (19.7.5a):

$$\|h\|_H \le \|\{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]\}^{-1}\|_{L(H)}$$

$$\|(\mathcal{L}_T^*)^{-1}\|_{L(L_2(0,T;U);H))}\|\mathcal{M}_T^*[\eta]h\|_{L_2(0,T;U)} \tag{19.7.6}$$

$$\le C_T\|\mathcal{M}_T^*[\eta]h\|_{L_2(0,T;U)}, \quad \forall\, h \in \mathcal{D}(\mathcal{L}_T^*) \subset H. \tag{19.7.7}$$

The inversion occurring in Eq. (19.7.5a) is the present counterpart of the inversion in Eq. (19.4.47) of Section 19.4. As there, it actually involves a double inversion, one internal and one external, each of which is required to be uniform in $\eta \in \mathcal{E}_T$ in order to obtain Eq. (19.7.7) from Eq. (19.7.6), by use also of Eq. (19.7.2).

Justification of inversion occurring in Eq. (19.7.5a) *for each* $\eta \in \mathcal{E}_T$ *and its limit version in Eq.* (19.7.5b): These two inversions are justified as in Proposition 19.4.3(c) in asserting the inversion (19.4.43) for $[I + \Lambda_T'(u)]^{-1}$ and its limit version in Eq. (19.4.44). Namely, they rely on the compactness property of the perturbations of the identity, that is

$$(\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] : \text{ compact } H \to H; \tag{19.7.8a}$$

$$(\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}(\mathcal{K}_T^0)^* : \text{ compact } H \to H, \tag{19.7.8b}$$

which are readily derived from Remark 19.4.3 [in the following order: (iv), (iii), (ii)], as well as the property in Eq. (19.7.2) for the (extended) operator $(\mathcal{L}_T^*)^{-1}$. The properties of Eq. (19.7.8a and b) are the counterpart versions of the properties of Eqs. (19.4.40a and b). In both scenarios, critical is the property of compactness of $\mathcal{L}^{\circledast}$ as in Remark 19.4.3(ii), due to (P.1) = Eq. (19.2.14).

Justification of uniform inversion from Eq. (19.7.6) *to Eq.* (19.7.7): To this end, we first invoke Lemma 19.4.4, in a version dual to Eq. (19.4.49), to assert that

$$\|(I - \mathcal{K}^{\circledast}[\eta])^{-1}\|_{L(\mathcal{E}_T)} \le \text{const., uniformly in } \eta \in \mathcal{E}_T, \tag{19.7.9}$$

and next we invoke Lemma 19.4.5 (in dual version) with, now

$$Z_1 \equiv H,\ Z_2 \equiv \mathcal{E}_T;\ Q^* \equiv (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast} : \text{ compact } Z_2 \equiv \mathcal{E}_T \to Z_1 \equiv H; \tag{19.7.10}$$

$$W_T[\eta] \equiv \mathcal{K}_T[\eta](I - \mathcal{K}[\eta])^{-1} : \text{ continuous } Z_2 \equiv \mathcal{E}_T \to Z_1 \equiv H; \tag{19.7.11}$$

$$Q^* W_T^*[\eta] \equiv (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta] \tag{19.7.12}$$

(counterpart of the setting: Eqs. (19.4.52), (19.4.53), (19.4.62), and (19.4.63) in Section 19.4). As to the verification of the assumptions of Lemma 19.4.5 in the present setting, we note that:

(a) the required weak convergence (see Eq. (19.4.55))

$$W_T\left[\eta_{n_k}\right] \equiv \mathcal{K}_T\left[\eta_{n_k}\right](I - \mathcal{K}\left[\eta_{n_k}\right])^{-1} \to W^0 \equiv \mathcal{K}_T^0(I - \mathcal{K}^0)^{-1} \text{ weakly from } \mathcal{E}_T \text{ to } H \tag{19.7.13}$$

—equivalent to weak convergence of corresponding adjoints—follows as in the case of Eq. (19.4.54): by virtue of combining property (P.2)(b), Eq. (19.2.29a) with property (P.3), Eq. (19.2.35a);

(b) whereas the injectivity of

$$I + Q^* W_T^*[\eta] = I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta],$$

and its limit version

$$I + Q^*(W^0)^* = I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}[I - (\mathcal{K}^0)^{\circledast}]^{-1}\left(\mathcal{K}_T^0\right)^*,$$

were already verified in part (i), see Eqs. (19.7.5a and b), by virtue of the properties of Eqs. (19.7.8a and b).

Thus, Lemma 19.4.5 (*in dual version*) applies and yields the counterpart of the uniform bound of Eq. (19.4.56),

$$\begin{aligned}\|\{I + Q^* W_T^*[\eta]\}^{-1}\|_{L(H)} &= \{I + (\mathcal{L}_T^*)^{-1}\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[\eta])^{-1}\mathcal{K}_T^*[\eta]\}^{-1}\|_{L(H)} \\ &\leq \text{const}_T, \text{ uniformly in } \eta \in \mathcal{E}_T. \end{aligned} \tag{19.7.14}$$

Then, Eq. (19.7.14), along with the boundedness in Eq. (19.7.2), justifies the passage from Eq. (19.7.6) to Eq. (19.7.7), as desired.

Thus, the desired uniform COI of Eq. (19.7.4) has been established here *directly.* This then yields an alternative proof of Theorem 19.1.1, via Proposition 19.5.1.

Part III: Proof of Theorem 19.1.1 through the Uniform COI of Eq. (19.5.5) = Eq. (19.5.10) via the Dual Uncontrolled Problem in Eq. (19.3.9) = Eq. (19.6.3)

19.8 A Comparison of Traces of χ and ϕ in $L_2(\Sigma_1)$ by Falling into Part I

According to the analysis of Sections 19.4 and 19.5, one strategy available for proving the exact controllability result of Theorem 19.1.1 on the semilinear problem in Eq. (19.1.1) consists in establishing the following Uniform COI—see Eq. (19.5.10)—for the problem in Eq. (19.3.9) = Eq. (19.6.3): there exists a constant $C_{Tr} > 0$, such that

$$C_{Tr}\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \leq \int_{\Sigma_1} \phi^2 \, d\Sigma_1, \tag{19.8.1}$$

uniformly in $q(t, x) = f'[\eta_1(t, x)] \in B(0, r) \subset L_\infty(Q)$, where $B(0, r)$ is a ball centered at the origin of radius $r > 0$ in $L_\infty(Q)$ [ultimately $\eta_1(t, x) = w(t, x)$] where C_{Tr} depends on T and r but not on the I.C. or on q running in $B(0, r)$. To prove Eq. (19.8.1), we return to the splitting $\phi = \psi + \chi$ in Eq. (19.6.36), where ψ and χ are solutions of the problems in Eq. (19.6.37). Next, by duality on assumption (C.1)—in the setting described by Eqs. (19.3.7) and (19.3.8)—with reference to the ψ-problem in Eq. (19.6.37), we obtain its COI by invoking Reference 27, Theorem 19.3.1, p. 271 and Theorem 19.2.1, p. 254: there exists a constant $c_T > 0$ independent on the I.C. such that

$$c_T\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \leq \int_{\Sigma_1} \psi^2 d\Sigma_1 = \int_{\Sigma_1} (\phi - \chi)^2 d\Sigma_1 \tag{19.8.2}$$

$$\leq 2\int_{\Sigma_1} \phi^2 d\Sigma_1 + 2\int_{\Sigma_1} \chi^2 d\Sigma_1. \tag{19.8.3}$$

Thus, to establish the desired COI of Eq. (19.8.1), it suffices—and indeed necessary as well: see Remark 19.8.1 below—to show that, there exists a constant $\text{const}_{Tr} > 0$ depending on T and r but independent on the I.C. or on $q \in B(0, r)$, such that

$$\int_{\Sigma_1} \chi^2 \, d\Sigma_1 \leq \text{const}_{Tr} \int_{\Sigma_1} \phi^2 \, d\Sigma_1, \tag{19.8.4}$$

uniformly on $q \in B(0, r) \subset L_\infty(Q)$.

A *direct* operator-theoretic proof of the inequality of Eq. (19.8.4), or its operator version of Eq. (19.8.7) below, will be given in Section 19.9. Here, in contrast, we provide an alternative operator-theoretic proof of the inequality of Eq. (19.8.4) by actually proving its *equivalent* version of Eq. (19.8.8) = Eq. (19.8.9) below, involving the quantity $[I + (\Lambda'_T[q])^*]$ that played a critical role in the analysis of Part I. This way, we shall fall into the technical apparatus of Part I and, in particular, appeal to Lemmas 19.4.4 and 19.4.5 to complete the present proof.

Proof of Eq. (19.8.4). By Proposition 19.6.2, Eqs. (19.6.44) and (19.6.43), we have that

$$\phi(t;\Phi_0)|_{\Sigma_1} = v_2(t;V_0)|_{\Sigma_1} \equiv \{\mathcal{M}_T^*[q]V_0\}(t), \quad V_0 = \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} \in H,\ \Phi_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \in Y; \tag{19.8.5}$$

$$\chi(t)|_{\Sigma_1} \equiv \{\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}_T^*[q]V_0\}(t). \tag{19.8.6}$$

Thus, proving the inequality of Eq. (19.8.4) is, via Eq. (19.8.5) and (19.8.6), reformulated as proving that

$$\|\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}_T^*[q]V_0\|_{L_2(\Sigma_1)} \le \text{const}_{Tr}\|\mathcal{M}_T^*[q]V_0\|_{L_2(\Sigma_1)}, \quad V_0 \in \mathcal{D}(\mathcal{L}_T^*) \subset H, \tag{19.8.7}$$

uniformly in $q \in B(0,r)$ and in $V_0 \in \mathcal{D}(\mathcal{L}_T^*)$. Moreover, we have seen in the proof of Proposition 19.5.1, that $(\mathcal{L}_T^{\#})^*$ is an isomorphism from $[\mathcal{N}(\mathcal{L}_T)]^\perp$ onto $\mathcal{D}(\mathcal{L}_T^*)$. Then the inequality of Eq. (19.8.7) is, in turn, equivalent to the following inequality: There exists a constant $C'_{Tr} > 0$ such that

$$\|\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}_T^*[q](\mathcal{L}_T^{\#})^*u\|_{L_2(\Sigma_1)} \le C'_{Tr}\|\mathcal{M}_T^*[q](\mathcal{L}_T^{\#})^*u\|_{L_2(\Sigma_1)}, \tag{19.8.8}$$

uniformly in $q \in B(0,r)$ and $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(0,T;L_2(\Gamma_1))$. But, by Eqs. (19.4.40a) and (19.4.39b), we can rewrite inequality of Eq. (19.8.8) as

$$\|(\Lambda'_T[q])^*u\|_{L_2(\Sigma_1)} \le C'_{Tr}\|[I + (\Lambda'_T[q])^*]u\|_{L_2(\Sigma_1)}, \tag{19.8.9}$$

uniformly in $q \in B(0,r)$ and $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp \subset L_2(\Sigma_1)$. We now establish Eq. (19.8.9), thereby completing the proof of Eq. (19.8.7), hence of Eq. (19.8.4). We write for $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$:

$$\begin{aligned} \|(\Lambda'_T[q])^*u\|_{L_2(\Sigma_1)} &= \|(I + \Lambda'_T[q])^*u - u\|_{L_2(\Sigma_1)} \\ &\le \|(I + \Lambda'_T[q])^*u\|_{L_2(\Sigma_1)} + \|u\|_{L_2(\Sigma_1)}. \end{aligned} \tag{19.8.10}$$

Moreover, from Section 19.4, Eq. (19.4.64) we have the inequality, same as Eq. (19.5.6)

$$\|u\|_{L_2(\Sigma_1)} \le \text{const}_{Tr}\|(I + \Lambda'_T[q])^*u\|_{L_2(\Sigma_1)}, \tag{19.8.11}$$

uniformly in $q \in B(0,r)$ and $u \in [\mathcal{N}(\mathcal{L}_T)]^\perp$. Substituting Eq. (19.8.11) into the RHS of Eq. (19.8.10) yields Eq. (19.8.9), as desired. Indeed, the inequality of Eq. (19.8.11) is guaranteed by inequality of Eq. (19.4.64) of Lemma 19.4.6 [which was proved by virtue of properties (P.1), (P.2), (P.3), (C.1), and (C.2) ultimately via Lemmas 19.4.4 and 19.4.5]. Thus, the inequality of Eq. (19.5.10) = Eq. (19.8.1) is established, and this, in turn, proves Theorem 19.1.1 by virtue of Proposition 19.5.2.

REMARK 19.8.1 We have seen above that the inequality of Eq. (19.8.4), used in Eq. (19.8.3), implies inequality (19.8.1). Conversely, here we show that the *inequality of Eq.* (19.8.1) *implies inequality of Eq.* (19.8.4).

In fact, we return to Eq. (19.6.39) and estimate via trace theory:

$$\|\chi\|_{L_2(\Sigma_1)} \le C\|\chi\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} \le c_T\|\phi(\,\cdot\,;\Phi_0)\|_{L_2(0,T;L_2(\Omega))} \tag{19.8.12}$$

$$\le C_{Tr}\|\{\phi_0,\phi_1\}\|_{Y=L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \tag{19.8.13}$$

uniformly in $q \in B(0, r)$, where in the last step we have invoked the regularity result of Eq. (19.3.11b) of Theorem 19.3.1. Thus, if Eq. (19.8.1) holds true, then Eq. (19.8.13) shows that Eq. (19.8.4) holds true as well.

REMARK 19.8.2 We have seen above that the inequality of Eq. (19.8.11), used in Eq. (19.8.10), implies the inequality of Eq. (19.8.9). Conversely, *the inequality* of Eq. (19.8.9) *implies the inequality* of Eq. (19.8.11), as it follows from

$$\|u\| = \|(I + \Lambda'_T[q])^* u - (\Lambda'_T[q])^* u\| \leq \|(I + \Lambda'_T[q])^* u\| + \|(\Lambda'_T[q])^* u\|, \qquad (19.8.14)$$

by substituting Eq. (19.8.9) in the right-hand side of Eq. (19.8.14).

Part IV: Another Proof of Theorem 19.1.1 via the Uniform COI of Eq. (19.5.10) = Eq. (19.8.1); by Showing Eq. (19.8.4) = Eq. (19.8.7) Directly

The present Part IV provides an alternative proof of the exact controllability Theorem 19.1.1, by again establishing the uniform COI of Eq. (19.5.10) = Eq. (19.8.1) for the dual uncontrolled ϕ-problem of Eq. (19.3.9) = Eq. (19.6.3). In common with Part III, this goal is achieved by proving the trace inequality of Eq. (19.8.4) = Eq. (19.8.7). Unlike Part III, however, the present proof of Eq. (19.8.4) = Eq. (19.8.7) is *direct*, while Section 19.8 of Part III established the *equivalent* inequality of Eq. (19.8.8) = Eq. (19.8.9). The present direct proof manages to use only one inversion—the counterpart of Lemma 19.4.4. By contrast, the double inversion of Lemma 19.4.5 is now entirely dispensed with. The present analysis still relies critically on suitable families of collectively compact operators, as in the context of Part I, Section 19.4, up to Lemma 19.4.4.

19.9 Direct Operator-Theoretic Proof of Trace Inequality of Eq. (19.8.4) = Eq. (19.8.7)

Orientation In the present section, we seek to prove *directly* the uniform inequality of Eq. (19.8.4):

$$\int_{\Sigma_1} \chi^2 d\Sigma_1 \leq \text{const}_{Tr} \int_{\Sigma_1} \phi^2 d\Sigma_1, \qquad (19.9.1)$$

rewritten in operator form, via Eq. (19.6.43) = Eqs. (19.8.5) and (19.6.44) = Eq. (19.8.6)—this was already done in Eq. (19.8.7)—as:

$$\|\mathcal{L}^{\circledast}(I - \mathcal{K}^{\circledast}[q])^{-1}\mathcal{K}_T^*[q]V_0\|_{L_2(\Sigma_1)} \leq \text{const}_{Tr}\|\mathcal{M}_T^*[q]V_0\|_{L_2(\Sigma_1)}, \qquad (19.9.2)$$

uniformly in $q \in B(0, r)$ [ball centered at the origin of radius $r > 0$ in $L_\infty(Q)$, see (19.A.20)] and $V_0 \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$, where ultimately $q(t, x) = f'(\eta_1(t, x))$. As we have seen in Section 19.8, the inequality of Eq. (19.9.1) = Eq. (19.9.2) is *equivalent* to the uniform COI of Eq. (19.5.5), which in turn yields the exact controllability result of Theorem 19.1.1 for the semilinear problem of Eq. (19.1.1), see Figure 19.5.1, by Proposition 19.5.1.

In Section 19.8, we have proved *not* the inequality of Eq. (19.9.1) = Eq. (19.9.2) directly, but rather its *equivalent* formulation of Eq. (19.8.8) = Eq. (19.8.9) via the isomorphism $(\mathcal{L}_T^{\#})^*$ between $[\mathcal{N}(\mathcal{L}_T)]^{\perp}$ and $\mathcal{D}(\mathcal{L}_T^*)$, which connects Eq. (19.9.2) with Eq. (19.8.8) = Eq. (19.8.9). This strategy had the advantage of making the present task of showing Eq. (19.9.1) = Eq. (19.9.2) fall into the setting of Section 19.4, when it came to establish the required inequality of Eq. (19.8.11). In fact, to show Eq. (19.8.11), all we needed in Section 19.8 was to invoke the inequality of Eq. (19.4.64) of Lemma 19.4.6, which was proved via Lemmas 19.4.4 and 19.4.5.

By contrast, in this section we are going to prove the inequality of Eq. (19.9.1) = Eq. (19.9.2) *directly,* without passing through the aforementioned isomorphism $(\mathcal{L}_T^{\#})^*$. This way, we shall provide a treatment that is different from, but partially parallel to, that of Section 19.4; more precisely, it will use the results on collectively compact families of operators in [1] behind Lemma 19.4.4, however, in a different topological setting.

The counterpart of Lemma 19.4.5 will now no longer be needed. Thus, only one uniform inversion is encountered, the counterpart of Lemma 19.4.4. The second uniform inversion of Lemma 19.4.5 in Part I is now entirely dispensed with.

Step 1. We return to Eq. (19.6.44) for η $[= q]$ equal to zero, whereby we obtain

$$\{\mathcal{M}_T^*[q=0]V_0\}(t) = \{\mathcal{L}_T^* V_0\}(t) = \psi(t;\Phi_0)|_{\Sigma_1}, \quad V_0 = \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix}, \tag{19.9.3}$$

where $\psi(t;\Phi_0)$, $\Phi_0 = [\phi_0, \phi_1] \in Y$, solves the problem of Eq. (19.6.37a-b-c) on the LHS. The formula of Eq. (19.9.3) (unperturbed case with $q \equiv 0$) is well known; see Reference 27, Eq. (19.3.15a), p. 273. As in Eq. (19.8.2), invoking Reference 27, Theorem 3.1, p. 271 and Theorem 2.1, p. 254, we have that under the setting described, for example, by Eqs. (19.3.6) to (19.3.8), the corresponding COI for the ψ-problem holds true: there exists a constant $c_T > 0$, independent on the I.C., such that on $\mathcal{D}(\mathcal{L}_T^*)$:

$$\left\| \mathcal{L}_T^* \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} \right\|^2_{L_2(\Sigma_1)} \equiv \int_0^T \int_{\Gamma_1} \psi^2 d\Sigma_1 \geq c_T \|\{\phi_0,\phi_1\}\|_Y^2 \equiv \|\{-\mathcal{A}^{-1}\phi_1, \phi_0\}\|_H^2, \tag{19.9.4}$$

$Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$.

The COI (19.9.4) says that the operator $\mathcal{L}_T^*$ is boundedly invertible on its range (Range $\mathcal{L}_T^*$) $= \mathcal{L}_T^* H$ in H. Because $\psi(t;\Phi_0)$ is in its range by Eq. (19.9.3), we conclude that

$$\begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} = (\mathcal{L}_T^*)^{-1}[\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}] \in H. \tag{19.9.5}$$

Hence, recalling from Eq. (19.2.2) A: topological isomorphism $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ onto $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, we obtain

$$\Phi_0 \equiv \begin{bmatrix} \phi_0 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} \begin{bmatrix} -\mathcal{A}^{-1}\phi_1 \\ \phi_0 \end{bmatrix} = A(\mathcal{L}_T^*)^{-1}[\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}] \in Y, \tag{19.9.6}$$

which gives the reconstruction map—from the Dirichlet trace on Σ_1 to its I.C.—of the solution to the ψ-problem of Eq. (19.6.37). From Eqs. (19.6.37) and (19.9.6), it follows via Eq. (19.2.3) that

$$\begin{aligned} \begin{bmatrix} \psi(t;\Phi_0) \\ \psi_t(t;\Phi_0) \end{bmatrix} &= e^{A(t-T)} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} C(t-T) & S(t-T) \\ -\mathcal{A}S(t-T) & C(t-T) \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \\ &= e^{A(t-T)} A(\mathcal{L}_T^*)^{-1}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) &\text{(19.9.7a)} \\ &\equiv \{\mathcal{O}(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1})\}(t) \in C[0,T];Y), &\text{(19.9.7b)} \end{aligned}$$

gives the reconstruction (or “observation” map) $\mathcal{O} \equiv \mathcal{O}_T$ from the Dirichlet trace on Σ_1 to the solution ψ of Eq. (19.6.37) in the interior.

To extract the first component $\psi(t;\Phi_0)$ from the identity of Eq. (19.9.7), we shall write

$$\begin{aligned} \psi(\,\cdot\,;\Phi_0) &= \mathcal{O}_1\left(\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\right) = \{\mathcal{O}[\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}]\}_1 \\ &= [C(\cdot - T) \quad S(\cdot - T)]A(\mathcal{L}_T^*)^{-1}\left[\psi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\right]. \end{aligned} \tag{19.9.8}$$

We recall from Eq. (19.4.8) that we are throughout considering $\mathcal{L}_T$ as an injective closed operator from $[\mathcal{N}(\mathcal{L}_T)]^\perp \cap \mathcal{D}(\mathcal{L}_T)$ onto H via Eq. (19.C.1) = Eq. (19.3.5). Hence, the range $\mathcal{L}_T^* H$ is dense in $[\mathcal{N}(\mathcal{L}_T)]^\perp$. But, by Eq. (19.9.4), $(\mathcal{L}_T^*)^{-1}$ is bounded on $\mathcal{L}_T^* H$, as noted above in Eq. (19.9.5). Hence, $(\mathcal{L}_T^*)^{-1}$ is bounded on $[\mathcal{N}(\mathcal{L}_T)]^\perp$. Hence,

$$(\mathcal{L}_T^*)^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^\perp; H); \quad A(\mathcal{L}_T^*)^{-1} \in L([\mathcal{N}(\mathcal{L}_T)]^\perp; Y); \tag{19.9.9}$$

$$\mathcal{O} \equiv \mathcal{O}_T \equiv e^{A(\cdot - T)} A(\mathcal{L}_T^*)^{-1} : \text{ continuous } [\mathcal{N}(\mathcal{L}_T)]^\perp \to C([0, T]; Y); \tag{19.9.10}$$

$$\mathcal{O}_1 \equiv \mathcal{O}_{1,T} \equiv (\mathcal{O}_T)_1 = [C(\cdot - T) \quad S(\cdot - T)] A(\mathcal{L}_T^*)^{-1}$$
$$: \text{ continuous } [\mathcal{N}(\mathcal{L}_T)]^\perp \to C([0, T]; L_2(\Omega)); \tag{19.9.11}$$

We next extend $\mathcal{O}$ (and $\mathcal{O}_1$) by zero on $\mathcal{N}(\mathcal{L}_T)$, so that

$$\mathcal{O}_{\text{ext}} = \begin{cases} \mathcal{O} & \text{on } [\mathcal{N}(\mathcal{L}_T)]^\perp \\ 0 & \text{on } \mathcal{N}(\mathcal{L}_T) \end{cases}; \quad \mathcal{O}_{1,\text{ext}} = \begin{cases} \mathcal{O}_1 & \text{on } [\mathcal{N}(\mathcal{L}_T)]^\perp \\ 0 & \text{on } \mathcal{N}(\mathcal{L}_T) \end{cases}. \tag{19.9.12}$$

A similar extension was performed in Eq. (19.7.2) of Section 19.7. This way, because by Eq. (19.9.3) $\psi(\cdot\,; \Phi_0)|_{\Sigma_1} \in \mathcal{L}_T^* H \subset [\mathcal{N}(\mathcal{L}_T)]^\perp$, we have $\mathcal{O}_{1,\text{ext}}(\psi(\cdot\,; \Phi_0)|_{\Sigma_1}) = \mathcal{O}_1(\psi(\cdot\,; \Phi_0)|_{\Sigma_1})$, given by Eq. (19.9.8), but we can also apply $\mathcal{O}_{1,\text{ext}}$ to both $\phi(\cdot\,; \Phi_0)|_{\Sigma_1}$ and $\chi(\cdot)|_{\Sigma_1}$.

Step 2. Lemma 19.9.1. *In the notation of Eqs.* (19.9.10) *to* (19.9.12)*, and with reference to* $\phi = \psi + \chi$ *in Eqs.* (19.6.36) *and* (19.6.37)*, we have:*

$$\chi(t) - \int_t^T S(\tau - t) q(\tau) \{\chi(\tau) + [\mathcal{O}_{1,\text{ext}}(\gamma_0 \chi(\cdot))](\tau)\}\, d\tau$$
$$= \int_t^T S(\tau - t) q(\tau) \left[\mathcal{O}_{1,\text{ext}}\left(\phi(\cdot\,; \Phi_0)|_{\Sigma_1}\right)\right](\tau)\, d\tau, \tag{19.9.13}$$

where γ_0 denotes the Dirichlet trace operator, $\gamma_0 : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$, $s > \frac{1}{2}$.

PROOF We return to Eq. (19.6.39) with $S(\cdot)$ odd and with $q(\tau) = f'(\eta_1(\tau))$, $v_2(t; V_0) \equiv \phi(t; \Phi_0)$ and obtain by using $\phi = \psi + \chi$ in Eq. (19.6.36) the extended reconstruction map $\mathcal{O}_{1,\text{ext}}$, which coincides with $\mathcal{O}_1$ on $\psi(\cdot\,; \Phi_0)|_{\Sigma_1}$, given by Eqs. (19.9.8) and (19.9.12):

$$\chi(t) = \int_T^t S(t - \tau) q(\tau) \phi(\tau; \Phi_0)\, d\tau$$
$$= \int_t^T S(\tau - t) q(\tau) \psi(\tau; \Phi_0)\, d\tau + \int_t^T S(\tau - t) q(\tau) \chi(\tau)\, d\tau \tag{19.9.14}$$

by Eq. (19.9.8)
$$= \int_t^T S(\tau - t) \mathcal{O}_{1,\text{ext}}\left[\psi(\cdot\,; \Phi_0)|_{\Sigma_1}\right](\tau) + \int_t^T S(\tau - t) q(\tau) \chi(\tau)\, d\tau$$
$$= \int_t^T S(\tau - t) q(\tau) \mathcal{O}_{1,\text{ext}}\left[\phi(\cdot\,; \Phi_0)|_{\Sigma_1}\right](\tau)\, d\tau$$
$$+ \int_t^T S(\tau - t) q(\tau) \chi(\tau)\, d\tau \tag{19.9.15}$$
$$- \int_t^T S(\tau - t) q(\tau) \mathcal{O}_{1,\text{ext}}\left(\chi(\cdot)|_{\Sigma_1}\right)(\tau)\, d\tau, \tag{19.9.16}$$

and Eq. (19.9.16) establishes Eq. (19.9.13). ∎

Step 3. Next, for $v \in L_2(0, T; L_2(\Omega))$, and $q \in L_\infty(Q)$ as usual, we introduce the operator

$$(\mathcal{T}[q]v)(t) = \int_t^T S(\tau - t)q(\tau)v(\tau)\,d\tau \in C\big([0, T]; \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\big), \tag{19.9.17}$$

continuously, see Eq. (19.2.3). By comparison with Eq. (19.2.22) [where $f'(\eta_1(\tau))$ is replaced now by $q(\tau)$], we see that: *the operator $\mathcal{T}[q]$ is the $L_2[0, T; L_2(\Omega)]$-adjoint of the first component of the operator $\mathcal{K}[q]$ in Eq.* (19.2.22). Then, in view of Eq. (19.9.17), we can rewrite the identity of Eq. (19.9.13) for $\chi \in C\{[0, T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}})\}$, see (C.13), as

$$\{I - \mathcal{T}[q] + \mathcal{T}[q]\mathcal{O}_{1,\text{ext}}\gamma_0\}\chi = \mathcal{T}[q]\mathcal{O}_{1,\text{ext}}\left[\phi(\,\cdot\,; \Phi_0)|_{\Sigma_1}\right]. \tag{19.9.18}$$

Step 4. To fit into the setting of the operator $\mathcal{K}[\eta]$ in Eq. (19.2.22), we complement (19.9.17) by introducing the notation

$$(\mathcal{T}[\eta_1]v)(t) = \int_t^T S(\tau - t)f'(\eta_1(\tau))v(\tau)\,d\tau \tag{19.9.19a}$$

$$: \text{ continuous } L_1(0, T; L_2(\Omega)) \to C\big([0, T]; \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\big) \tag{19.9.19b}$$

via the regularity of Eq. (19.2.3) with η_1 (parameter) $\in L_2[0, T; L_2(\Omega)]$ and with f subject to Eq. (19.1.2). We next collect relevant properties of the operator $\mathcal{T}[\eta_1]$ [cf., property (P.2) ((19.2.27)–(19.2.29)) for the operator $\mathcal{K}[\eta]$ in Eq. (19.2.22).

PROPOSITION 19.9.2

With reference to $\mathcal{T}[\eta_1]$ in Eq. (19.9.19), *the following properties hold true:*

Not only do we have that for any $\epsilon > 0$,

$$\mathcal{T}[\eta_1] : \textit{compact } L_2(0, T; L_2(\Omega)) \to L_2\big(0, T; H^{1-\epsilon}_{\Gamma_0}(\Omega)\big) \textit{ for each } \eta_1 \in L_2(0, T; L_2(\Omega)), \tag{19.9.20}$$

but, in addition, we have

(a)

$$\begin{cases} \textit{with } \mathcal{T}[\eta_1] \textit{ viewed as in Eq. (19.9.20), the family of operators} \\ \{\mathcal{T}[\eta_1]\}_{\eta_1 \in L_2(0,T;L_2(\Omega))} \textit{ is collectively compact: that is, Reference 1, p. 4,} \\ \textit{the set union } \bigcup_{\eta_1} \mathcal{T}[\eta_1]\ \{\textit{unit ball of } L_2(0, T; L_2(\Omega))\} \textit{ is a precompact} \\ \textit{set of } L_2\big[0, T; H^{1-\epsilon}_{\Gamma_0}(\Omega)\big], \end{cases} \tag{19.9.21}$$

where the union of the image of the unit ball of $L_2(0, T; L_2(\Omega))$ under the operator $\mathcal{T}[\eta_1]$ is taken over all $\eta_1 \in L_2(0, T; L_2(\Omega))$. Moreover, $\mathcal{T}[\eta_1]$ satisfies the following additional property.

(b) For any sequence $\eta_{1n} \in L_2(0, T; L_2(\Omega))$, we can extract a subsequence η_{1n_k} such that

$$\mathcal{T}\left[\eta_{1n_k}\right] \to \mathcal{T}^0 \textit{ strongly, from } L_2(0, T; L_2(\Omega)) \textit{ to } L_2\big(0, T; H^{1-\epsilon}_{\Gamma_0}(\Omega)\big), \tag{19.9.22}$$

where $\mathcal{T}^0$ is given by

$$(\mathcal{T}^0 v)(t) = \int_t^T S(\tau - t)f_0(\tau)v(\tau)\,d\tau$$

$$: \textit{ continuous } L_1(0, T; L_2(\Omega)) \to C\big([0, T]; \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\big), \tag{19.9.23}$$

for some suitable $f_0 \in L_\infty(\mathbb{R})$, possibly depending on the subsequence.

PROOF The proof is identical to that of property (P.2) for the operator $\mathcal{K}[\eta]$ in Eq. (19.2.22), except on a different topological setting. We shall use Aubin's lemma [2]. From Eq. (19.9.19) we readily estimate via Eqs. (19.2.3) and Eq. (19.1.2),

$$\begin{aligned}\|\mathcal{T}[\eta_1]v\|_{L_2(0,T;\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} &\leq \sqrt{T}\|\mathcal{T}[\eta_1]v\|_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} \\ &\leq C_T\|v\|_{L_2(0,T;L_2(\Omega))}, \quad \text{uniformly in } \eta_1 \in L_2[0,T;L_2(\Omega)]. \end{aligned} \tag{19.9.24}$$

Next, for the time derivative of $\mathcal{T}[\eta_1]$ in Eq. (19.9.19):

$$\left(\frac{d}{dt}\mathcal{T}[\eta_1]v\right)(t) = -\int_t^T C(\tau - t) f'(\eta_1(\tau))v(\tau)\, d\tau, \tag{19.9.25}$$

we similarly estimate via Eqs. (19.2.3) and (19.1.2),

$$\left\|\frac{d}{dt}\mathcal{T}[\eta_1]v\right\|_{L_2(0,T;L_2(\Omega))} \leq C_T\|v\|_{L_2(0,T;L_2(\Omega))}, \quad \text{uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)). \tag{19.9.26}$$

Because the injection $H^1_{\Gamma_0}(\Omega) \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \hookrightarrow H^{1-\epsilon}_{\Gamma_0}(\Omega)$, $\epsilon > 0$, is compact, Aubin's lemma [2] applies and yields Eq. (19.9.20) and, in fact, Eq. (19.9.21), because the estimates Eqs. (19.9.24) and (19.9.26) are *uniform* in $\eta_1 \in L_2(0,T;L_2(\Omega))$.

To prove part (b), we proceed as in Eq. (19.2.29) through Eq. (19.2.32). Given $\eta_{1n} \in L_2(0,T; L_2(\Omega))$, we have $|f'(\eta_{1n})| \leq$ const by Eq. (19.1.2), so that by Alaoglou's theorem we can extract a subsequence η_{1n_k} such that $f'(\eta_{1n_k}) \to$ some f_0 in $L_\infty(\mathbb{R})$-weak star. One then defines the operator $\mathcal{T}^0$ as in (19.9.23). We further obtain as in Reference 29, Proposition 3.3(b), p. 129 that

$$\mathcal{A}^{\frac{1}{2}}\mathcal{T}\left[\eta_{1n_k}\right] \to \mathcal{A}^{\frac{1}{2}}\mathcal{T}^0 \text{ weakly in } L_2(0,T;L_2(\Omega)), \tag{19.9.27}$$

as well as

$$\frac{d}{dt}\mathcal{T}\left[\eta_{1n_k}\right] \to \frac{d}{dt}\mathcal{T}^0 \text{ weakly in } L_2(0,T;L_2(\Omega)). \tag{19.9.28}$$

Then, as a consequence of the weak convergence of Eq. (19.9.27) (which takes care of the "space" coordinates) and of Eq. (19.9.28) (which takes care of the "time" coordinate), along with the compactness of $\mathcal{A}^{-\epsilon}$ from $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ to $\mathcal{D}(\mathcal{A}^{\frac{1}{2}-\epsilon})$, we deduce the strong convergence of Eq. (19.9.22). Proposition 19.9.2 is proved. ∎

Consequences of Proposition 19.9.2

As a consequence of the collectively compact property of Eq. (19.9.21) and of the strong convergence of Eq. (19.9.22) (or else again by Aubin's lemma applied to $\mathcal{T}^0$), we obtain that

$$\mathcal{T}^0: \text{ compact } L_2(0,T;L_2(\Omega)) \to L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right). \tag{19.9.29}$$

Moreover, we have that the integral operator $\mathcal{T}^0$ satisfies

$$[I - \mathcal{T}^0]^{-1} \in L(L_2\big(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\big), \quad \epsilon > 0. \tag{19.9.30}$$

In fact, it suffices by Eq. (19.9.29) to establish that $[I - \mathcal{T}^0]$ is injective: $[I - \mathcal{T}^0]v = 0 \Rightarrow v = 0$, which is true because this leads by differentiating via Eq. (19.9.23) twice that $\ddot{v} = \mathcal{A}v - f_0 v$, $v(T) = \dot{v}(T) = 0$, hence $v = 0$. (Or else, one proceeds as in obtaining Eq. (19.4.2) from Eq. (19.4.3)). The counterpart of Lemma 19.4.4 is

PROPOSITION 19.9.3
Not only do we have

$$(I - \mathcal{T}[\eta_1])^{-1} \in L\big(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\big), \quad \epsilon > 0. \tag{19.9.31}$$

for each $\eta_1 \in L_2(0,T;L_2(\Omega))$ fixed, but moreover,

(a₁)

$$\|(I - \mathcal{T}[\eta_1])^{-1}\|_{L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)))} \leq const_\epsilon, \quad \textit{uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)). \quad (19.9.32)$$

(a_2) Let η_{1n_k} be a subsequence of a given arbitrary sequence $\eta_{1n} \in L_2(0,T;L_2(\Omega))$, such that

$$\mathcal{T}\left[\eta_{1n_k}\right] \to \mathcal{T}^0 \textit{ (given by (19.9.23)) strongly in } L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right), \quad (19.9.33)$$

as guaranteed a-fortiori *by Eq.* (19.9.22)*, whereby then $\mathcal{T}^0$ is compact on $L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$, as* a-fortiori *guaranteed by Proposition 19.8.2(b), Eq.* (19.9.29)*. Then, not only $(I-\mathcal{T}^0)^{-1} \in L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)))$ as already noted in Eq.* (19.9.30)*, but moreover,*

$$\left(I - \mathcal{T}\left[\eta_{1n_k}\right]\right)^{-1} \to (I-\mathcal{T}^0)^{-1} \textit{ strongly in } L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right). \quad (19.9.34)$$

PROOF Same as the proof of Lemma 19.4.4. The proof relies on Proposition 8.2—counterpart of property (P.2), Eqs. (19.2.27) to (19.2.29)—which then allows the application of a standard result ([1], Theorem 1.6, p. 6) on the family of collectively compact operators $\{\mathcal{T}[\eta_1]\}$ on $L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$, *a-fortiori* from Eq. (19.9.21). Indeed, the strong convergence of Eq. (19.9.33) guaranteed *a-fortiori* by Eq. (19.9.22), once combined with the existence of $[I-\mathcal{T}^0]^{-1}$ already established in Eq. (19.9.30) implies via ([1], Theorem 1.6, p. 6) that

$$\left(I - \mathcal{T}^0\left[\eta_{1n_k}\right]\right)^{-1} \in L\left(L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right)\right), \quad (19.9.35)$$

for all k sufficiently large, and moreover, that the strong convergence in Eq. (19.9.34) takes place.

Part (a_2) is proved and this establishes part (a_1) because the sequence $\{\eta_{1n}\}$ was arbitrary. ∎

We next introduce the family of operators

$$\Pi[\eta_1] \equiv \mathcal{T}[\eta_1](I + \mathcal{O}_{1,\text{ext}}\gamma_0), \quad \eta_1 \in L_2(0,T;L_2(\Omega)) \quad (19.9.36a)$$

$$: \text{ continuous } L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right) \to C\left([0,T];\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right), \quad \epsilon > 0, \quad (19.9.36b)$$

where $\mathcal{O}_{1,\text{ext}}$ is defined in Eq. (19.9.12) and γ_0 is the Dirichlet trace operator introduced below Eq. (19.9.13).

The regularity in Eq. (19.9.36b) of $\Pi[\eta_1]$ is justified as follows. First, the Dirichlet trace operator is continuous, in fact, compact, from $H^{1-\epsilon}_{\Gamma_0}(\Omega)$ to $L_2(\Gamma)$. Next, by Eq. (19.9.11), $\mathcal{O}_{1,\text{ext}}$ is continuous from $L_2(0,T;U) \to C([0,T];L_2(\Omega))$, $U = L_2(\Gamma_1)$, so that

$$\mathcal{O}_{1,\text{ext}}\gamma_0 : \text{ continuous } L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right) \to C([0,T];L_2(\Omega)). \quad (19.9.37)$$

Finally, Eq. (19.9.37) combined with Eq. (19.9.19b) yields Eq. (19.9.36b).

PROPOSITION 19.9.4

The family of operators $\{\Pi[\eta_1]\}$ in Eq. (19.9.36) *has the same properties on the space $L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$ as the family of operators $\{\mathcal{T}[\eta_1]\}$. Namely, not only*

(i)

$$\Pi[\eta_1] : \textit{ compact } L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right) \to \textit{ itself}, \quad (19.9.38)$$

but, in addition,

(ii)

$$\begin{cases} \{\Pi[\eta_1]\}_{\eta_1 \in L_2(0,T;L_2(\Omega))} \textit{ is a collectively compact family} \\ \textit{of operators on } L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right). \end{cases} \quad (19.9.39)$$

(iii) *For any sequence $\eta_{1n} \in L_2(0,T;L_2(\Omega))$, we can extract a subsequence η_{1n_k} such that*

$$\Pi\left[\eta_{1n_k}\right] \to \Pi^0 \equiv \mathcal{T}^0(I+\mathcal{O}_{1,\text{ext}}\gamma_0) \text{ strongly in } L_2\big(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\big)\,, \tag{19.9.40}$$

with $\mathcal{T}^0$ (depending on the subsequence) that can be taken to be the operator in Eq. (19.9.23).

(iv)

$$\Pi^0 : \text{ compact } L_2\big(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\big) \to \text{itself.} \tag{19.9.41}$$

(v)

$$[I-\Pi^0]^{-1} \in L\big(L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right)\big)\,. \tag{19.9.42}$$

(vi)

$$(I-\Pi[\eta_1])^{-1} \in L\big(L_2\left(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\right)\big), \quad \text{for each fixed } \eta_1 \in L_2(0,T;L_2(\Omega)). \tag{19.9.43}$$

(vii) *Let η_{1n_k} be a subsequence of a given arbitrary sequence $\eta_{1n} \in L_2(0,T;L_2(\Omega))$, so that the strong convergence of Eq.* (19.9.40) *takes place, as guaranteed by part (iii). Then, not only Eq.* (19.9.42) *holds true, but, moreover,*

$$\left(I-\Pi\left[\eta_{1n_k}\right]\right)^{-1} \to (I-\Pi^0)^{-1} \text{ strongly in } L_2\big(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\big)\,. \tag{19.9.44}$$

(viii) *Finally, the following uniform bound holds true:*

$$\|(I-\Pi[\eta_1])^{-1}\|_{L(L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))} \le const_\epsilon, \quad \text{uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)). \tag{19.9.45}$$

PROOF It suffices to invoke the regularity property Eq. (19.9.37) in the definition of $\Pi[\eta_1]$ in Eq. (19.9.36a) which yielded Eq. (19.9.36b). Then all the arguments employed for the operator family $\{\mathcal{T}[\eta_1]\}$ in Propositions 19.9.2 and 19.9.3 carry over verbatim to the family $\{\Pi[\eta_1]\}$ in Eq. (19.9.36). □

Step 5. We now prove the desired estimate of Eq. (19.9.1).

PROPOSITION 19.9.5

With reference to the solutions χ and ϕ of problems in Eqs. (19.6.37 right-hand side) *and* (19.6.3), *with $q(t,x)$ replaced by $f'(\eta_1(t,x))$, the following inequality holds true: there exists a constant $C_T > 0$, independent of the I.C. $\Phi_0 \in Y$ and of $\eta_1 \in L_2(0,T;L_2(\Omega))$, such that for $\epsilon > 0$,*

$$\|\chi|_{\Sigma_1}\|_{L_2(\Sigma_1)} \le c\|\chi\|_{L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))} \le C_T\|\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\|_{L_2(\Sigma_1)}. \tag{19.9.46}$$

PROOF By virtue of the definition of Eq. (19.9.36), we return to Eq. (19.9.18) and rewrite it (we are replacing q with $f'(\eta_1)$) as

$$(I-\Pi[\eta_1])\chi = \mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}(\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) \in C\big([0,T];\mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big)\big), \tag{19.9.47}$$

for each $\eta_1 \in L_2(0,T;L_2(\Omega))$. By Proposition 19.9.3(vi), Eq. (19.9.43), we can perform the inversion in Eq. (19.9.47) in $L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$, thus obtaining

$$\chi = (I-\Pi[\eta_1])^{-1}\mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}(\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}) \in L_2\big(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega)\big). \tag{19.9.48}$$

In Eq. (19.9.47), we now recall the critical uniform bound Eq. (19.9.44) of Proposition 19.9.3(viii) for $(I - \Pi[\eta_1])^{-1}$ as well as the uniform bound

$$\|\mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}\|_{L(L_2(\Sigma_1);C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}}))} \leq \text{const}_T, \quad \text{uniformly in } \eta_1 \in L_2(0,T;L_2(\Omega)), \tag{19.9.49}$$

which follows by combining the regularity of $\mathcal{O}_{1,\text{ext}}$ in Eq. (19.9.11) with the uniform estimate of Eq. (19.9.24) for $\mathcal{T}[\eta_1]$. Setting for notational convenience $X \equiv L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))$, we then obtain from Eq. (19.9.48) via Eqs. (19.9.45) and (19.9.49)

$$\|\chi\|_{L_2(0,T;H^{1-\epsilon}_{\Gamma_0}(\Omega))} \leq \left\|(I - \Pi[\eta_1])^{-1}_{L(X)}\right\| \mathcal{T}[\eta_1]\mathcal{O}_{1,\text{ext}}\|_{L(L_2(\Sigma_1);X)} \|\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\|_{L_2(\Sigma_1)} \tag{19.9.50}$$

$$\leq C_T \|\phi(\,\cdot\,;\Phi_0)|_{\Sigma_1}\|_{L_2(\Sigma_1)}, \tag{19.9.51}$$

uniformly in $\eta_1 \in L_2(0,T;L_2(\Omega))$. This proves the RHS inequality in Eq. (19.9.46). The LHS inequality in Eq. (19.9.46) is a direct application of trace theory. Proposition 19.9.5 is proved. ∎

Finally, the inequality of Eq. (19.9.1) is then a rereading of the inequality of Eq. (19.9.45) with $q(t,x) = f'(\eta_1(t,x)) \in L_\infty(Q)$. The inequality of Eq. (19.9.1) is thus proved. As a consequence, Theorem 19.1.1 is established via Figure 19.5.1.

19.10 Comparison between the Two Strategies: Part I vs. Part II, Part III, and Part IV

Section 19.5—condensed in Figure 19.5.1—summarizes the approach of the present paper (which is closely based on the original abstract approach in Reference 29) in the study of exact controllability of semilinear problems at least in its Part I. The necessary and sufficient condition of Proposition 19.4.1(a) is then proposed to be tested via one of two different sufficient conditions (which are, in fact, related by Propositions 19.4.2 and 19.4.3), both *with operator-theoretic emphasis*: one that seeks the uniform global invertibility of Eq. (19.1.3) = Eq. (19.4.64) = Eq. (19.5.7) through an analysis of the controlled problem (Part I) and one that seeks the uniform COI of Eq. (19.5.10) through an analysis of the dual uncontrolled problem of Eq. (19.3.9) = Eq. (19.6.3) = Eq. (19.8.4) (Part II, Part III, and Part IV). Refer to Propositions 19.5.1 and 19.5.2. In this section, we perform a comparison between these two basic routes on a few different dynamics.

Case 1. The Problem of Eq. (19.1.1). Semilinear Wave Equation with Neumann Boundary Control $u \in L_2(0,T;L_2(\Gamma_1))$ on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$

The (global) exact controllability result of Theorem 19.1.1 was obtained in this paper in four ways: through the abstract operator approach of either Part I (Sections 19.2–19.4), Part II (Sections 19.6 and 19.7), Part III (Section 19.8), or else Part IV (Section 19.9). A comparison between the two approaches may be in order. The common thread is represented by uniform operator inversions, which are obtained by identifying suitably collectively compact families of operators, whereby then the basic theory thereof [1] may be applied.

Case 2. Second-Order Hyperbolic Equations on a Riemannian Manifold with Dirichlet Boundary Control

Model Let M be a finite-dimensional Riemannian manifold with metric $g(\,\cdot\,,\,\cdot\,) = \langle\,\cdot\,,\,\cdot\,\rangle$ and squared norm $|X|^2 = g(X,X)$. Let Ω be an open bounded connected compact set of M with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Let ν denote the outward unit normal field along the boundary Γ. Furthermore, we denote by Δ_g the Laplace (Laplace-Beltrami) operator on the manifold M, and we denote the

Levi-Civita connection on M by D. In this section, we consider the following Riemannian wave equation with energy level (H^1)-terms on Ω:

$$\begin{cases} w_{tt} = \Delta_g w + F(w) + f(w) & \text{in } Q = (0, T] \times \Omega; & (19.10.1a) \\ w(0, \cdot) = w_0, \ w_t(0, \cdot) = w_1 & \text{in } \Omega; & (19.10.1b) \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0 = (0, T] \times \Gamma_0; & (19.10.1c) \\ w|_{\Sigma_1} = u & \text{in } \Sigma_1 = (0, T] \times \Gamma_1. & (19.10.1d) \end{cases}$$

The function F is assumed to be of the following form:

$$F(w) = \langle P(t, x), Dw \rangle + p_1(t, x) w_t + p_0(t, x) w, \tag{19.10.2a}$$

where $Dw = \nabla_g w$ is a vector field, $Dw \in \mathcal{X}(M) =$ the set of all vector fields on M; p_0, p_1 are functions on $[0, T] \times \Omega$; finally, $P(t, \quad)$ is a vector field on M, $t > 0$. We assume

$$p_0 \in L_{n+1}(Q); \quad p_1 \in L_\infty(Q), \ P \in L_\infty(0, T; \Lambda), \tag{19.10.2b}$$

[10]. Thus, *a-fortiori*, there exists a constant $C_T > 0$ such that

$$|F(w)|^2 \le C_T \left\{ |Dw|^2 + w_t^2 + w^2 \right\}, \quad x, t \in Q, \text{ a.e.} \tag{19.10.2c}$$

Two vertical bars $|\cdot|$ denote the norm in the tensor space T_x or its completion $L^2(\Omega, T)$ [10]. The nonlinear function $f(r)$ satisfies the same assumption of Eq. (19.1.2) as in the Neumann problem of Eq. (19.1.1).

Special case in $\mathbb{R}^n$ A relevant special case on a bounded open domain Ω of $\mathbb{R}^n$ is given when $\mathcal{A}$ is an elliptic second-order differential operator in $\mathbb{R}^n$:

$$\mathcal{A}w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial w}{\partial x_j} \right), \quad x = [x_1, \dots, x_n], \tag{19.10.3a}$$

with real coefficients $a_{ij} = a_{ji}$ of class C^1, satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \ge a \sum_{i=1}^{n} \xi_i^2, \quad x \in \Omega, \tag{19.10.3b}$$

for some positive $a > 0$. Accordingly, we may obtain that

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j > 0, \quad \forall\, x \in \mathbb{R}^n, \quad \xi = [\xi_1, \xi_2, \dots, \xi_n] \in \mathbb{R}^n, \quad \xi \neq 0. \tag{19.10.3c}$$

Let $A(x)$ and $G(x)$ be the $n \times n$ coefficient matrix and its inverse

$$A(x) = (a_{ij}(x)); \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)), \quad i, j = 1, \dots, n; \quad x \in \mathbb{R}^n, \tag{19.10.4}$$

both positive definite for any $x \in \mathbb{R}^n$, by the assumption of Eq. (19.10.3c).

Riemannian metric Let $\mathbb{R}^n$ be endowed with the usual topology and $x = [x_1, x_2, \dots, x_n]$ be the natural coordinate system. For each $x \in \mathbb{R}^n$, define the inner product and the norm on the tangent space $\mathbb{R}^n_x \equiv \mathbb{R}^n$ by

$$g(X, Y) = \langle X, Y \rangle_g = \sum_{i,j=1}^{n} g_{ij}(x) \alpha_i \beta_j; \tag{19.10.5a}$$

$$|X|_g = \{\langle X, X\rangle_g\}^{\frac{1}{2}}; \quad X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}; \quad Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \tag{19.10.5b}$$

Then, $(\mathbb{R}^n, g)$ is a Riemannian manifold with Riemannian metric g, denoted by $g = \sum_{ij=1}^{n} g_{ij}\, dx_i\, dx_j$. We have

$$\mathcal{A}w = \Delta_g w + Dw. \tag{19.10.6}$$

Thus, the case of a variable coefficient operator $\mathcal{A}$ as in Eq. (19.10.3a) in $\mathbb{R}^n$ can be included in the Riemannian wave equation model of Eq. (19.10.1a).

One could also start with a differential operator such as $\mathcal{A}$ of the form of Eq. (19.10.3a) except this time as defined on M and, by a similar change of metric using Eq. (19.10.4), end up with the canonical form of Eq. (19.10.1a).

Main assumption As in [59], we assume that there exists a function $d : \bar{\Omega} \Rightarrow \mathbb{R}$ of class C^3 that is strictly convex in the metric g. This means that the Hessian $D^2 d$ (a two-order tensor) satisfies $D^2 d(X, X) > 0$, $\forall\, x \in \bar{\Omega}$, $\forall\, X \in M_x$. By translation, we can always make $d(x)$ positive on $\bar{\Omega}$. Moreover, because $\bar{\Omega}$ is compact, we can always obtain that

$$D^2 d(X, X) \equiv \langle D_x(Dd), X\rangle_g \geq 2\rho |X|_g^2, \quad \forall\, x \in \Omega,\ \forall\, X \in M_x, \tag{19.10.7}$$

for some $\rho > 0$, $\min_{\bar{\Omega}} d(x) > 0$. Accordingly, we define the time T_0 by

$$T_0 = 2\left(\frac{\max_{\bar{\Omega}} d(x)}{\rho}\right)^{\frac{1}{2}}, \quad \rho \text{ as in (19.10.7).} \tag{19.10.8}$$

REMARK 19.10.1 Several approaches are available to provide a metric $g = \Sigma g_{ij}\, dx_i\, dx_j$—hence coefficients $a_{ij}(x)$ via Eq. (19.10.4)—which satisfies the main assumption of Eq. (19.10.6). See Reference 9, Section 4 for a review.

Approaches include the Riemannian distance function (theorem of R. E. Greene and H. Wu), the Hessian comparison theorem, curvature flow methods, etc.

To simplify the exposition, we shall also *assume* that $d(x)$ has no critical point on $\bar{\Omega}$

$$\inf_{x\in\Omega} |Dd| > 0, \tag{19.10.9}$$

though this assumption can eventually be dispensed with, see Reference 59, Section 10. We define the following subset of the boundary $\Gamma = \partial\Omega$:

$$\Gamma_0 = \left\{ x \in \Gamma : \langle Dd(x), \nu(x)\rangle = \frac{\partial d(x)}{\partial \nu} \leq 0 \right\}; \quad \Gamma_1 = \Gamma \backslash \Gamma_0. \tag{19.10.10}$$

Uniform Continuous Observability Inequality of the Corresponding Linearized System We shall state below the required "uniform continuous observability inequality" for the following "dual" problem corresponding to the linearized version of the problem of Eq. (19.10.1):

$$\begin{cases} \phi_{tt} = \Delta_g \phi + \langle Q(t,x), D\phi\rangle + q_1(t,x)\phi_t + q_0(t,x)\phi & \text{in } Q; \quad (19.10.11a) \\ \phi(T,\cdot) = \phi_0,\ \phi_t(T,\cdot) = \phi_1 & \text{in } \Omega; \quad (19.10.11b) \\ \phi|_\Sigma \equiv 0 & \text{in } \Sigma, \quad (19.10.11c) \end{cases}$$

where Q, q_1, q_0 satisfy regularity assumptions as P, p_1, p_0, in Eq. (19.10.2b), respectively. The term $f'(w(t,x)) \in L_\infty(Q)$, which arises in the linearization of Eq. (19.10.1a) with f subject to Eq. (19.1.2), is incorporated in the term q_0.

The following result provides the key sought-after version, as it applies to the present case of Eq. (19.10.1), of the "uniform observability property" required by Eq. (19.5.10).

THEOREM 19.10.1

Assume hypotheses of Eqs. (19.10.7) *and* (19.10.9) *on the strictly convex function $d(x)$. Let $T > T_0$, where T_0 is defined in Eq.* (19.10.8). *Then, with reference to Eq.* (19.10.11), *the following continuous observability inequality holds true: There exists a constant $C(r) > 0$, such that*

$$\|\{\phi_0, \phi_1\}\|^2_{H^1_0(\Omega)\times L_2(\Omega)} \leq C(r) \int_0^T \int_{\Gamma_1} \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Sigma_1, \tag{19.10.12}$$

where $C(r)$ is given by

$$C(r) = ce^{cr^2} \text{ for some } c = C(T, \Omega);\; r = \|q_0\|_{L_{n+1}(Q)} + \|q_1\|_{L_\infty(Q)} + \|\mathcal{Q}\|_{L_\infty(0,T;\Lambda)}. \tag{19.10.13}$$

Theorem 19.10.1 was proved in Reference 59, Theorem 8.1, p. 357. It generalized previous versions with constant coefficient principal part on a Euclidean domain [20, 60, 38].

Semilinear System As already noted before its statement, Theorem 19.10.1 provides the present version of the sufficient condition of Eq. (19.5.10) (Uniform Continuous Observability Inequality for the linearized problem corresponding to the original semilinear system of Eq. (19.10.1) in the present Dirichlet boundary control case). The technical work that goes into proving the estimate of Eq. (19.10.12)—and its predecessors in the Euclidean setting in References 20, 38, and 60 (based on pointwise Carleman estimates with no lower-order terms)—replaces entirely the approaches of Part II and Part III in the Dirichlet versions. An estimate of Eq. (19.10.12) *is* the uniform COI of Eq. (19.5.5) in our present Dirichlet case. Therefore, by the counterpart of Proposition 19.5.1 or 19.5.2, as applied to the present Dirichlet problem of Eq. (19.10.1) (where $\Lambda'_T(q)$ is still compact), we obtain the desired (global) exact controllability of the system of Eq. (19.10.1).

THEOREM 19.10.2

Let Eq. (19.1.2) *be fulfilled for the function f in Eq.* (19.10.1a). *Assume the hypotheses of Eqs.* (19.10.7) *and* (19.10.9) *on the strictly convex function $d(x)$. Let $T > T_0$, where T_0 is defined in Eq.* (19.10.8). *Then the problem of Eq.* (19.10.1) *is (globally) exactly controllable on the state space $H \equiv L_2(\Omega) \times H^{-1}(\Omega)$ within the class of $L_2(0, T; L_2(\Gamma_1))$-controls. Explicitly: given $T > 0$, the I.C. $\{w_0, w_1\} \in H$, the target state $\{\tilde{w}_0, \tilde{w}_1\} \in H$, there exists a control $u \in L_2(0, T; L_2(\Gamma_1))$, such that the corresponding solution of the problem of Eq.* (19.10.1) *satisfies $\{w(T, \cdot), w_t(T, \cdot)\} = \{\tilde{w}_0, \tilde{w}_1\}$.*

Theorem 19.10.1 generalizes Reference 29, Theorem 19.1.1, case $\gamma = 0$ to which it reduces in the case of $\Delta_g = \Delta$ (that is, constant coefficient principal part on a Euclidean domain: $-a_{ij} = \delta_{ij}$ in Eq. (19.10.3a) with no energy-level terms $p_1 \equiv P \equiv 0$).

As mentioned in the introduction, this basic case $\gamma = 0$ is excluded from the results of Reference 61 and also of Reference 60, Theorem 4.2, p. 831, as the latter falls into the former.

REMARK 19.10.2 No compactness-uniqueness argument is involved in obtaining the estimate of Eq. (19.10.12). On the contrary, Eq. (19.10.12) is used (and has been used in this paper) to claim a global uniqueness result for the overdetermined problem of Eqs. (19.10.11a–c) plus $\frac{\partial \phi}{\partial \nu}|_{\Sigma_1} \equiv 0$. The proof of Eq. (19.10.12) is based on pointwise Carleman estimates with *no* lower-order terms, in the presence of low regularity of the coefficients, as it is pursued in References 20, 60, and 38 in the Euclidean setting and extended further in the Riemannian setting in [59]. This approach is more refined than (by now) classical multipliers' approaches. The resulting control-theoretic *a-priori* inequalities obtained through this pointwise Carleman estimate approach have the definite advantage of coming with an explicit estimate of the final constants involved. This, despite the low regularity of the coefficients, in particular $L_\infty(Q)$-potentials q. This is a new feature over much of the control-theoretic literature since the early 1980s [4, 27, 39, 58]. Thus, compactness-uniqueness arguments are dispensed with. There is, however, at present a big difference between the Dirichlet and the Neumann boundary control cases for wave equations. In the Dirichlet boundary control case of the present Section 19.9, this point-wise Carleman estimate approach yields the estimate of Eq. (19.10.12), which is *precisely* the required COI for the problem of Eq. (19.10.11), in particular, with potential $q_0 \in L_\infty(Q)$. By contrast, in the Neumann boundary control case of the present paper—say Eq. (19.3.9) = Eq. (19.6.3) = Eq. (19.A.4)—this approach does yield an estimate, whereby the $H = \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega)$—norm of the I.C. $\{\phi_0, \phi_1\}$ is bounded above by boundary traces of ϕ and ϕ_t in $L_2(\Sigma_1)$ [38]:

$$\|\{\phi_0, \phi_1\}\|^2_{\mathcal{D}(A^{\frac{1}{2}})\times L_2(\Omega)} \leq C_T \int_{\Sigma_1} [\phi^2 + \phi_t^2]\, d\Sigma_1. \tag{19.10.14}$$

But this estimate is *not* the required COI of Eq. (19.5.10) = Eq. (19.8.1). It is *equivalent* to it *but only if* $q_t \in L_\infty(Q)$, see Remark 19.A.1, which is not an acceptable assumption to treat the semilinear problem (1.1). That is why, in the Neumann case of the present paper, we had to resort to different approaches, described in Parts I, II, III, and IV.

Appendix 19.A: Preliminaries on the Linearized Problem. Main Result

Controlled Dynamics

As in Section 19.1, we let $\{\Omega, \Gamma_0, \Gamma_1\}$ be a given triple, $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Omega \in \mathbb{R}^n$ a bounded domain, $n \geq 2$; $\Gamma_0 \neq \emptyset$. Consider the following wave mixed problem:

$$\begin{cases} y_{tt} = \Delta y + q(t,x)y & \text{in } Q = (0,T] \times \Omega; & \text{(19.A.1a)} \\ y(0,\,\cdot\,) = y_0,\ y_t(0,\,\cdot\,) = y_1 & \text{in } \Omega; & \text{(19.A.1b)} \\ y|_{\Sigma_0} = 0;\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = u & \text{in } \Sigma_i = (0,T] \times \Gamma_i,\ i = 0, 1. & \text{(19.A.1c)} \end{cases}$$

We note that Eq. (19.A.1c) (LHS) is a technical condition made to invoke Poincaré inequality; it can be eliminated by using the equivalence in Reference 38, Eq. (19.6.1).

Notation

Throughout Appendices 19.A through C, y denotes the solution of Eq. (19.A.1). By contrast, in Parts I and II, to conform to Reference 29, we defined y as $y = \{w, w_t\}$ where w is the solution of Eq. (19.1.1). No confusion is likely to arise, as the Appendices are essentially independent of Parts I and II in this regard.

Case of Fixed "Potential" *q*

We assume that $q(t,x)$ is fixed and satisfies

$$q(t,x) \in L_\infty(Q). \tag{19.A.2}$$

The problem Eq. (19.A.1) coincides with the ζ-problem of Eq. (19.2.37) of Section 19.2, for the choice $q = f'(\eta_1(\,\cdot\,))$, where eventually $\eta_1 = w(t,x) \in C([0,T]; H^1_{\Gamma_0}(\Omega))$, see regularity statement in Step 1 at the beginning of the proof of Section 19.4. It is convenient to study problem of Eq. (19.A.1) as a prototype of the linearization of Eq. (19.2.37) of the original semilinear problem of Eq. (19.1.1), with a fully general $L_\infty(Q)$-potential.

REMARK 19.A.1 The assumption of Eq. (19.A.2) is made with the aim of solving the exact controllability problem for semilinear models. In contrast, if we assume $q_t(t,x) \in L_\infty(Q)$, the approach in Reference 27 would apply. The approach that follows is dictated by the assumption of Eq. (19.A.2).

LEMMA 19.A.1

Exact controllability of the problem of Eq. (19.A.1) on the state space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ within the class of $L_2(\Sigma_1) = L_2[0,T; L_2(\Gamma_1)]$-controls holds true if and only if the following COI is satisfied: there exists a constant $C_{T,q} > 0$, depending only on T and q but not on the I.C., such that

$$\int_{\Sigma_1} \phi^2 d\Sigma_1 \ge C_{T,q} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[H^1_{\Gamma_0}(\Omega)]'} \tag{19.A.3}$$

(whenever the left-hand side is finite), where ϕ solves the following homogeneous backward problem:

$$\begin{cases} \phi_{tt} = \Delta\phi + q(t,x)\phi & \text{on } Q; & (19.A.4a) \\ \phi(T,\cdot) = \phi_0 \in L_2(\Omega),\ \phi_t(T,\cdot) = \phi_1 \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]' & \text{on } \Omega; & (19.A.4b) \\ \phi|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 & \text{on } \Sigma_i = (0,T] \times \Gamma_i. & (19.A.4c) \end{cases}$$

The regularity of the problem of Eq. (19.A.4) is stated in Theorem 19.3.1 and proved in Theorem 19.C.1 of Appendix 19.C.

Notation

The problem of Eq. (19.A.4) coincides with the problem of Eq. (19.3.9) = Eq. (19.6.3). As in Eqs. (19.2.2) and (19.2.4), we set

$$H^1_{\Gamma_0}(\Omega) = \{f \in H^1(\Omega) :\ f|_{\Gamma_0} = 0\} = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \text{ (equivalent norms)}; \tag{19.A.5}$$

$$\mathcal{A}f = -\Delta f,\ \mathcal{D}(\mathcal{A}) = \left\{ f \in H^1(\Omega) :\ f|_{\Gamma_0} = 0,\ \left.\frac{\partial f}{\partial\nu}\right|_{\Gamma_1} = 0 \right\}; \tag{19.A.6}$$

$$\mathcal{A}^{-1} \in \mathcal{L}(L_2(\Omega)) \text{ (because of } \Gamma_0 \ne \emptyset);\ H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega);\ Y \equiv L_2(\Omega) = \left[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\right]'; \tag{19.A.7}$$

$$\left(H^1_{\Gamma_0}(\Omega)\right)' = \left[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\right]' \text{ (duality w.r.t. } L_2(\Omega) \text{ as a pivot space).} \tag{19.A.8}$$

PROOF OF LEMMA 19.A.1. By usual duality.

Step 1. Consistently with the notation in Eq. (19.3.15) or Eq. (19.4.33) for the abstract linearized z-problem of Eq. (19.2.18)—corresponding to the linearized ζ-problem in Eq. (19.2.37) in PDE terms; that is, the y-problem Eq. (19.A.1) with $q = f'(\eta_1(\cdot))$—we introduce the input → solution operator $\mathcal{M}_T \equiv \mathcal{M}_T[q]$ (*unbounded*, densely defined, *closed*), see Eq. (19.3.3) for $\mathcal{L}_T$, that is, for $q \equiv 0$:

$$\mathcal{M}_T u = \{y(T), y_t(T)\} \in H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega) \quad \text{for } u \in \mathcal{D}(\mathcal{M}_T) = \mathcal{D}(\mathcal{L}_T) \subset L_2(\Sigma_1). \tag{19.A.9}$$

The required exact controllability property of Eq. (19.A.1) on $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ at time $t = T$ within the class of $L_2[0,T; L_2(\Gamma_1)]$-controls u is *reformulated* as the property that the operator $\mathcal{M}_T$ in Eq. (19.A.9) be surjective:

$$\mathcal{M}_T : L_2(\Sigma_1) \supset \mathcal{D}(\mathcal{M}_T) \overset{\text{onto}}{\longrightarrow} H \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega). \tag{19.A.10}$$

This ontoness property for the closed operator $\mathcal{M}_T$ is, in turn, equivalent to the property that its (Hilbert space) adjoint $\mathcal{M}_T^* : H \supset \mathcal{D}(\mathcal{M}_T^*) \to L_2(\Sigma_1)$ has a continuous inverse (see Reference 56, p. 235):

$$\left\| \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|_{L_2(\Sigma_1)} \geq C_T \|\{h_0, h_1\}\|^2_{\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega)}, \quad \forall \{h_0, h_1\} \in \mathcal{D}(\mathcal{M}_T^*) \subset H. \tag{19.A.11}$$

(This is the "continuous observability" inequality in this case.) We shall see now that the inequality of Eq. (19.A.3) is nothing but a PDE reformulation of inequality Eq. (19.A.11). (We have $(\mathcal{M}_T u, z)_{\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega)} = (u, \mathcal{M}_T^* z)_{L_2(\Sigma)}$, consistently with the notation $*$ in Remark 19.4.1.)

Step 2. We shall show that $\mathcal{M}_T^* \equiv \mathcal{M}_T^*[q]$ satisfies

$$\left(\mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)(t) = \phi(t; \phi_0, \phi_1)|_{\Gamma_1}; \ \phi_0 = h_1 \in L_2(\Omega); \phi_1 = -\mathcal{A} h_0 \in \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'. \tag{19.A.12}$$

To prove Eq. (19.A.12), we set $y_0 = y_1 = 0$ in Eq. (19.A.1b), multiply the y-problem in Eq. (19.A.1) by ϕ (solution of Eq. (19.A.4)), and integrate by parts (Green's second theorem) to get

$$\int_\Omega \int_0^T y_{tt} \phi \, dQ = \int_0^T \int_\Omega \Delta y \phi \, dQ + \iint_Q q y \phi \, dQ. \tag{19.A.13}$$

In the $L_2(\Omega)$-inner product, we obtain, using Eqs. (19.A.4a), (19.A.1c, left-hand side), and (19.A.4c, right-hand side):

$$\begin{aligned} &[y_t(T), \phi(T)] - [y(T), \phi_t(T)] - (y_1, \phi_0) + [y_0, \phi_t(0)] + \int_Q y \phi_{tt} \, dQ \\ &= \int_Q y \Delta \phi \, dQ + \int_\Sigma \frac{\partial y}{\partial \nu} \phi \, d\Sigma - \int_\Sigma y \frac{\partial \phi}{\partial \nu} d\Sigma + \int_Q y q \phi \, dQ, \end{aligned}$$

or, as $y_0 = y_1 = 0$; $\phi(T) = \phi_0$, $\phi_t(T) = \phi_1$ via Eq. (19.A.1c, right-hand side) and Eq. (19.A.4c, left-hand side):

$$(y_t(T), \phi_0)_{L_2(\Omega)} - (y(T), \phi_1)_{L_2(\Omega)} = \int_{\Sigma_1} u \phi \, d\Sigma_1, \tag{19.A.14}$$

or

$$\left(\begin{bmatrix} y(T) \\ y_t(T) \end{bmatrix}, \begin{bmatrix} -\phi_1 \\ \phi_0 \end{bmatrix} \right)_{L_2(\Omega) \times L_2(\Omega)} = (u, \phi|_{\Gamma_1})_{L_2(\Sigma_1)}. \tag{19.A.15}$$

On the other hand, by definition of $\mathcal{M}_T$ and $\mathcal{M}_T^*$, we have for $\{h_0, h_1\} \in \mathcal{D}(\mathcal{M}_T^*) \subset H \equiv \mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega)$:

$$\begin{aligned} \left(\mathcal{M}_T u, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega)} &= \left(\begin{bmatrix} y(T) \\ y_t(T) \end{bmatrix}, \begin{bmatrix} \mathcal{A} h_0 \\ h_1 \end{bmatrix} \right)_{L_2(\Omega) \times L_2(\Omega)} \\ &= \left(u, \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{L_2(\Sigma_1)}. \end{aligned} \tag{19.A.16}$$

By comparison between Eqs. (19.A.15) and (19.A.16), we conclude that Eq. (19.A.12) holds true, as desired.

Step 3. Thus, by Eq. (19.A.12), we have

$$\left\| \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|^2_{L_2(\Sigma_1)} = \int_{\Sigma_1} \phi^2(t) d\Sigma_1; \tag{19.A.17}$$

$$\begin{aligned} \|\{h_0, h_1\}\|^2_{\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right) \times L_2(\Omega)} &= \left\|\left\{\mathcal{A}^{\frac{1}{2}} h_0, h_1\right\}\right\|^2_{L_2(\Omega)\times L_2(\Omega)} = \left\|\left\{\mathcal{A}^{-\frac{1}{2}} \phi_1, \phi_0\right\}\right\|^2_{L_2(\Omega)\times L_2(\Omega)} \\ &= \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)]'}, \end{aligned} \tag{19.A.18}$$

and then the necessary and sufficient condition (COI) of Eq. (19.A.11) reads off as

$$\int_{\Sigma_1} \phi^2(t; \phi_0, \phi_1)|_{\Gamma_1} d\Sigma_1 \geq C_T \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)]'}, \tag{19.A.19}$$

where $\phi_0 = h_1 \in L_2(\Omega)$, $\phi_1 = -\mathcal{A} h_0 \in [\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)]'$. Thus, Lemma 19.A.1 is established.

Thus, by duality, the *crux of the matter* is to prove the validity of the inequality of Eq. (19.A.3). The latter requires, in general, geometric conditions on $\{\Omega, \Gamma_0, \Gamma_1\}$. One illustration of this is given by the setting given in Eqs. (19.3.6) to (19.3.8) of Section 19.3. We have the following:

THEOREM 19.A.2

(E.C. of Eq. (19.A.1) *for fixed "potential" q.) Assume Eq.* (19.A.2) *on q and, moreover, let the triple* $\{\Omega, \Gamma_0, \Gamma_1\}$ *satisfy the geometrical assumptions Eqs.* (19.3.6) *and* (19.3.7) *of Section 19.3. Let* $T > T_0$ *with* T_0 *be defined by Eq.* (19.3.8). *Then:*

(1) The COI of Eq. (19.A.3) *holds true. For a fixed* $q \in L_\infty(Q)$*, the constant* $C_{T,q}$ *in Eq.* (19.A.3) *depends on T as well as on the* $\|q\|_{L_\infty(Q)}$*-norm of q:* $C_{T,\|q\|_{L_\infty(Q)}}$.

(2) Hence, by Lemma 19.A.1, the y-problem Eq. (19.A.1) *is exactly controllable on* $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$ *by means of* $L_2(0, T; L_2(\Gamma_1))$*-controls.*

This result is included *a-fortiori* in either one of Parts I through IV. Refer to Figure 19.5.1.

Case of "Potential" q Varying in a Ball

As seen in the discussion leading to the Uniform COI of Eq. (19.5.10), as well as at the outset of Part III, in the study of exact controllability for the corresponding semilinear problem of Eq. (19.1.1), we need to have "control" on the dependence of the above constant $C_{T,\|q\|_{L_\infty(Q)}}$, when q varies in a ball of $L_\infty(Q)$ of preassigned radius, as to establish Eq. (19.5.10). In particular, we need to ascertain that such constant $C_{T,\|q\|_\infty}$ depends on only the radius of such ball. For $r > 0$, define

$$B(0, r) = \{p \in L_\infty(Q) : \ \|p\|_{L_\infty(Q)} \leq r\}. \tag{19.A.20}$$

Then we have the following

THEOREM 19.A.3

(E.C. of Eq. (19.A.1) *for potential q varying on a ball of radius r, as in Eq.* (19.A.20).) *Consider the problem of Eq.* (19.A.1) *where the potential q varies in the* $L_\infty(Q)$*-ball* $B(0, r)$ *defined in Eq.* (19.A.20). *Assume the validity of the COI of Eq.* (19.A.3) *for some* $q \in B(0, r)$ *and for* $T > T_0$ *defined in Eq.* (19.3.8). *Then, the constant* $C_{T,q}$ *in Eq.* (19.A.3) *can be chosen as to depend only on*

the radius $r > 0$ *in Eq.* (19.A.20). *More precisely, we have that: There exists a constant* $C_{T,r} > 0$ *such that the inequality of Eq.* (19.A.3) *can be written as*

$$C_{T,r}\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \le \int_{\Sigma_1} \phi^2 d\Sigma_1, \text{ uniformly in all } q \in B(0,r) \subset L_\infty(Q). \quad (19.A.21)$$

This is the uniform of COI Eq. (19.5.10).

This result is established in any of Parts I through IV. Refer to Figure 19.5.1.

REMARK 19.A.2 It is well known (see References 58 and 27 and Appendix) that the constant occurring in a COI is related to the minimal norm among the steering controls. Thus, Theorem 19.A.3 says that the minimal energy (norm) $L_2(0, T; L_2(\Gamma_1))$ among all steering controls is uniformly bounded with respect to potentials q in a fixed $L_\infty(Q)$-ball.

Appendix B: Lemma 19.A.1 Revisited via an Evolution Operator Approach

The goal of this section is to *re-prove* Lemma 19.A.1. In the process, we shall obtain the formula of Eq. (19.B.13) for $\mathcal{M}_T^* = \mathcal{M}_T^*[q]$, which was used in Section 19.6, Eq. (19.6.4).

Instead of a PDE-approach as in Appendix A, we follow here an evolution operator strategy (the time-dependent counterpart of the semigroup or cosine operator strategy in References 27, 29, and 58). Introduce the operator

$$\mathbb{A}(t) = \begin{bmatrix} 0 & I \\ -\mathcal{A} + q(t,\cdot) & 0 \end{bmatrix} = A + F'[q]; \quad \mathcal{D}[\mathbb{A}(t)] \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big), \quad (19.B.1)$$

where $\mathcal{A}$ is defined in Eq. (19.A.6) and A is defined in Eq. (19.2.2). One needs to invoke a generation result from the general theory (Reference 48, Sections 19.7.3 and 19.7.7): There exists an evolution operator $U(t,s) \in Y \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega)$, possessing the following properties:

$$U(t,\tau)U(\tau,s) = U(t,s), \; 0 \le s \le \tau \le t; \quad (19.B.2)$$

$$\frac{\partial U(t,s)}{\partial t} = \mathbb{A}(t)U(t,s); \quad (19.B.3a)$$

$$\frac{\partial U^*(t,s)}{\partial t} = U^*(t,s)\mathbb{A}^*(t) \text{ (adjoint in the } H \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega)\text{-topology)}; \quad (19.B.3b)$$

$$\frac{\partial U(t,s)}{\partial s} = -U(t,s)\mathbb{A}(s); \quad (19.B.3c)$$

$$\frac{\partial U^*(t,s)}{\partial s} = -\mathbb{A}^*(s)U^*(t,s). \quad (19.B.3d)$$

Moreover, as usual, the problem in Eq. (19.A.1) for y may be written abstractly as follows. Recall the ("Neumann") map N in Eq. (19.2.5)

$$h = Ng \Longleftrightarrow \left\{ \Delta h = 0 \text{ in } \Omega;\; h|_{\Gamma_0} = 0 \text{ in } \Gamma_0;\; \frac{\partial h}{\partial \nu}\bigg|_{\Gamma_1} = g \text{ in } \Gamma_1 \right\}. \quad (19.B.4)$$

Thus, rewrite the problem in Eq. (19.A.1) as

$$\begin{cases} y_{tt} = \Delta(y - Nu) + qy & \text{in } Q; \quad \text{or recalling (19.A.6), (B.4) :} \\ [y - Nu]_{\Gamma_0} = 0 & \text{in } \Sigma_0; \quad y_{tt} = -\mathcal{A}(y - Nu) + qy \in L_2(\Omega) \\ & \qquad\qquad \text{or} \\ \frac{\partial}{\partial \nu}[y - Nu]_{\Gamma_1} = 0 & \text{in } \Sigma_1; \quad y_{tt} = -\mathcal{A}y + \mathcal{A}Nu + qy \in [\mathcal{D}(\mathcal{A})]'. \end{cases} \tag{19.B.5}$$

Thus, the second-order/first-order abstract models of the problem in Eq. (19.A.1) are

$$y_{tt} = (-\mathcal{A} + q)y + \mathcal{A}Nu \in [\mathcal{D}(\mathcal{A})]'; \tag{19.B.6}$$

or via Eq. (19.B.1), $\tilde{y} = [y, y_t]$:

$$\tilde{y}_t = \frac{d}{dt}\begin{bmatrix} y \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A} + q & 0 \end{bmatrix}\begin{bmatrix} y \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix} = \mathbb{A}(t)\tilde{y} + \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix} \tag{19.B.7a}$$

$$\tilde{y}_t = \mathbb{A}(t)\tilde{y} + Bu, \quad Bu = \begin{bmatrix} 0 \\ \mathcal{A}Nu \end{bmatrix}; \ \mathbb{A}(t) = A + F'[q], \ \text{via (2.7)}, \tag{19.B.7b}$$

B as in Eq. (19.2.6). As usual, we have (Reference 27, Lemma 3.2, p. 272)

$$(Bu, h)_H = (u, B^*h)_{L_2(\Gamma)} = (u, N^*\mathcal{A}h_2)_{L_2(\Gamma)}; \tag{19.B.8a}$$

$$B^*h = B^*\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = N^*\mathcal{A}h_2 = \begin{cases} 0 & \text{on } \Gamma_0 \\ h_2|_{\Gamma_1} & \text{on } \Gamma_1 \end{cases}; \quad h_2 \in \mathcal{D}(\mathcal{A}) \tag{19.B.8b}$$

[the validity of Eq. (19.B.8) can be extended to, say, all $f \in H^{\frac{3}{2}+\epsilon}(\Omega)$ s.t. $\frac{\partial f}{\partial \nu}|_{\Gamma_1} = 0, \ f|_{\Gamma_0} = 0$]. The map $\mathcal{M}_T$ introduced in Eq. (19.A.9) can now be made explicit by means of the 'variation of constant' formula:

$$\begin{bmatrix} y(T) \\ y_t(T) \end{bmatrix} = \mathcal{M}_T u = \int_0^T U(T, t)Bu(t)\, dt = \int_0^T U(T, t)\begin{bmatrix} 0 \\ \mathcal{A}Nu(t) \end{bmatrix} dt. \tag{19.B.9}$$

By using Eq. (19.B.9), we now pick $\{h_0, h_1\} \in H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$ and compute $\left(\mathcal{M}_T^*\left[\begin{smallmatrix} h_0 \\ h_1 \end{smallmatrix}\right]\right)(t)$, as required by Eq. (19.A.11) We intend to reobtain Eq. (19.A.12). [For the present approach in the time-independent case $q(t, x) = q(x)$, see Reference 27, p. 271–273. We compute via Eq. (19.B.9),

$$\left(\mathcal{M}_T u, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}\right)_{H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)} = \left(\int_0^T U(T, t)Bu(t)\, dt, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}\right)_H$$

$$= \int_0^T \left(u(t), B^*U^*(T, t)\begin{bmatrix} h_0 \\ h_1 \end{bmatrix}\right)_{L_2(\Gamma_1)} \tag{19.B.10a}$$

$$
\begin{aligned}
\text{(by (19.B.7b))} \quad &= \left(\int_0^T U(T,t) \begin{bmatrix} 0 \\ \mathcal{A} N u(t) \end{bmatrix} dt, \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)} \\
&= \int_0^T \left(\begin{bmatrix} 0 \\ \mathcal{A} N u(t) \end{bmatrix}, U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)} \\
&= \int_0^T \left(u(t), N^* \mathcal{A} \left[U^*(T,t) \begin{Bmatrix} h_0 \\ h_1 \end{Bmatrix} \right]_2 \right)_{L_2(\Gamma_1)} dt \\
&= \left(u, \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)_{L_2(\Sigma_1)},
\end{aligned} \tag{19.B.10b}
$$

where $\left[U^*(T,t)\left[\begin{smallmatrix} h_0 \\ h_1 \end{smallmatrix}\right]\right]_2$ denotes the second component of the 2-component vector $U^*(T,t)\left[\begin{smallmatrix} h_0 \\ h_1 \end{smallmatrix}\right]$. Thus, Eq. (19.B.10) gives

$$
\left(\mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)(t) = B^* U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = B^* V(t; V_0) \tag{19.B.11a}
$$

$$
\text{(by [19.B.8b])} \quad = N^* \mathcal{A} \left\{ U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\}_2 = \left\{ U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\}_2 \Big|_{\Sigma_1}, \tag{19.B.11b}
$$

recalling that $N^* \mathcal{A}$ is the Dirichlet trace on Γ_1, by Eq. (19.B.8b). Here we have set

$$
\begin{bmatrix} v_1(t; V_0) \\ v_2(t; V_0) \end{bmatrix} \equiv V(t; V_0) \equiv U^*(T; t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}; \text{ thus } V_0 \equiv V(T) = [h_0, h_1] \in H \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega), \tag{19.B.12}
$$

so that Eq. (19.B.11b) is rewritten as

$$
\left(\mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right)(t) = B^* V(t; V_0) = v_2(t; V_0)|_{\Sigma_1}. \tag{19.B.13}
$$

REMARK 19.B.1 Notice that, consistently with Remark 19.4.1, the symbol $*$ for an adjoint always refer to the space $H \equiv \mathcal{D}\big(\mathcal{A}^{\frac{1}{2}}\big) \times L_2(\Omega)$. Thus, $(Bu, h)_H = (u, B^* h)_{L_2(\Gamma)}, h \in \mathcal{D}(B^*) \subset H$, whereas $\mathcal{M}_T^*$, $U^*(T,t)$ are adjoints in H. Similarly, the operator $\mathbb{A}^*(t)$ in Eq. (19.B.3b and d).

Recalling Eq. (19.B.3d), we obtain by Eq. (19.B.12),

$$
V_t(t; V_0) = U_t^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = -\mathbb{A}^*(t) U^*(T,t) \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = -\mathbb{A}^*(t) V(t; V_0). \tag{19.B.14}
$$

From $\mathbb{A}(t)$ in Eqs. (19.B.1) and (19.B.7b), we readily compute that the $H \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)$-adjoint of $\mathbb{A}(t)$ is

$$
\mathbb{A}^*(t) = \begin{bmatrix} 0 & -I + \mathcal{A}^{-1}(q\,\cdot) \\ \mathcal{A} & 0 \end{bmatrix} = A^* + (F'[q])^*; \; \mathcal{D}[\mathbb{A}^*(t)] = \mathcal{D}(\mathbb{A}(t)), \tag{19.B.15a}
$$

or

$$\mathbb{A}^*(t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 + \mathcal{A}^{-1}(qx_2) \\ \mathcal{A}x_1 \end{bmatrix}; \tag{19.B.15b}$$

Thus, we rewrite Eq. (19.B.14) for $V(t;V_0) = \begin{bmatrix} v_1(t;V_0) \\ v_2(t;V_0) \end{bmatrix}$ by virtue of Eq. (19.B.15) as

$$\begin{cases} v_{1t} = v_2 - \mathcal{A}^{-1}(qv_2) \\ v_{2t} = -\mathcal{A}v_1; & \Rightarrow v_{2tt} = -\mathcal{A}v_2 + qv_2; \\ v_{2t}(T) = -\mathcal{A}v_1(T) = -\mathcal{A}h_0 \in \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]', \text{ by (19.B.12)} \end{cases} \tag{19.B.16}$$

In conclusion, $v_2(t;V_0)$ solves the problem (same as Eq. (19.3.9) or Eq. (19.6.3))

$$\begin{cases} v_{2tt} = \Delta v_2 + q(t,x)v_2 & \text{on } Q; \\ v_2(T) = h_1 \in L_2(\Omega);\ v_{2t}(T) = -\mathcal{A}h_0 \in \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]' & \text{on } \Omega; \\ v_2|_{\Sigma_0} \equiv 0,\ \left.\dfrac{\partial v_2}{\partial \nu}\right|_{\Sigma_1} \equiv 0 & \text{on } \Sigma_i; \end{cases} \tag{19.B.17}$$

whose regularity is given by Theorem 19.3.1. Thus, via Eq. (19.B.13) we obtain

$$\left\| \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|^2_{L_2(\Sigma_1)} \equiv \int_{\Sigma_1} v_2^2(t;V_0)\, d\Sigma; \tag{19.B.18}$$

whereas via Eq. (19.B.17) we obtain

$$\begin{aligned} \|\{h_0,h_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)} &\equiv \left\|\left\{\mathcal{A}^{\frac{1}{2}}h_0, v_2(T)\right\}\right\|^2_{L_2(\Omega)\times L_2(\Omega)} = \left\|\left\{\mathcal{A}^{-\frac{1}{2}}v_{2t}(T), v_2(T)\right\}\right\|^2_{L_2(\Omega)\times L_2(\Omega)} \\ &= \|\{v_2(T), v_{2t}(T)\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \end{aligned} \tag{19.B.19}$$

Then the COI of Eq. (19.A.11):

$$\left\| \mathcal{M}_T^* \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} \right\|^2_{L_2(\Sigma_1)} \geq C_T \|\{h_0,h_1\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)}$$

is now rewritten via Eqs. (19.B.18) and (19.B.19) as

$$\int_0^T \int_{\Gamma_1} v_2^2 d\Sigma \geq C_T \|\{v_2(T), v_{2t}(T)\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \tag{19.B.20}$$

where v_2 solves the problem of Eq. (19.B.17). Comparing Eqs. (19.B.17) and (19.B.20) with Eqs. (19.A.4) and (19.A.3), we see that the v_2-problem of Eq. (19.B.17) is *precisely* the ϕ-problem in Eqs. (19.A.4) and (19.B.20) is precisely Eq. (19.A.3). Thus, we have *re-proved Lemma 19.A.1.*

Appendix C: Proof of Theorem 19.3.1: Regularity of the ϕ-Problem of Eq. (19.3.9) = Eq. (19.6.3) = Eq. (19.A.4) (Same as the v_2-Problem of Eq. (19.B.17)) with I.C. in $Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$

Actually, because the ϕ-problem is time reversible, we may take the Initial Conditions $\{\phi_0,\phi_1\} \in Y \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$ at $t = 0$, rather than $t = T$. We shall prove Theorem 19.3.1, relabeled here as

THEOREM 19.C.1

With reference to the ϕ-problem of Eq. (19.3.9) = *Eq.* (19.6.3) = *Eq.* (19.A.4), *with q as in Eq.* (19.3.10) = *Eq.* (19.A.2) *and Initial Conditions $\{\phi_0, \phi_1\}$ at $t = 0$, we have*

$$\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]' \Rightarrow \{\phi, \phi_t\} \in C\left([0, T]; L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'\right), \quad (19.C.1)$$

continuously, where the constant of continuity depends explicitly on the $\|q\|_{L_\infty(Q)}$-norm as in Eq. (19.C.16).

One may provide several more or fewer standard proofs: by the evolution operator $U(t, s)$ as in Appendix 19.B, by fixed point (contraction), etc. Here, for completeness, we provide a proof that has the advantage of showing the *dependence on q of certain estimates*, see Eq. (19.C.13), to be needed later in various contexts.

PROOF OF THEOREM 19.C.1

Step 1. See Eq. (19.C.13). As in Eq. (19.6.36), we split the ϕ-problem in Eq. (19.3.9) = Eq. (19.6.3) = Eq. (19.A.4) (with I.C. at $t = 0$) as the sum of the less regular problem ψ and the more regular problem χ by setting

$$\phi = \psi + \chi, \quad (19.C.2)$$

where ψ and χ are solutions of the following problems:

$$\begin{cases} \psi_{tt} = \Delta\psi; \\ \psi(0, \cdot) = \phi_0, \ \psi_t(0, \cdot) = \phi_1; \\ \psi|_{\Sigma_0} \equiv 0, \ \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\Sigma_1} \equiv 0; \end{cases} \quad \begin{cases} \chi_{tt} = \Delta\chi + q(t, x)\phi = \Delta\chi + q(t, x)\chi + q(t, x)\psi \quad \text{in } Q; \\ \chi(0, \cdot) = \chi_t(0, \cdot) = 0 \quad \text{in } \Omega; \\ \chi|_{\Sigma_0} \equiv 0; \ \left.\dfrac{\partial\chi}{\partial\nu}\right|_{\Sigma_1} \equiv 0 \quad \text{in } \Sigma_i. \end{cases} \quad (19.C.3)$$

The idea of the proof is that by using the unperturbed ψ-problem, we can shift the topologies up or down. For the pure wave equation in ψ, the regularity result

$$\{\phi_0, \phi_1\} \in Y \equiv L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]' \Rightarrow \{\psi, \psi_t\} \in C\left([0, T]; L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]'\right) \quad (19.C.4)$$

continuously, is standard. It can be shown in various ways, for example, by using $\psi(t) = C(t)\psi_0 + S(t)\psi_1$, where $C(\cdot)$ is the "cosine" operator generated by $-\mathcal{A}$, and $S(t)$ the corresponding "sine" operator, see Eq. (19.2.3).

Step 2. Regarding the χ-problem in the RHS of Eq. (19.C.3), a *more regular* result is actually true. Namely, setting

$$E(t) \equiv \int_\Omega \left[|\nabla\chi(t)|^2 + \chi_t^2(t)\right] d\Omega \text{ norm-equivalent to } \|\{\chi(t), \chi_t(t)\}\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)}, \quad (19.C.5)$$

by the Poincaré inequality ($\Gamma_0 \neq \emptyset$), we shall first prove that

$$E(t) \equiv \int_\Omega \left[|\nabla\chi(t)|^2 + \chi_t^2(t)\right] d\Omega \le C_T e^{M_q t} \|q\|_{L_\infty(Q)} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \quad (19.C.6a)$$

$$M_q \equiv \max\{c_P\|q\|_{L_\infty(Q)}, 2\|q\|_{L_\infty(Q)}\}, \quad c_P = \text{constant in Poincaré inequality}. \quad (19.C.6b)$$

In fact, multiplying the χ-problem by χ_t and integrating by parts, we obtain, as usual, using the homogeneous I.C. and denoting $\|q\|_\infty = \|q\|_{L_\infty(Q)}$:

$$\begin{aligned}\int_\Omega [\chi_t^2(t) + |\nabla\chi(t)|^2] d\Omega &= 2\int_0^t\int_\Omega q(\cdot,\,\cdot)\chi\chi_t dQ + 2\int_0^t\int_\Omega q(\cdot,\,\cdot)\psi\chi_t dQ \\ &\leq \|q\|_\infty \int_0^t\int_\Omega \left[\chi^2(\tau) + \chi_t^2(\tau)\right] d\Omega\, d\tau \\ &\quad + \|q\|_\infty \int_0^t\int_\Omega \left[\psi^2(\tau) + \chi_t^2(\tau)\right] d\Omega\, d\tau. \end{aligned} \tag{19.C.7}$$

We now invoke the Poincaré inequality: $\int_\Omega \chi^2 d\Omega \leq c_P \int_\Omega |\nabla\chi|^2 d\Omega$, which is valid because we have assumed $\Gamma_0 \neq \emptyset$. Thus, Eq. (19.C.7) becomes

$$\begin{aligned}\int_\Omega \left[\chi_t^2(t) + |\nabla\chi(t)|^2\right] d\Omega &\leq c_P\|q\|_\infty \int_0^t\int_\Omega |\nabla\chi(\tau)|^2 d\Omega\, d\tau \\ &\quad + 2\|q\|_\infty \int_0^t\int_\Omega \chi_t^2(\tau) d\Omega\, d\tau + \|q\|_\infty \int_0^t\int_\Omega \psi^2(\tau) d\Omega\, d\tau. \end{aligned} \tag{19.C.8}$$

Setting $M_q \equiv \max\{c_P\|q\|_\infty, 2\|q\|_\infty\}$ and invoking a weaker L_2-regularity result included in conclusion Eq. (19.C.4) in the last term in Eq. (19.C.8), we obtain

$$\begin{aligned}\int_\Omega \left[\chi_t^2(t) + |\nabla\chi(t)|^2\right] d\Omega &\leq C_T\|q\|_\infty \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} \\ &\quad + M_q\int_0^t\int_\Omega \left[\chi_t^2(\tau) + |\nabla\chi(\tau)|^2\right] d\Omega\, d\tau; \end{aligned} \tag{19.C.9}$$

or via Eq. (19.C.5),

$$E(t) \leq C_T\|q\|_\infty \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'} + M_q\int_0^t E(\tau)\, d\tau. \tag{19.C.10}$$

Application of the Gronwall's inequality on Eq. (19.C.10) yields

$$E(t) \equiv \int_\Omega \left[|\nabla\chi(t)|^2 + \chi_t^2(t)\right] d\Omega \leq C_T e^{M_q t}\|q\|_\infty \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \tag{19.C.11}$$

which is precisely Eq. (19.C.6a).

Step 3. From the point-wise estimate of Eq. (19.C.11) = Eq. (19.C.6a), we then obtain

$$\sup_{0\leq t\leq T}\int_\Omega \left[|\nabla\chi(t)|^2 + \chi_t^2(t)\right] d\Omega \leq C_T e^{M_q T}\|q\|_{L_\infty(Q)} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \tag{19.C.12}$$

Because of norm equivalence noted in Eq. (19.C.5) (via $\Gamma_0 \neq \emptyset$), we finally obtain from Eq. (19.C.12) (first in $L_\infty(0, T;\, \cdot)$, then by approximation in $C([0, T];\, \cdot)$)

$$\|\{\chi, \chi_t\}\|^2_{C([0,T];\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega))} \leq C_T e^{M_q T}\|q\|_{L_\infty(Q)} \|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}, \tag{19.C.13}$$

which shows the desired regularity property

$$\{\phi_0, \phi_1\} \in L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]' \Rightarrow \{\chi, \chi_t\} \in C([0, T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Omega)), \tag{19.C.14}$$

with an explicit dependence on $\|q\|_{L_\infty(Q)}$ of the constant of continuity.

Step 4. Then statements Eq. (19.C.4) and Eq. (19.C.14), combined in Eq. (19.C.2), yield, *a-fortiori*

$$\{\phi_0, \phi_1\} \in L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)\right]' \Rightarrow \{\phi, \phi_t\} \in C\left([0, T]; L_2(\Omega) \times \left[\mathcal{D}\left(\mathcal{A}^{\frac{1}{2}}\right)'\right]\right), \tag{19.C.15}$$

which is the desired conclusion of Eq. (19.C.1). Actually Eqs. (19.C.4) and (19.C.13) give the more precise result

$$\|\{\phi, \phi_t\}\|^2_{C([0,T];L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')} \leq C_T(1 + e^{M_q T}\|q\|_{L_\infty(Q)})\|\{\phi_0, \phi_1\}\|^2_{L_2(\Omega)\times[\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'}. \quad (19.C.16)$$

Theorem 19.C.1 is proved. ∎

That is, Theorem 19.3.1 (= Theorem 19.C.1) of Section 19.3 is proved.

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Chapter 20

Carleman Estimates for the Three-Dimensional Nonstationary Lamé System and Application to an Inverse Problem

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Abstract In this chapter, we establish Carleman estimates for the three-dimensional isotropic nonstationary Lamé system with homogeneous Dirichlet boundary conditions. Using this estimate, we prove the uniqueness and stability in determining spatially varying density and two Lamé coefficients by a single measurement of solution over $(0, T) \times \omega$, where $T > 0$ is sufficiently large and the subdomain ω satisfies a geometric condition.

20.1 Introduction

This chapter is concerned with Carleman estimates for the three-dimensional nonstationary isotropic Lamé system with homogeneous Dirichlet boundary condition and their application to an inverse problem of determining spatially varying density and the Lamé coefficients by a single interior measurement of the solution.

We consider the three-dimensional isotropic nonstationary Lamé system with homogeneous Dirichlet boundary condition $\mathbf{u}|_{\partial\Omega} = 0$:

$$(P\mathbf{u})(x_0, x') \equiv \rho(x')\partial_{x_0}^2 \mathbf{u}(x_0, x') - (L_{\lambda,\mu}\mathbf{u})(x_0, x') = \mathbf{f}(x_0, x'),$$
$$x \equiv (x_0, x') \in Q \equiv (0, T) \times \Omega, \tag{20.1}$$

where

$$(L_{\lambda,\mu}\mathbf{v})(x') \equiv \mu(x')\Delta\mathbf{v}(x') + [\mu(x') + \lambda(x')]\nabla_{x'}\mathrm{div}\, \mathbf{v}(x')$$
$$+ [\mathrm{div}\, \mathbf{v}(x')]\nabla_{x'}\lambda(x') + \left[\nabla_{x'}\mathbf{v} + (\nabla_{x'}\mathbf{v})^T\right] \nabla_{x'}\mu(x'), \quad x' \in \Omega. \tag{20.2}$$

Throughout this paper, $\Omega \subset \mathbb{R}^3$ is a bounded domain whose boundary $\partial\Omega$ is of class C^3, x_0 and $x' = (x_1, x_2, x_3)$ denote the time variable and the spatial variable, respectively, and $\mathbf{u} = (u_1, u_2, u_3)^T$ is displacement at (x_0, x') where $\cdot^T$ denotes the transpose of matrices, E_k is the $k \times k$ unit matrix, and

$$\partial_{x_j}\phi = \phi_{x_j} = \frac{\partial \phi}{\partial x_j}, \quad j = 0, 1, 2, 3.$$

We set $\nabla_{x'}\mathbf{v} = (\partial_{x_k} v_j)_{1\le j,k\le 3}$ for a vector function $\mathbf{v} = (v_1, v_2, v_3)^T$, and $\nabla_{x'}\phi = (\partial_{x_1}\phi, \partial_{x_2}\phi, \partial_{x_3}\phi)^T$ for a scalar function ϕ. Henceforth, ∇ means $\nabla_x = (\partial_{x_0}, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ if we do not specify. Moreover, the coefficients ρ, λ, μ satisfy

$$\rho, \lambda, \mu \in C^2(\overline{\Omega}), \quad \rho(x') > 0, \mu(x') > 0, \ \lambda(x') + \mu(x') > 0 \quad \text{for } x' \in \overline{\Omega}. \tag{20.3}$$

The Carleman estimate is an L^2-inequality on solutions to a partial differential equation and involves a large parameter and a special weight function. The Carleman estimate was introduced by Carleman [13] for proving the unique continuation for an elliptic equation, and general theories have been developed for single partial differential equations (e.g., see Reference 17). Moreover, the Carleman estimates have been effectively applied to the following problems:

1. **Energy estimate called "observability inequality"**: Cheng et al. [14], Kazemi and Klibanov [37], Klibanov and Malinsky [41], Lasiecka and Triggiani [44], Lasiecka et al. [45], Lasiecka et al. [46, 47], Tataru [53, 54], and Triggiani and Yao [56].

2. **Exact controllability and related control problems:** Bellassoued [4–6], Imanuvilov [19], Imanuvilov and Yamamoto [27, 28], Lasiecka and Triggiani [44], Lasiecka et al. [45–47], and Tataru [54].

3. **Inverse problems of determining functions in partial differential equations by a finite number of overlateral boundary data:** See Bukhgeim and Klibanov [12] as a pioneering paper. There are extensive references, and we will give them in Section 20.3.

Thus, it is first important to establish a Carleman estimate, which depends on types of partial differential equations under consideration. Especially for a single partial differential equation, the general theory for Carleman estimates has been well developed (e.g., see References 17, 33, and 34). In particular, for a single hyperbolic equation, see Imanuvilov [20]. However, for systems of partial differential equations where the principal terms are coupled, the results are still restricted because of the intrinsic difficulty. The most general result for such a system is the Carleman-type estimate obtained in the proof of the Carderon uniqueness theorem (see, e.g., References 15 and 60).

The nonstationary isotropic Lamé system is basic in the theory of elasticity, and unfortunately it does not satisfy all the conditions of the Calderon uniqueness theorem. In the existing papers, Carleman estimates for the Lamé system have been proved mainly for functions with compact supports (e.g., Eller et al. [16], Ikehata et al. [18], Imanuvilov et al. [21], and Isakov [32]). Because of the restriction that $\mathbf{u}$ under consideration should have compact support, for the observability inequalities and the inverse problems, we have to take Cauchy data $\mathbf{u}$ and $\nabla\mathbf{u}$ on the whole boundary $(0, T) \times \partial\Omega$ or $\mathbf{u}$ in a neighborhood of $\partial\Omega$ over $(0, T)$. Because we need not take Cauchy data on $(0, T) \times \partial\Omega$ or in such a neighborhood for the wave equation (e.g., see Lions [49] for the observability inequality, and Imanuvilov and Yamamoto [24, 26] for the inverse problem for a single hyperbolic equation), we can naturally expect similar results also for the nonstationary isotropic Lamé system.

In the two-dimensional case, we have recently established Carleman estimates for $\mathbf{u}$ without compact supports to apply them to an inverse problem of determining the density and two Lamé coefficients:

1. Imanuvilov and Yamamoto [30] for the case of the Dirichlet boundary condition

2. Imanuvilov and Yamamoto [31] for the case of the stress boundary condition

In this chapter, we will prove Carleman estimates in the case where the spatial dimension is three and $\mathbf{u}$ satisfies the homogeneous Dirichlet boundary condition and apply them to an inverse problem of determining ρ, λ, and μ by an interior measurement after suitably choosing single initial data. The three-dimensional case is handled similarly to the two-dimensional case [30], but the treatment should be modified.

We refer to Imanuvilov and Yamamoto [29] concerning the stationary isotropic Lamé system and Isakov et al. [35] and Lin and Wang [48] concerning the Lamé system with residual stress, which causes anisotropicity.

This chapter is composed of seven sections. In Section 20.2, we state Carleman estimates (Theorems 20.1–20.3) for functions that do not necessarily have compact supports but satisfy the homogeneous Dirichlet boundary condition on $(0, T) \times \partial\Omega$. Theorem 20.1 is a Carleman estimate whose right-hand side is estimated in H^1-space. Theorems 20.2 and 20.3 are Carleman estimates, respectively, with right-hand sides in L^2-space and in H^{-1}-space, and are proved from Theorem 20.1 by the same method used in Reference 30. In Section 20.3, we will apply the H^{-1}-Carleman estimate (Theorem 20.3) and prove the uniqueness and the conditional stability in the inverse problem with a single interior measurement. In Sections 20.4 to 20.7, we prove Theorem 20.1.

NOTATIONS $H^{1,s}(Q)$ is the Sobolev space of scalar-valued functions equipped with the norm

$$\|u\|_{H^{1,s}(Q)} = \sqrt{\int_Q (|\nabla u|^2 + s^2 u^2)\, dx},$$

$\mathbf{H}^{1,s}(Q) = H^{1,s}(Q) \times \cdots \times H^{1,s}(Q)$ is the corresponding space of vector-valued functions. Henceforth we set

$$i = \sqrt{-1}, \quad D_{x_j} = \frac{1}{i}\partial_{x_j}, \quad j = 0, 1, 2, 3,$$

and $\overline{c}$ denotes the complex conjugate of $c \in \mathbb{C}$. By $\mathcal{L}(X, Y)$ we denote the Banach space of all the linear bounded operators defined on a Banach space X to another Banach space Y. We set

$$\xi = (\xi_0, \xi_1, \xi_2, \xi_3), \quad \xi' = (\xi_0, \xi_1, \xi_2), \quad \zeta = (s, \xi_0, \xi_1, \xi_2).$$

By $\mathcal{O}(\delta)$ we denote the conic neighborhood of a point ζ^*:

$$\mathcal{O}(\delta) = \left\{ \zeta;\ \left| \frac{\zeta}{|\zeta|} - \zeta^* \right| \le \delta \right\}.$$

20.2 Carleman Estimates for the Three-Dimensional Nonstationary Lamé System

Let us consider the three-dimensional Lamé system

$$P\mathbf{u}(x_0, x') \equiv \rho(x')\partial_{x_0}^2 \mathbf{u}(x_0, x') - (L_{\lambda,\mu}\mathbf{u})(x_0, x') = \mathbf{f}(x_0, x') \quad \text{in } Q, \tag{20.4}$$

$$\mathbf{u}|_{(0,T)\times\partial\Omega} = 0, \quad \mathbf{u}(T, x') = \partial_{x_0}\mathbf{u}(T, x') = \mathbf{u}(0, x') = \partial_{x_0}\mathbf{u}(0, x') = 0, \tag{20.5}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\mathbf{f} = (f_1, f_2, f_3)^T$ are vector-valued functions, and the partial differential operator $L_{\lambda,\mu}$ is defined by Eq. (20.2). The coefficients $\rho, \lambda, \mu \in C^2(\overline{\Omega})$ are assumed to satisfy Eq. (20.3). Let $\omega \subset \Omega$ be an arbitrarily fixed subdomain (not necessarily connected).

Let $\vec{n}(x') = [n_1(x'), n_2(x'), n_3(x')]$ and $\vec{t}(x')$, respectively, denote the outward unit normal vector and a unit tangential vector to $\partial\Omega$ at x' and set $\frac{\partial v}{\partial \vec{n}} = \nabla_{x'} v \cdot \vec{n}$ and $\frac{\partial v}{\partial \vec{t}} = \nabla_{x'} v \cdot \vec{t}$. Set

$$Q_\omega = (0, T) \times \omega.$$

We set

$$\begin{cases} p_1(x, \xi) = \rho(x')\xi_0^2 - \mu(x')\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right), \\ p_2(x, \xi) = \rho(x')\xi_0^2 - [\lambda(x') + 2\mu(x')]\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) \end{cases} \tag{20.6}$$

for $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$, and $\nabla_\xi = (\partial_{\xi_0}, \partial_{\xi_1}, \partial_{\xi_2}, \partial_{\xi_3})$. For arbitrary smooth functions $\varphi(x, \xi)$ and $\psi(x, \xi)$, we define the Poisson bracket by the formula

$$\{\varphi, \psi\} = \sum_{j=0}^{3} \left(\partial_{\xi_j}\varphi\right)\left(\partial_{x_j}\psi\right) - \left(\partial_{\xi_j}\psi\right)\left(\partial_{x_j}\varphi\right).$$

We set $\langle a, b\rangle = \sum_{k=1}^{3} a_k \overline{b}_k$ for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in \mathbb{C}^3$.

We assume that the density ρ; the Lamé coefficients λ, μ; and the domains Ω, ω satisfy the following condition (cf. Reference 17).

CONDITION 20.1 *There exists a function $\psi \in C^3(\overline{Q})$ such that $|\nabla_x \psi| \neq 0$ on $\overline{Q \setminus Q_\omega}$, and*

$$\{p_k, \{p_k, \psi\}\}(x, \xi) > 0, \quad \forall k \in \{1, 2\} \tag{20.7}$$

if $(x, \xi) \in (\overline{Q \setminus Q_\omega}) \times (\mathbb{R}^4 \setminus \{0\})$ satisfies $p_k(x, \xi) = \langle \nabla_\xi p_k, \nabla \psi\rangle = 0$ and

$$\frac{1}{2is}\{p_k(x, \xi - is\nabla\psi(x)), p_k(x, \xi + is\nabla\psi(x))\} > 0, \quad \forall k \in \{1, 2\} \tag{20.8}$$

if $(x, \xi, s) \in (\overline{Q \setminus Q_\omega}) \times (\mathbb{R}^4 \setminus \{0\}) \times (\mathbb{R}_+ \setminus \{0\})$ satisfies

$$p_k(x, \xi + is\nabla\psi(x)) = \langle \nabla_\xi p_k(x, \xi + is\nabla\psi(x)), \nabla\psi(x)\rangle = 0.$$

On the lateral boundary, we assume that

$$\sqrt{\rho}|\psi_{x_0}| < \frac{\mu}{\sqrt{\lambda + 2\mu}}\left|\frac{\partial\psi}{\partial\vec{t}}\right| + \frac{\sqrt{\mu}\sqrt{\lambda + \mu}}{\sqrt{\lambda + 2\mu}}\left|\frac{\partial\psi}{\partial\vec{n}}\right| \text{ for any unit tangential vector } \vec{t}(x'),$$

$$x' \in \overline{\partial\Omega \setminus \partial\omega}, \ p_1(x, \nabla\psi) < 0 \quad \forall x \in \overline{(0, T) \times (\partial\Omega \setminus \partial\omega)} \quad \text{and} \quad \left.\frac{\partial\psi}{\partial\vec{n}}\right|_{(0,T)\times(\overline{\partial\Omega\setminus\partial\omega})} < 0. \tag{20.9}$$

Let $\psi(x)$ be the weight function in Condition 20.1. Using this function, we introduce the function $\phi(x)$ by

$$\phi(x) = e^{\tau\psi(x)}, \quad \tau > 1, \tag{20.10}$$

where the parameter $\tau > 0$ will be fixed below. Denote

$$\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} = \int_Q \left(\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\left|\partial_x^\alpha \mathbf{u}\right|^2 + s|\nabla \mathrm{rot}\, \mathbf{u}|^2 + s^3|\mathrm{rot}\, \mathbf{u}|^2 + s|\nabla \mathrm{div}\, \mathbf{u}|^2 + s^3|\mathrm{div}\, \mathbf{u}|^2\right) e^{2s\phi} dx,$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\alpha_j \in \mathbb{N}_+ \cup \{0\}$, $j \in \{0, 1, 2, 3\}$, $\partial_x^\alpha = \partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$.

Now we state our Carleman estimates as main results.

THEOREM 20.1
Let $\mathbf{f} \in \mathbf{H}^1(Q)$ *and let the function* ψ *satisfy Condition 20.1, and* (20.3) *hold true. Then there exists* $\widehat{\tau} > 0$ *such that for any* $\tau > \widehat{\tau}$*, there exists* $s_0 = s_0(\tau) > 0$ *such that for any solution* $\mathbf{u} \in \mathbf{H}^1(Q) \cap L^2[0, T; \mathbf{H}^2(\Omega)]$ *to the* (20.4) *and* (20.5)*, the following estimate holds true:*

$$\|\mathbf{u}\|^2_{Y(\phi,Q)} \equiv \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} + s \left\| \frac{\partial \mathbf{u}}{\partial \vec{n}} e^{s\phi} \right\|^2_{\mathbf{H}^1[(0,T)\times\partial\Omega]} + s \left\| \frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2} e^{s\phi} \right\|^2_{\mathbf{L}^2[(0,T)\times\partial\Omega]}$$
$$\leq C(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}), \quad \forall s \geq s_0(\tau), \tag{20.11}$$

where the constant $C = C(\tau) > 0$ *is independent of* s.

Next we formulate Carleman estimates where norms of the function $\mathbf{f}$ are taken, respectively, in $\mathbf{L}^2(Q)$ and $L^2[0, T; \mathbf{H}^{-1}(\Omega)]$. In particular, the latter Carleman estimate is used in Section 20.3 for obtaining our stability result in the inverse problem.

In addition to Condition 20.1, we assume that

$$\partial_{x_0}\psi(T, x') < 0, \quad \partial_{x_0}\psi(0, x') > 0, \quad \forall x' \in \overline{\Omega}. \tag{20.12}$$

We have the following.

THEOREM 20.2
Let $\mathbf{f} \in \mathbf{L}^2(Q)$ *and let us assume* (20.3) *and* (20.12) *and Condition 20.1. Then there exists* $\widehat{\tau} > 0$ *such that for any* $\tau > \widehat{\tau}$*, there exists* $s_0 = s_0(\tau) > 0$ *such that for any solution* $\mathbf{u} \in \mathbf{H}^1(Q)$ *to* (20.4) *and* (20.5)*, the following estimate holds true:*

$$\int_Q (|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi}dx$$
$$\leq C\left(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + \int_{Q_\omega} (|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi}dx\right), \quad \forall s \geq s_0(\tau), \tag{20.13}$$

where the constant $C = C(\tau) > 0$ *is independent of* s.

THEOREM 20.3
Let $\mathbf{f} = \mathbf{f}_{-1} + \sum_{j=0}^{3} \partial_{x_j}\mathbf{f}_j$ *with* $\mathbf{f}_{-1} \in L^2[0, T; \mathbf{H}^{-1}(\Omega)]$ *and* $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in \mathbf{L}^2(Q)$*, and let us assume* (20.3) *and* (20.12) *and Condition 20.1. Then there exists* $\widehat{\tau} > 0$ *such that for any* $\tau > \widehat{\tau}$*, there exists* $s_0 = s_0(\tau) > 0$ *such that for any solution* $\mathbf{u} \in \mathbf{L}^2(Q)$ *to* (20.4) *and* (20.5)*, the following estimate holds true:*

$$\int_Q |\mathbf{u}|^2 e^{2s\phi}dx$$
$$\leq C\left(\|\mathbf{f}_{-1}e^{s\phi}\|^2_{L^2[0,T;\boldsymbol{H}^{-1}(\Omega)]} + \sum_{j=0}^{3} \|\mathbf{f}_j e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + \int_{Q_\omega} |\mathbf{u}|^2 e^{2s\phi}dx\right), \quad \forall s \geq s_0(\tau), \tag{20.14}$$

where the constant $C = C(\tau) > 0$ *is independent of* s.

In Theorems 20.2 and 20.3, the solution $\mathbf{u}$ is defined by the transposition method (e.g., Reference 49). On the basis of Theorem 20.1, we can prove Theorems 20.2 and 20.3 exactly in the same way as the corresponding theorems in Reference 30, and it suffices to prove only Theorem 20.1.

20.3 Inverse Problem of Determining the Density and the Lamé Coefficients by a Single Measurement

Recall that the differential operator $L_{\lambda,\mu}$ is defined by (20.2). We assume (20.3) for ρ, λ, μ. By $\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)(x)$, we denote the sufficiently smooth solution to

$$\rho(x')\left(\partial_{x_0}^2 \mathbf{u}\right)(x) = (L_{\lambda,\mu}\mathbf{u})(x), \quad x \in Q, \tag{20.15}$$

$$\mathbf{u}(x) = \eta(x), \quad x \in (0, T) \times \partial\Omega, \tag{20.16}$$

$$\mathbf{u}(T/2, x') = \mathbf{p}(x'), \quad (\partial_{x_0}\mathbf{u})(T/2, x') = \mathbf{q}(x'), \quad x' \in \Omega, \tag{20.17}$$

with given η, $\mathbf{p}$, and $\mathbf{q}$. Let $\omega \subset \Omega$ be a suitably given subdomain.

In this section, we discuss the following.

INVERSE PROBLEM *Let $\mathbf{p}_j, \mathbf{q}_j, \eta_j$, $1 \le j \le \mathcal{N}$, be appropriately given. Then determine* $\lambda(x')$, $\mu(x')$, $\rho(x')$, and $x' \in \Omega$, by

$$\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}_j, \mathbf{q}_j, \eta_j)(x), \quad x \in Q_\omega \equiv (0, T) \times \omega. \tag{20.18}$$

In particular, we are concerned with the stability of the mapping

$$\{\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}_j, \mathbf{q}_j, \eta_j)|_{Q_\omega}\}_{1 \le j \le \mathcal{N}} \longrightarrow \{\lambda, \mu, \rho\}.$$

This formulation of the inverse problem is based on finite measurements. The research originated with Bukhgeim and Klibanov [12] where a Carleman estimate and an integral inequality with the weight function are combined to solve the inverse problem. As detailed accounts for such methodology, see References 33, 34, 40, and 42. Moreover, according to equations, we refer to the following works:

1. Baudouin and Puel [3] and Bukhgeim [10] for an inverse problem of determining potentials in Schrödinger equations

2. Imanuvilov and Yamamoto [23], Isakov [33, 34], and Klibanov [40] for the corresponding inverse problems for parabolic equations

3. Bellasssoued [7, 8], Bellassoued and Yamamoto [9], Bukhgeim et al. [11], Imanuvilov and Yamamoto [24–26] (especially for conditional stability), Isakov [32–34], Isakov and Yamamoto [36], Khaĭdarov [38, 39], Klibanov [40], Puel and Yamamoto [50, 51], and Yamamoto [59] for inverse problems of determining coefficients in scalar hyperbolic equations

4. Amirov [1] for an inverse problem of an ultrahyperbolic equation

As for the inverse problem of determining some (or all) of λ, μ, and ρ, we can refer to Isakov [32], Ikehata et al. [18], Imanuvilov et al. [21], and Imanuvilov and Yamamoto [30]:

1. Reference 32 established the uniqueness in determining a single coefficient $\rho(x')$, using four measurements (i.e., $\mathcal{N} = 4$).

2. Reference 18 decreased the number $\mathcal{N}$ of measurements to three for determining ρ.

3. Reference 21 proved conditional stability and the uniqueness in the determination of the three functions $\lambda(x')$, $\mu(x')$, $\rho(x')$, $x' \in \Omega$, with two measurements (i.e., $\mathcal{N} = 2$).

In all three papers [32, 18, 21], the authors have to assume that $\partial\omega \supset \partial\Omega$ because the technique based on Carleman estimates required that $\mathbf{u}$ has a compact support in Q. In the two-dimensional case, Reference 30 reduced $\mathcal{N} = 2$ to $\mathcal{N} = 1$ (i.e., a single measurement) in determining all of λ, μ, ρ with more general ω and established conditional stability. As for other inverse problems for the Lamé systems, see Yakhno [57].

In this section, we will prove the conditional stability, which is a three-dimensional version of that in Reference 30. As for the two-dimensional Lamé system with stress boundary condition, in Reference 31 a similar inverse problem is discussed by a single measurement.

To formulate our main result, we will introduce notations and an admissible set of unknown parameters $\lambda, \mu,$ and ρ. Similarly to inverse hyperbolic problems, we have to assume that the observation subdomain ω should satisfy a geometric condition and the observation time T has to be sufficiently large, which is a natural consequence of the hyperbolicity of the governing partial differential equation. First we formulate the geometric condition. Henceforth, we set $(x', y') = \sum_{j=1}^{3} x_j y_j$ for $x' = (x_1, x_2, x_3)$ and $y' = (y_1, y_2, y_3)$. Let a subdomain $\omega \subset \Omega$ satisfy

$$\partial\omega \supset \{x' \in \partial\Omega;\ ((x' - y'), \vec{n}(x')) \geq 0\} \tag{20.19}$$

with some $y' \notin \overline{\Omega}$.

REMARK The condition in (20.19) is the same condition that yields the observability inequality for the wave equation $\partial_{x_0}^2 - \Delta$ if the observation time T is larger than $2\sup_{x'\in\Omega} |x' - y'|$ (e.g., Section 2 of Chapter 7 in Reference 49). Moreover, if (20.19) holds and $T > 0$ is sufficiently large, then ω and T satisfy the geometric optics condition in Reference 2.

Denote

$$d = \left(\sup_{x'\in\Omega} |x' - y'|^2 - \inf_{x'\in\Omega} |x' - y'|^2 \right)^{\frac{1}{2}}. \tag{20.20}$$

Next we define an admissible set of unknown coefficients λ, μ, and ρ. Let $M_0 > 0$, $0 < \theta_0 \leq 1$, and $\theta_1 > 0$ be arbitrarily fixed, and let us introduce the conditions on a function β:

$$\begin{cases} \beta(x') \geq \theta_1 > 0, & x' \in \overline{\Omega}, \\ \|\beta\|_{C^3(\overline{\Omega})} \leq M_0, & \dfrac{(\nabla_{x'}\beta(x'), (x' - y'))}{2\beta(x')} \leq 1 - \theta_0, \quad x' \in \overline{\Omega\setminus\omega}. \end{cases} \tag{20.21}$$

For fixed functions a, b, η on $\partial\Omega$ and $\mathbf{p}, \mathbf{q}$ in Ω and a fixed constant $M_1 > 0$, we set

$$\begin{aligned} \mathcal{W} = \mathcal{W}_{M_0,M_1,\theta_0,\theta_1,a,b} = \Bigg\{ &(\lambda, \mu, \rho) \in (C^3(\overline{\Omega}))^3;\ \lambda = a, \mu = b \quad \text{on } \partial\Omega, \\ &\frac{\lambda + 2\mu}{\rho}, \frac{\mu}{\rho} \text{ satisfy Eq. (20.21)},\ \|\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)\|_{W^{7,\infty}(Q)} \leq M_1, \\ &\frac{\min\{\mu^2(x'), \mu(x')(\lambda + \mu)(x')\}}{\rho(x')(\lambda + 2\mu)(x')} \geq \theta_1 \quad \text{on } \overline{\Omega} \Bigg\}. \end{aligned} \tag{20.22}$$

REMARK If λ, μ, and ρ are sufficiently close to positive constant functions, then $(\lambda, \mu, \rho) \in \mathcal{W}$. This suggests that $\mathcal{W}$ contains sufficiently many (λ, μ, ρ).

It is rather restrictive that $\frac{\lambda+2\mu}{\rho}$ and $\frac{\mu}{\rho}$ should satisfy (20.21), which is one possible sufficient condition for the pseudoconvexity (i.e., Condition 20.1). We can relax Condition (20.21) to a more generous condition that can be related with a necessary condition for a Carleman estimate, and we refer to Imanuvilov et al. [22], where a scalar hyperbolic equation is discussed, but the modification to the Lamé system is straightforward. Such a relaxed condition guarantees that the geodesics that are generated by the hyperbolic equations with principal symbol (20.6) cannot remain on the level sets given by the weight function ϕ. In particular, by Reference 22, we can replace the Condition (20.21) by one such that the Hessians

$$\left[\partial_{x_j}\partial_{x_k}\left(\frac{\rho}{\mu}\right)^{\frac{1}{2}}\right]_{1\le j,k\le 2}, \quad \left[\partial_{x_j}\partial_{x_k}\left(\frac{\rho}{\lambda+2\mu}\right)^{\frac{1}{2}}\right]_{1\le j,k\le 2}$$

are nonnegative and $\left|\nabla\left(\frac{\rho}{\mu}\right)\right| \neq 0$ and $\left|\nabla\left(\frac{\rho}{\lambda+2\mu}\right)\right| \neq 0$ on $\overline{\Omega}$.

Next we choose $\theta > 0$ such that

$$\theta + \frac{M_0 d}{\sqrt{\theta_1}}\sqrt{\theta} < \theta_0\theta_1, \quad \theta_1 \inf_{x'\in\Omega} |x'-y'|^2 - \theta d^2 > 0. \tag{20.23}$$

Here we note that because $y' \notin \overline{\Omega}$, such $\theta > 0$ exists. Let E_3 be the 3×3 identity matrix. We note that $(L_{\lambda,\mu}\mathbf{p})(x')$ is a three-column vector for three-column vector $\mathbf{p}$. Moreover by $\{\mathbf{a}\}_j$ we denote the matrix (or vector) obtained from $\mathbf{a}$ after deleting the j-th row, and $\langle A\rangle_j$ is the matrix that is obtained from A by deleting the j-th column of A. Furthermore, we assume that

$$x_1 - y_1 \neq 0 \quad \text{for any } (x_1, x_2, x_3) \in \overline{\Omega}. \tag{20.24}$$

Now we are ready to state the following theorem.

THEOREM 20.4

Let (λ, μ, ρ) be an arbitrary element of $\mathcal{W}$. For $\mathbf{p} = (p_1, p_2, p_3)^T$ and $\mathbf{q} = (q_1, q_2, q_3)^T$, we assume that there exist $j_1, j_2, j_3 \in \{1,2,3,4,5,6\}$ such that

$$det\begin{Bmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & [div\,\mathbf{p}(x')]E_3 & \left\{\nabla_{x'}\mathbf{p}(x') + [\nabla_{x'}\mathbf{p}(x')]^T\right\}(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & [div\,\mathbf{q}(x')]E_3 & \left\{\nabla_{x'}\mathbf{q}(x') + [\nabla_{x'}\mathbf{q}(x')]^T\right\}(x'-y') \end{Bmatrix}_{j_1} \neq 0, \quad \forall x' \in \overline{\Omega}, \tag{20.25}$$

$$det\begin{Bmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & \nabla_{x'}\mathbf{p}(x') + [\nabla_{x'}\mathbf{p}(x')]^T & (div\,\mathbf{p})(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & \nabla_{x'}\mathbf{q}(x') + [\nabla_{x'}\mathbf{q}(x')]^T & (div\,\mathbf{q})(x'-y') \end{Bmatrix}_{j_2} \neq 0, \quad \forall x' \in \overline{\Omega}, \tag{20.26}$$

$$det\begin{Bmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & [div\,\mathbf{p}(x')]\langle E_3\rangle_1 & \left\langle \nabla_{x'}\mathbf{p}(x') + [\nabla_{x'}\mathbf{p}(x')]^T\right\rangle_1 \\ (L_{\lambda,\mu}\mathbf{q})(x') & [div\,\mathbf{q}(x')]\langle E_3\rangle_1 & \left\langle \nabla_{x'}\mathbf{q}(x') + [\nabla_{x'}\mathbf{q}(x')]^T\right\rangle_1 \end{Bmatrix}_{j_3} \neq 0, \quad \forall x' \in \overline{\Omega}, \tag{20.27}$$

and that

$$T > \frac{2}{\sqrt{\theta}}d. \tag{20.28}$$

Then there exist constants $\kappa = \kappa(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) \in (0,1)$ and $C_1 = C_1(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) > 0$ such that

$$\begin{aligned} &\|\widetilde{\lambda} - \lambda\|_{L^2(\Omega)} + \|\widetilde{\mu} - \mu\|_{L^2(\Omega)} + \|\widetilde{\rho} - \rho\|_{H^{-1}(\Omega)} \\ &\quad \le C_1 \|\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta) - \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)\|^{\kappa}_{H^4[0,T;\boldsymbol{L}^2(\omega)]} \end{aligned}$$

for any $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}) \in \mathcal{W}$.

Our stability and uniqueness result requires only one measurement: $\mathcal{N} = 1$. In the case where $x_k - y_k \neq 0$ for $k = 2$ or 3, the conclusion is true if we replace (20.27) by

$$\det \begin{Bmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & [\operatorname{div}\mathbf{p}(x')]\langle E_3\rangle_k & \langle \nabla_{x'}\mathbf{p}(x') + [\nabla_{x'}\mathbf{p}(x')]^T\rangle_k \\ (L_{\lambda,\mu}\mathbf{q})(x') & [\operatorname{div}\mathbf{q}(x')]\langle E_3\rangle_k & \langle \nabla_{x'}\mathbf{q}(x') + [\nabla_{x'}\mathbf{q}(x')]^T\rangle_k \end{Bmatrix}_{j_3} \neq 0, \quad \forall x' \in \overline{\Omega}.$$

For the determination of the three coefficients by a single measurement, we have to choose initial data that satisfy strong conditions in Eqs. (20.25) to (20.27), which do not generically hold, and we should satisfy them artificially and *a posteriori*. Moreover, as the following example shows, we can take such $\mathbf{p}$ and $\mathbf{q}$.

Example of Ω, p, and q meeting (20.25) to (20.27) For simplicity, we assume that $y' = (0, 0, 0)$, $\overline{\Omega}$ does not intersect any of the planes $\{x_1 = 0\}$, $\{x_2 = 0\}$, $\{x_3 = 0\}$, and $\{x_1 + x_3 = 0\}$, and λ, and μ are positive constants. Noting that the fifth columns of the matrices in (20.25) and (20.26) have $x' - y'$ as factors, we will take quadratic functions in x'. For example, we take

$$\mathbf{p}(x') = \begin{pmatrix} 0 \\ x_1x_2 \\ 0 \end{pmatrix}, \quad \mathbf{q}(x') = \begin{pmatrix} x_2^2 \\ 0 \\ x_2^2 \end{pmatrix}.$$

Then, choosing $j_1 = j_2 = j_3 = 6$, we can verify that (20.25) to (20.27) are all satisfied. We set

$$\psi(x) = |x' - y'|^2 - \theta\left(x_0 - \frac{T}{2}\right)^2, \quad \phi(x) = e^{\tau\psi(x)}, \quad x = (x_0, x') \in Q. \tag{20.29}$$

By $y' \notin \overline{\Omega}$, we note that $|\nabla_{x'}\psi| \neq 0$, $x \in \overline{Q}$.

First, in terms of Eqs. (20.19), (20.22), and (20.23), we can prove the following lemma in the same way as in Reference 30.

LEMMA 20.1
Let $(\lambda, \mu, \rho) \in \mathcal{W}$, and let us assume (20.23) *and* (20.28). *Then, for sufficiently large $\tau > 0$, the function ψ given by* (20.29) *satisfies Condition 20.1 and* (20.12). *Therefore, the conclusion of Theorem 20.3 holds, and the constants $C_1(\tau)$, $\widehat{\tau}$, and $s_0(\tau)$ in* (20.14) *can be taken independently of $(\lambda, \mu, \rho) \in \mathcal{W}$.*

Next we consider a first-order partial differential operator

$$(P_0 g)(x') = \sum_{j=1}^{3} p_{0,j}(x')\partial_{x_j} g(x'),$$

where $p_{0,j} \in C^1(\overline{\Omega})$, and $j = 1, 2, 3$. Then, by integration by parts, we can directly prove two Carleman estimates for P_0 (see Reference 30 for the proof).

LEMMA 20.2
We assume that

$$\sum_{j=1}^{3} p_{0,j}(x')\partial_{x_j}\phi(T/2, x') > 0, \quad x' \in \overline{\Omega}. \tag{20.30}$$

Then there exists a constant $\tau_0 > 0$ *such that for all* $\tau > \tau_0$*, there exist* $s_0 = s_0(\tau) > 0$ *and* $C_2 = C_2(s_0, \tau_0, \Omega, \omega) > 0$ *such that*

$$\int_\Omega s^2|g|^2 e^{2s\phi(T/2,x')}dx' \le C_2 \int_\Omega |P_0 g|^2 e^{2s\phi(T/2,x')}dx'$$

for all $s > s_0$ *and* $g \in H^1(\Omega)$ *satisfying*

$$g = 0 \quad on \left\{x' \in \partial\Omega;\ \sum_{j=1}^{3} p_{0,j}(x')n_j(x') \ge 0\right\}.$$

LEMMA 20.3

We assume that

$$\sum_{j=1}^{3} p_{0,j}(x')\partial_{x_j}\phi(T/2, x') \ne 0, \quad x' \in \overline{\Omega}. \tag{20.31}$$

Then the conclusion of Lemma 20.2 is true for all $s > s_0$ *and* $g \in H_0^1(\Omega)$.

Now we proceed to

PROOF OF THEOREM 20.1 The proof is done by modifying the argument in Imanuvilov and Yamamoto [30]. We can separate $\partial\Omega$ into two relatively open subsets Γ_1 and Γ_2 such that

$$\begin{cases} \overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega,\ n_1(x') \le 0 \text{ for } x' \in \overline{\Gamma_1},\ n_1(x') \ge 0 \text{ for } x' \in \overline{\Gamma_2}, \\ \text{and for any } x' = (x_1, x_2, x_3) \in \overline{\Omega}, \text{ there exists a unique point } \widetilde{x}' = (\widetilde{x_1}, x_2, x_3) \in \overline{\Gamma_1} \\ \text{such that the segment connecting } x' \text{ and } \widetilde{x}' \text{ is on } \overline{\Omega}. \end{cases} \tag{20.32}$$

In fact, we can choose straight lines parallel to the x_1-axis that divide Ω into parts $\Omega_1, \dots, \Omega_m$ such that

$$\Omega_j = \{x';\ \gamma_{1j}(x_2, x_3) < x_1 < \gamma_{2j}(x_2, x_3), \quad (x_2, x_3) \in \mathcal{D}_j\}$$

where $\mathcal{D}_j$ is a domain in $\mathbb{R}^2$ and γ_{1j}, γ_{2j} are continuous functions on $\overline{\mathcal{D}_j}$. We set

$$\Gamma_1 = \bigcup_{j=1}^{m} \{x';\ x_1 = \gamma_{1j}(x_2, x_3), \quad (x_2, x_3) \in \mathcal{D}_j\}$$

and $\Gamma_2 = \partial\Omega \setminus \overline{\Gamma_1}$. Then we can easily see that Condition (20.22) is satisfied.

By Eq. (20.22), for any $x' = (x_1, x_2, x_3) \in \overline{\Omega}$, we can prove that there exists a unique point $(\gamma(x_2, x_3), x_2, x_3) \in \Gamma_1$. By (20.24), $x_1 - y_1 < 0$ for any $x' \in \overline{\Omega}$ or $x_1 - y_1 > 0$ for any $x' \in \overline{\Omega}$. First let $x_1 - y_1 < 0$. We set

$$F(x_1, x_2, x_3) = \int_{\gamma(x_2,x_3)}^{x_1} f(\xi, x_2, x_3)d\xi, \quad x' \in \overline{\Omega}. \tag{20.33}$$

Then

$$\frac{\partial F}{\partial x_1}(x') = f(x'), \quad x' \in \overline{\Omega}. \tag{20.34}$$

On the other hand, if $x_1 - y_1 > 0$, then instead of Γ_1, we take $(\gamma(x_2, x_3), x_2, x_3) \in \Gamma_2$ in Eq. (20.33), and we can argue similarly to the case of $x_1 - y_1 < 0$. Therefore, we will exclusively assume that $x_1 - y_1 < 0$.

Henceforth, for simplicity, we set

$$\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta), \quad \mathbf{v} = \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)$$

and

$$\mathbf{y} = \mathbf{u} - \mathbf{v}, \quad f = \rho - \widetilde{\rho}, \quad g = \lambda - \widetilde{\lambda}, \quad h = \mu - \widetilde{\mu}.$$

Then

$$\widetilde{\rho}\partial_{x_0}^2 \mathbf{y} = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{y} + \mathbb{G}\mathbf{u} \quad \text{in } Q, \tag{20.35}$$

$$\mathbf{y}\left(\frac{T}{2}, x'\right) = \partial_{x_0}\mathbf{y}\left(\frac{T}{2}, x'\right) = 0, \quad x' \in \Omega, \tag{20.36}$$

and

$$\mathbf{y} = 0 \quad \text{on } (0, T) \times \partial\Omega. \tag{20.37}$$

Here we set

$$\begin{aligned} \mathbb{G}\mathbf{u}(x) \;=\; & -\partial_{x_1} F(x')\partial_{x_0}^2 \mathbf{u}(x) + (g+h)(x')\nabla_{x'}(\operatorname{div}\mathbf{u})(x) + h(x')\Delta\mathbf{u}(x) \\ & + (\operatorname{div}\mathbf{u})(x)\nabla_{x'} g(x') + \left\{\nabla_{x'}\mathbf{u}(x) + [\nabla_{x'}\mathbf{u}(x)]^T\right\} \nabla h(x'). \end{aligned} \tag{20.38}$$

By (20.28), we have the inequality $\frac{\theta T^2}{4} > d^2$. Therefore, by (20.20) and definition (20.29) of the function ϕ, we have

$$\phi(T/2, x') \geq d_1, \quad \phi(0, x') = \phi(T, x') < d_1, \quad x' \in \overline{\Omega}$$

with

$$d_1 = \exp(\tau \inf_{x' \in \Omega} |x' - y'|^2). \tag{20.39}$$

Thus, for a given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that

$$\phi(x) \geq d_1 - \varepsilon, \quad x \in \left[\frac{T}{2} - \delta, \frac{T}{2} + \delta\right] \times \overline{\Omega} \tag{20.40}$$

and

$$\phi(x) \leq d_1 - 2\varepsilon, \quad x \in ([0, 2\delta] \cup [T - 2\delta, T]) \times \overline{\Omega}. \tag{20.41}$$

To apply Lemma 20.1, it is necessary to introduce a cutoff function χ satisfying $0 \leq \chi \leq 1$, $\chi \in C^\infty(\mathbb{R})$ and

$$\chi = \begin{cases} 0 & \text{on } [0, \delta] \cup [T - \delta, T], \\ 1 & \text{on } [2\delta, T - 2\delta]. \end{cases} \tag{20.42}$$

In the sequel, $C_j > 0$, denote generic constants depending on $s_0, \tau, M_0, M_1, \theta_0, \theta_1, \eta, \Omega, T, y', \omega, \chi$, and $\mathbf{p}$, $\mathbf{q}$, ε, δ but independent of $s > s_0$. Setting $\mathbf{z}_1 = \chi\partial_{x_0}^2\mathbf{y}$, $\mathbf{z}_2 = \chi\partial_{x_0}^3\mathbf{y}$, and $\mathbf{z}_3 = \chi\partial_{x_0}^4\mathbf{y}$, we have

$$\begin{cases} \widetilde{\rho}\partial_{x_0}^2 \mathbf{z}_1 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_1 + \chi\mathbb{G}\left(\partial_{x_0}^2\mathbf{u}\right) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^3\mathbf{y} + \widetilde{\rho}\left(\partial_{x_0}^2\chi\right)\partial_{x_0}^2\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2 \mathbf{z}_2 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_2 + \chi\mathbb{G}\left(\partial_{x_0}^3\mathbf{u}\right) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^4\mathbf{y} + \widetilde{\rho}\left(\partial_{x_0}^2\chi\right)\partial_{x_0}^3\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2 \mathbf{z}_3 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_3 + \chi\mathbb{G}\left(\partial_{x_0}^4\mathbf{u}\right) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^5\mathbf{y} + \widetilde{\rho}\left(\partial_{x_0}^2\chi\right)\partial_{x_0}^4\mathbf{y} \quad \text{in } Q. \end{cases} \tag{20.43}$$

Henceforth we set

$$\mathcal{E} = \int_{Q_\omega} \left(\left|\partial_{x_0}^2 \mathbf{y}\right|^2 + \left|\partial_{x_0}^3 \mathbf{y}\right|^2 + \left|\partial_{x_0}^4 \mathbf{y}\right|^2 \right) e^{2s\phi} dx.$$

Noting that $\mathbf{u} \in W^{7,\infty}(Q)$, in view of (20.42) and Lemma 20.1, we can apply Theorem 20.3 to (20.43), so that

$$\begin{aligned}
\sum_{j=2}^{4} \int_Q \left|\partial_{x_0}^j \mathbf{y}\right|^2 \chi^2 e^{2s\phi} dx &\le C_3 \left(\|F e^{s\phi}\|^2_{L^2(Q)} + \|g e^{s\phi}\|^2_{L^2(Q)} + \|h e^{s\phi}\|^2_{L^2(Q)} \right) \\
&+ C_3 \sum_{j=3}^{5} \left\| (\partial_{x_0}\chi) \left(\partial_{x_0}^j \mathbf{y}\right) e^{s\phi} \right\|^2_{L^2[0,T;\mathbf{H}^{-1}(\Omega)]} \\
&+ C_3 \sum_{j=2}^{4} \left\| \left(\partial_{x_0}^2 \chi\right) \left(\partial_{x_0}^j \mathbf{y}\right) e^{s\phi} \right\|^2_{L^2[0,T;\mathbf{H}^{-1}(\Omega)]} + C_3 \mathcal{E} \\
&\le C_4 \left(\|F e^{s\phi}\|^2_{L^2(Q)} + \|g e^{s\phi}\|^2_{L^2(Q)} + \|h e^{s\phi}\|^2_{L^2(Q)} \right) + C_4 e^{2s(d_1 - 2\varepsilon)} + C_4 \mathcal{E} \qquad (20.44)
\end{aligned}$$

for all large $s > 0$. On the other hand,

$$\begin{aligned}
&\int_\Omega \left| \left(\partial_{x_0}^2 \mathbf{y}\right)(T/2, x') \right|^2 e^{2s\phi(T/2,x')} dx' \\
&= \int_0^{T/2} \frac{\partial}{\partial x_0} \left(\int_\Omega \left| \left(\partial_{x_0}^2 \mathbf{y}\right)(x_0, x') \right|^2 \chi^2(x_0) e^{2s\phi} dx' \right) dx_0 \\
&= \int_0^{T/2} \int_\Omega 2 \left(\partial_{x_0}^3 \mathbf{y}, \partial_{x_0}^2 \mathbf{y} \right) \chi^2 e^{2s\phi} dx \\
&+ 2s \int_0^{T/2} \int_\Omega \left|\partial_{x_0}^2 \mathbf{y}\right|^2 \chi^2 (\partial_{x_0}\phi) e^{2s\phi} dx + \int_0^{T/2} \int_\Omega \left|\partial_{x_0}^2 \mathbf{y}\right|^2 \left[\partial_{x_0}(\chi^2)\right] e^{2s\phi} dx \\
&\le C_5 \int_Q s\chi^2 \left(\left|\partial_{x_0}^3 \mathbf{y}\right|^2 + \left|\partial_{x_0}^2 \mathbf{y}\right|^2 \right) e^{2s\phi} dx + C_5 e^{2s(d_1 - 2\varepsilon)}.
\end{aligned}$$

Therefore, (20.44) yields

$$\begin{aligned}
&\int_\Omega \left| \left(\partial_{x_0}^2 \mathbf{y}\right)(T/2, x') \right|^2 e^{2s\phi(T/2,x')} dx' \\
&\quad \le C_6 s \int_Q (|F|^2 + |g|^2 + |h|^2) e^{2s\phi} dx + C_6 s e^{2s(d_1 - 2\varepsilon)} + C_6 s \mathcal{E}
\end{aligned}$$

for all large $s > 0$. Similarly we can estimate $\int_\Omega \left| \left(\partial_{x_0}^3 \mathbf{y}\right)(T/2, x') \right|^2 e^{2s\phi(T/2,x')} dx'$ to obtain

$$\begin{aligned}
&\int_\Omega \left(\left| \left(\partial_{x_0}^2 \mathbf{y}\right)(T/2, x') \right|^2 + \left| \left(\partial_{x_0}^3 \mathbf{y}\right)(T/2, x') \right|^2 \right) e^{2s\phi(T/2,x')} dx' \\
&\quad \le C_6 s \int_Q (|F|^2 + |g|^2 + |h|^2) e^{2s\phi} dx + C_6 s e^{2s(d_1 - 2\varepsilon)} + C_6 s \mathcal{E} \qquad (20.45)
\end{aligned}$$

for all large $s > 0$.

Now we will consider first-order partial differential equations satisfied by h, g, and F. That is, by (20.35), and (20.36) and $\mathbf{u}, \mathbf{v} \in W^{7,\infty}(Q)$, we have

$$\widetilde{\rho} \partial_{x_0}^2 \mathbf{y} \left(\frac{T}{2}, x' \right) = \mathbb{G} \mathbf{u} \left(\frac{T}{2}, x' \right), \quad \widetilde{\rho} \partial_{x_0}^3 \mathbf{y} \left(\frac{T}{2}, x' \right) = \mathbb{G} \partial_{x_0} \mathbf{u} \left(\frac{T}{2}, x' \right). \qquad (20.46)$$

For simplicity, we set

$$\begin{cases} \mathbf{a} = \begin{pmatrix} -\frac{1}{\rho} L_{\lambda,\mu}\mathbf{p} \\ -\frac{1}{\rho} L_{\lambda,\mu}\mathbf{q} \end{pmatrix}, \\ \mathbf{b}_1 = \begin{pmatrix} \operatorname{div}\mathbf{p} \\ 0 \\ 0 \\ \operatorname{div}\mathbf{q} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ \operatorname{div}\mathbf{p} \\ 0 \\ 0 \\ \operatorname{div}\mathbf{q} \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ \operatorname{div}\mathbf{p} \\ 0 \\ 0 \\ \operatorname{div}\mathbf{q} \end{pmatrix}, \\ (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) = \begin{pmatrix} \nabla\mathbf{p} + (\nabla\mathbf{p})^T \\ \nabla\mathbf{q} + (\nabla\mathbf{q})^T \end{pmatrix}, \\ \mathbf{G} = \begin{pmatrix} \widetilde{\rho}\partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2}, x'\right) - (g+h)\nabla_{x'}(\operatorname{div}\mathbf{p}) - h\Delta\mathbf{p} \\ \widetilde{\rho}\partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2}, x'\right) - (g+h)\nabla_{x'}(\operatorname{div}\mathbf{q}) - h\Delta\mathbf{q} \end{pmatrix} \quad \text{on } \overline{\Omega}. \end{cases} \tag{20.47}$$

Then we can rewrite (20.46) as

$$\mathbf{a}\partial_{x_1} F + \mathbf{b}_1\partial_{x_1} g + \mathbf{b}_2\partial_{x_2} g + \mathbf{b}_3\partial_{x_3} g = \mathbf{G} - \mathbf{d}_1\partial_{x_1} h - \mathbf{d}_2\partial_{x_2} h - \mathbf{d}_3\partial_{x_3} h.$$

Therefore, for $j_1 \in \{1, 2, 3, 4, 5, 6\}$, we have

$$\begin{aligned} &\{\mathbf{a}\}_{j_1}\partial_{x_1} F + \{\mathbf{b}_1\}_{j_1}\partial_{x_1} g + \{\mathbf{b}_2\}_{j_1}\partial_{x_2} g + \{\mathbf{b}_3\}_{j_1}\partial_{x_3} g \\ &\quad = \{\mathbf{G}\}_{j_1} - \{\mathbf{d}_1\}_{j_1}\partial_{x_1} h - \{\mathbf{d}_2\}_{j_1}\partial_{x_2} h - \{\mathbf{d}_3\}_{j_1}\partial_{x_3} h, \quad \text{on } \overline{\Omega}. \end{aligned} \tag{20.48}$$

Equation (20.48) is a system of five linear equations with respect to four unknowns $\partial_{x_1} F$, $\partial_{x_1} g$, $\partial_{x_2} g$, $\partial_{x_3} g$, and so for the existence of solutions, we need the consistency of the coefficients, that is,

$$\det\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G} - \mathbf{d}_1\partial_{x_1} h - \mathbf{d}_2\partial_{x_2} h - \mathbf{d}_3\partial_{x_3} h\}_{j_1} = 0 \quad \text{on } \overline{\Omega},$$

that is,

$$\sum_{k=1}^{3} \det\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_k\}_{j_1}\partial_{x_k} h = \det\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G}\}_{j_1} \quad \text{on } \overline{\Omega} \tag{20.49}$$

by the linearity of the determinant. In terms of Condition (20.25) and $h \equiv \mu - \widetilde{\mu} = 0$ on $\partial\Omega$, considering (20.49) as a first-order partial differential operator in h, we can apply Lemma 20.3, so that

$$\begin{aligned} s^2 \int_\Omega |h|^2 e^{2s\phi(T/2,x')} dx' &\le C_7 \left\| \det\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G}\}_{j_1} e^{s\phi(T/2,\cdot)} \right\|_{L^2(\Omega)}^2 \\ &\le C_8 \int_\Omega \left[\left| \partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2}, x'\right) \right|^2 + \left| \partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2}, x'\right) \right|^2 \right] e^{2s\phi(T/2,x')} dx' \\ &\quad + C_8 \int_\Omega (|g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' \end{aligned} \tag{20.50}$$

in view of (20.47). Similarly to (20.48), we rewrite (20.46) and, by (20.26) we can similarly deduce

$$s^2 \int_\Omega |g|^2 e^{2s\phi(T/2,x')} dx' \le C_9 \int_\Omega \left[\left| \partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left| \partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 \right] e^{2s\phi(T/2,x')} dx'$$
$$+ C_9 \int_\Omega (|g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' \tag{20.51}$$

for all large $s > 0$. By (20.50) and (20.51), for sufficiently large $s > 0$, we have

$$s^2 \int_\Omega (|g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx'$$
$$\le C_{10} \int_\Omega \left[\left| \partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left| \partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 \right] e^{2s\phi(T/2,x')} dx'. \tag{20.52}$$

Finally, replacing j_1 by $j_3 \in \{1, 2, 3, 4, 5, 6\}$, we consider (20.48) as a system of five linear equations with respect to four unknowns $\partial_{x_2} g$, $\partial_{x_3} g$, $\partial_{x_2} h$, $\partial_{x_3} h$. By the condition for the existence of solutions, we have

$$\det\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{G} - \mathbf{a}\partial_{x_1} F - \mathbf{b}_1 \partial_{x_1} g - \mathbf{d}_1 \partial_{x_1} h\}_{j_3} = 0$$

on $\overline{\Omega}$. Therefore

$$-\partial_{x_1}(e_1 F + e_2 g + e_3 h) + (\partial_{x_1} e_1) F$$
$$= -(\partial_{x_1} e_2) g - (\partial_{x_1} e_3) h - \det\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{G}\}_{j_3}$$

on $\overline{\Omega}$. Here we set

$$e_1 = \det\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{a}\}_{j_3},$$

$$e_2 = \det\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{b}_1\}_{j_3},$$

and

$$e_3 = \det\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_1\}_{j_3}.$$

In Lemma 20.2, we consider the case of $p_{0,1} = -1$ and $p_{0,2} = p_{0,3} = 0$. By (20.32) and (20.33) and $g = h = 0$ on $\partial\Omega$, we see that if $-n_1(x') = \sum_{j=1}^3 p_{0,j}(x') n_j(x') \ge 0$, then $(F + g + h)(x') = 0$. Moreover, by $x_1 - y_1 < 0$ for $x' \in \overline{\Omega}$, condition in (20.30) is satisfied. Consequently, choosing $s > 0$ sufficiently large and using (20.52), by Lemma 20.3 and (20.27), we obtain

$$s^2 \int_\Omega |F|^2 e^{2s\phi(T/2,x')} dx'$$
$$\le C_{11} \int_\Omega \left[\left| \partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left| \partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 \right] e^{2s\phi(T/2,x')} dx' \tag{20.53}$$

for all large $s > 0$. Consequently, substituting (20.52) and (20.53) into (20.45) and using $\phi(T/2, x') \ge \phi(x_0, x')$ for $(x_0, x') \in Q$, we obtain

$$\int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx'$$
$$\le \frac{C_{12} T}{s} \int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' + \frac{C_{12}}{s} e^{2s(d_1 - 2\varepsilon)} + \frac{C_{12}}{s} \mathcal{E}$$

for all large $s > 0$. Taking $s > 0$ sufficiently large and noting $e^{2s\phi(T/2,x')} \geq e^{2sd_1}$ for $x' \in \overline{\Omega}$, we obtain

$$\int_\Omega (|F|^2 + |g|^2 + |h|^2)\,dx' \leq C_{13}e^{-4s\varepsilon} + C_{13}e^{2sC_{14}}\mathcal{E} \tag{20.54}$$

for all large $s > s_0$: a constant that is dependent on τ, but independent of s. Next we take in (20.54) instead of the constant C_{13} the constant $C_{13}e^{2s_0C_{14}}$. Now this inequality holds true for all $s > 0$.

Now we choose $s > 0$ such that $e^{2sC_{14}}\mathcal{E} = e^{-4s\varepsilon}$, that is,

$$s = -\frac{1}{4\varepsilon + 2C_{14}} \ln \mathcal{E}.$$

Here we may assume that $\mathcal{E} < 1$ and so $s > 0$. Then it follows from (20.54) that

$$\int_\Omega (|F|^2 + |g|^2 + |h|^2)\,dx' \leq 2C\mathcal{E}^{\frac{4\varepsilon}{4\varepsilon+2C}}.$$

By definition (20.33) of F, we have

$$\int_\Omega frdx' = \int_\Omega (\partial_{x_1}F)rdx' = \int_\Omega F(\partial_{x_1}r)\,dx'$$

for all $r \in H_0^1(\Omega)$ by integration by parts. Hence we can directly verify that $\|f\|_{H^{-1}(\Omega)} \leq C\|F\|_{L^2(\Omega)}$, so that the proof of Theorem 20.4 is complete.

20.4 Proof of Theorem 20.1

Without loss of generality, we may assume that $\rho \equiv 1$. Otherwise we introduce new coefficients $\mu_1 = \mu/\rho, \lambda_1 = \lambda/\rho$ to argue similarly. We can directly verify that the functions rot**u** and div**u** satisfy the equations

$$\partial_{x_0}^2 \mathrm{rot}\mathbf{u} - \mu\Delta\mathrm{rot}\mathbf{u} = m_1, \quad \partial_{x_0}^2 \mathrm{div}\mathbf{u} - (\lambda + 2\mu)\Delta\mathrm{div}\mathbf{u} = m_2 \qquad \text{in } Q, \tag{20.55}$$

where

$$m_1 = K_1\mathrm{rot}\mathbf{u} + K_2\mathrm{div}\mathbf{u} + \mathcal{K}_1\mathbf{u} + \mathrm{rot}\,\mathbf{f}, \quad m_2 = K_3\mathrm{rot}\mathbf{u} + K_4\mathrm{div}\mathbf{u} + \mathcal{K}_2\mathbf{u} + \mathrm{div}\,\mathbf{f},$$

and $K_j, \mathcal{K}_k$ are first-order differential operators with L^∞ coefficients. Thanks to Condition 20.1 on the weight function ψ, there exists $\widehat{\tau}$ such that for all $\tau > \widehat{\tau}$, we have (see Reference 52):

$$s\|(\mathrm{rot}\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\|(\mathrm{div}\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \leq C_1\left(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}[(0,T)\times\partial\Omega]}\right.$$
$$\left. + s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{\mathbf{L}^2[(0,T)\times\partial\Omega]} + \|\mathbf{u}\|^2_{\mathcal{B}(Q_\omega)}\right), \quad \forall s \geq s_0(\tau), \tag{20.56}$$

where the constant C_1 is independent of s. To estimate the $H^1(Q)$-norm of the function **u**, we rewrite (20.1) in the form

$$\rho\partial_{x_0}^2\mathbf{u} - \mu\Delta\mathbf{u} = \mathbf{F}, \quad \mathbf{u}|_{\partial\Omega} = 0,$$

where

$$\mathbf{F} = \mathbf{f} + [\lambda(x') + \mu(x')]\nabla_{x'}\mathrm{div}\,\mathbf{u}(x) + [\mathrm{div}\,\mathbf{u}(x)]\nabla_{x'}\lambda(x') + [\nabla_{x'}\mathbf{u} + (\nabla_{x'}\mathbf{u})^T]\nabla_{x'}\mu(x').$$

Thanks to Condition 20.1 we can apply the Carleman estimate in Reference 20 to this hyperbolic equation

$$\begin{aligned} s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} &\le C_2\left[\|\mathbf{F}e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q_\omega)}\right] \\ &\le C_2\big[\|(\mathrm{div}\,\mathbf{u})e^{s\phi}\|^2_{L^2(Q)} + \|(\nabla_{x'}\mathrm{div}\,\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + \|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \\ &\quad + \|\mathbf{f}e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q_\omega)}\big]. \end{aligned}$$

This estimate and inequality (20.56) imply

$$\begin{aligned} s^2\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\|(\mathrm{rot}\,\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\|(\mathrm{div}\,\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \le C_2\Bigg\{\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \\ + s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}[(0,T)\times\partial\Omega]} + s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{\mathbf{L}^2[(0,T)\times\partial\Omega]} + \|\mathbf{u}\|^2_{\mathcal{B}(Q_\omega)}\Bigg\}, \quad \forall s \ge s_0. \qquad (20.57) \end{aligned}$$

Next we estimate the second derivatives of the function $\mathbf{u}$.

Denote $\mathrm{rot}\,\mathbf{u} = \mathbf{y}$. Using a well-known formula: $\mathrm{rot}\,\mathrm{rot} = -\Delta_{x'} + \nabla_{x'}\mathrm{div}$, we obtain

$$-\Delta_{x'}\mathbf{u} = \mathrm{rot}\,\mathbf{y} - \nabla_{x'}\mathrm{div}\,\mathbf{u} \quad \text{in } \Omega, \quad \mathbf{u}|_{\partial\Omega} = 0.$$

Using the standard *a priori* estimate for the above Dirichlet problem for the Poisson equation, we have:

$$\sum_{j,k=1}^{3} \|(\partial_{x_j}\partial_{x_k}\mathbf{u})e^{s\phi}\|_{\mathbf{L}^2(Q)} \le C_2\left[s\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)} + \|(\mathrm{div}\,\mathbf{u})e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)} + \|(\mathrm{rot}\mathbf{u})e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)}\right].$$

By (20.57) one can estimate the left-hand side of this inequality by the right-hand side of (20.56).

Next, using this estimate and (20.1), we obtain the estimate for the norm $\|(\partial^2_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ via the right-hand side of (20.56). Finally, we obtain the estimate for $\|(\partial_{x_0}\partial_{x_j}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ and $s^2\|(\partial_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ by an interpolation argument. Therefore, combining these estimates with (20.56) and (20.57), we have

$$\begin{aligned} \|\mathbf{u}\|^2_{Y(\phi,Q)} \le C_3\Bigg\{\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}[(0,T)\times\partial\Omega]} + s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{\mathbf{L}^2[(0,T)\times\partial\Omega]} \\ + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\Bigg\}, \quad \forall s \ge s_0(\tau), \qquad (20.58) \end{aligned}$$

where the constant C_3 is independent of s.

Now we need to estimate the boundary integrals of the right-hand side of (20.58). In order to do that, it is convenient to use another weight function φ such that $\varphi|_{(0,T)\times\partial\Omega} = \phi|_{(0,T)\times\partial\Omega}$ and $\varphi(x) < \phi(x)$ for all x in a small neighborhood of $(0,T)\times\partial\Omega$. We construct such a function φ locally near the boundary $\partial\Omega$:

$$\varphi(x) = e^{\tau\widetilde{\psi}(x)}, \quad \widetilde{\psi}(x) = \psi(x) - \frac{1}{N^2}\ell_1(x') + N\ell_1^2(x'),$$

where $N > 0$ is a large positive parameter, and $\ell_1 \in C^3(\overline{\Omega})$ is a function such that

$$\ell_1(x') > 0, \quad \forall x' \in \Omega, \quad \ell_1|_{\partial\Omega} = 0, \ \nabla_{x'}\ell_1|_{\partial\Omega} \ne 0.$$

Denote $\Omega_{\frac{1}{N^2}} = \{x' \in \Omega;\ \mathrm{dist}\,(x', \partial\Omega) \le \frac{1}{N^2}\}$. Obviously there exists $N_0 > 0$ such that

$$\varphi(x) < \phi(x), \quad \forall x \in [0,T] \times \Omega_{\frac{1}{N^2}}, \quad N \in (N_0, \infty).$$

The following lemma plays a key role in our proof.

LEMMA 20.4
Under the conditions of Theorem 20.1, there exists $\widehat{\tau} > 0$ such that for all $\tau > \widehat{\tau}$, there exists $s_0(\tau) > 0$ such that

$$\|\mathbf{u}\|^2_{Y(\varphi,Q)} + \sqrt{N} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|} \left\| \left(\partial_x^\alpha \mathbf{u}\right) e^{s\varphi} \right\|^2_{\mathbf{L}^2(Q)} \le C_4 \Big[\|\mathbf{f} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{u}\|^2_{\mathcal{B}(\varphi, Q_\omega)} \Big], \quad \forall s \ge s_0(\tau, N),\ \mathit{supp}\,\mathbf{u} \subset [0,T] \times \overline{\Omega_{\frac{1}{N^2}}}, \tag{20.59}$$

where the constant C_4 is independent of s and N.

We will postpone the proof of Lemma 20.4, and by means of this lemma we continue the proof of Theorem 20.1. Let us fix the parameter N such that (20.59) holds true. We take $\widetilde{\delta} \in (0, \frac{1}{N^2})$ sufficiently small such that

$$\phi(x) > \varphi(x), \quad \forall x \in [0,T] \times \overline{\Omega_{\widetilde{\delta}} \backslash \Omega_{\widetilde{\delta}/2}}. \tag{20.60}$$

We consider a cutoff function $\widetilde{\theta} \in C^3(\overline{\Omega}_{\widetilde{\delta}})$ such that $\widetilde{\theta}|_{\Omega_{\widetilde{\delta}/2}} = 1$ and $\widetilde{\theta}|_{\Omega_{\widetilde{\delta}} \backslash \Omega_{3\widetilde{\delta}/4}} = 0$. The function $\widetilde{\theta}\mathbf{u}$ satisfies the equation

$$\begin{gathered} P(\widetilde{\theta}\mathbf{u}) = \widetilde{\theta}\mathbf{f} + [P, \widetilde{\theta}]\mathbf{u}, \quad \widetilde{\theta}\mathbf{u}|_{(0,T)\times\partial\Omega} = 0, \\ (\widetilde{\theta}\mathbf{u})(0,\cdot) = (\widetilde{\theta}\mathbf{u})_{x_0}(0,\cdot) = (\widetilde{\theta}\mathbf{u})(T,\cdot) = (\widetilde{\theta}\mathbf{u})_{x_0}(T,\cdot) = 0. \end{gathered} \tag{20.61}$$

Applying the Carleman estimate of (20.59) to (20.61), and using the fact that $(\varphi - \phi)|_{(0,T)\times\partial\Omega} = 0$, we obtain

$$s \left\| \frac{\partial \mathbf{u}}{\partial \vec{n}} e^{s\phi} \right\|^2_{\mathbf{H}^{1,s}[(0,T)\times\partial\Omega]} + s \left\| \frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2} e^{s\phi} \right\|^2_{\mathbf{L}^2[(0,T)\times\partial\Omega]} \le C_5 \big\{ \|\mathbf{f} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)} + \|[P, \widetilde{\theta}]\mathbf{u} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi, Q_\omega)} \big\}, \quad \forall s \ge s_0(\tau). \tag{20.62}$$

Because the supports of the coefficients of the commutator $[P, \widetilde{\theta}]$ are in $[0,T] \times \overline{\Omega_{\widetilde{\delta}} \backslash \Omega_{\widetilde{\delta}/2}}$ by (20.60), we have

$$\|[P, \widetilde{\theta}]\mathbf{u} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)} \le C_6 \left[\sum_{|\alpha|=0}^{2} s^{3-2|\alpha|} \left\| \left(\partial_x^\alpha \mathbf{u}\right) e^{s\phi} \right\|^2_{\mathbf{L}^2(Q)} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi, Q_\omega)} \right]. \tag{20.63}$$

Combining (20.62) and (20.63), we obtain

$$\begin{aligned} & s \left\| \frac{\partial \mathbf{u}}{\partial \vec{n}} e^{s\phi} \right\|^2_{\mathbf{H}^{1,s}[(0,T)\times\partial\Omega]} + s \left\| \frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2} e^{s\phi} \right\|^2_{\mathbf{L}^2[(0,T)\times\partial\Omega]} \\ & \le C_7 \left[\|\mathbf{f} e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + \sum_{|\alpha|=0}^{2} s^{3-2|\alpha|} \left\| \left(\partial_x^\alpha \mathbf{u}\right) e^{s\phi} \right\|^2_{\mathbf{L}^2(Q)} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi, Q_\omega)} \right], \quad \forall s \ge s_0(\tau). \end{aligned} \tag{20.64}$$

Finally we will estimate the surface integrals at the right-hand side of (20.58) by the right-hand side of (20.64). In the new inequality, the term

$$\sum_{|\alpha|=0}^{2} s^{3-2|\alpha|} \left\| \left(\partial_x^\alpha \mathbf{u}\right) e^{s\phi} \right\|^2_{\mathbf{L}^2(Q)},$$

which appears at the right-hand side, can be absorbed by $\|\mathbf{u}\|^2_{Y(\phi,Q)}$. Thus, the proof of Theorem 20.1 is complete.

The rest of the paper is devoted to the proof of the Lemma 20.4.

PROOF OF LEMMA 20.4 First we note that, thanks to the large parameter N, it suffices to prove (20.59) only locally by assuming

$$\operatorname{supp} \mathbf{u} \subset B_\delta \cap \left([0,T] \times \overline{\Omega}_{\frac{1}{N^2}}\right),$$

where B_δ is the ball of the radius $\delta > 0$ centered at some point $y^* \in [0,T] \times \partial\Omega$. In the case of $B_\delta \cap [(0,T) \times \partial\Omega] = \emptyset$, we can prove the lemma in a usual way for a function with compact support (see, e.g., Reference 17). Without loss of generality, we may assume that $y^* = (y_0^*, 0, 0, 0)$. Moreover, the parameter $\delta > 0$ can be chosen arbitrarily small. Assume that near $(0,0,0)$, the boundary $\partial\Omega$ is locally given by the equation $x_3 - \ell(x_1, x_2) = 0$. Furthermore, because the function $\widetilde{\mathbf{u}} = \mathcal{O}\mathbf{u}(x_0, \mathcal{O}^{-1}x')$ satisfies system of (20.4) and (20.5) with $\widetilde{\mathbf{f}} = \mathcal{O}\mathbf{f}(x_0, \mathcal{O}^{-1}x')$ for any orthogonal matrix $\mathcal{O}$, we may assume that

$$\left(\frac{\partial \ell}{\partial x_1}(0,0), \frac{\partial \ell}{\partial x_2}(0,0)\right) = 0. \tag{20.65}$$

Next we make the change of variables $y_1 = x_1$, $y_2 = x_2$, and $y_3 = x_3 - \ell(x_1, x_2)$. We set $y_0 = x_0$, $y = (y_0, y_1, y_2, y_3)$, and $y' = (y_1, y_2, y_3)$. By $A(y, D)$ denote the Laplace operator after the change of variables. One can check that the principal symbol of this operator is equal to $a(y,\xi) = -\xi_1^2 - \xi_2^2 - |G|^2\xi_3^2 + 2(\nabla_{y'}\ell, \xi)\xi_3$, $|G| = \sqrt{1 + |\nabla \ell|^2}$. In the new coordinates, the Lamé system has the form

$$\begin{aligned} \mathbb{P}(y,D)\mathbf{u} = \; & D_{y_0}^2 \mathbf{u} - \mu A(y,D)\mathbf{u} \\ & - (\lambda+\mu)\left(\nabla_{y'} - \nabla_{y'}\ell \frac{\partial}{\partial y_3}\right)\left[\operatorname{div}\mathbf{u} - \left(\frac{\partial \mathbf{u}}{\partial y_3}, \nabla_{y'}\ell\right)\right] \\ & + \widetilde{K}_1 \mathbf{u} = -\mathbf{f}, \end{aligned} \tag{20.66}$$

where we use the same notations $\mathbf{u}$, $\mathbf{f}$ after the change of variables and $\widetilde{K}_1$ is the partial differential operator of the first order. Denote by (z_1, z_2, z_3) and z_4 the functions rot $\mathbf{u}$ and div $\mathbf{u}$ in the y coordinate. These functions satisfy the equations

$$P_\mu(y,D)z_j = D_0^2 z_j - \mu A(y,D) z_j = m_j \quad j \in \{1,2,3\}, \tag{20.67}$$

$$P_{\lambda+2\mu}(y,D)z_4 = D_0^2 z_4 - (\lambda+2\mu)A(y,D)z_4 = m_4. \tag{20.68}$$

Here we set $\mathbf{w} = (\mathbf{w}', w_4)$ where

$$\mathbf{w}' = (\operatorname{rot}\mathbf{u})e^{s\varphi}, \quad w_4 = (\operatorname{div}\mathbf{u})e^{s\varphi} \quad \text{in the } y\text{-coordinate},$$

$$\mathbf{w}'_\nu = \chi_\nu(s, D')\mathbf{w}' \equiv \int_{\mathbb{R}^3} \chi_\nu(s,\xi')\widehat{\mathbf{w}}'(\xi_0, \xi_1, \xi_2, y_3) e^{i(y_0\xi_0 + y_1\xi_1 + y_2\xi_2)} d\xi_0 d\xi_1 d\xi_2,$$

where $\widehat{\mathbf{w}}'$ is the Fourier transform of $\mathbf{w}'$ with respect to the variables (y_0, y_1, y_2).

We consider a finite covering of the unit sphere $S^3 \equiv \{(s, \xi_0, \xi_1, \xi_2); s^2 + \xi_0^2 + \xi_1^2 + \xi_2^2 = 1\}$. That is, $S^3 \subset \cup_{\nu=1}^{K(\delta_1)}\{(s, \xi_0, \xi_1, \xi_2) \in S^3; |\zeta - \zeta_\nu^*| < \delta_1\}$ where $\zeta_\nu^* \in S^3$, and by $\{\chi_\nu(\zeta)\}_{1\le\nu\le K(\delta_1)}$ we denote the corresponding partition of unity: $\sum_{\nu=1}^{K(\delta_1)} \chi_\nu(\zeta) = 1$ for any $\zeta \in S^3$ and supp $\chi_\nu \subset \{\zeta \in S^3; |\zeta - \zeta_\nu^*| < \delta_1\}$. Henceforth, we extend χ_ν to the set $\{\zeta; |\zeta| > 1\}$ as the homogeneous function of the order zero such that $\chi_\nu \in C^\infty(\mathbb{R}^3)$ and

$$\operatorname{supp} \chi_\nu \subset \mathcal{O}(\delta_1) \equiv \left\{\zeta;\ \left|\frac{\zeta}{|\zeta|} - \zeta_\nu^*\right| < \delta_1\right\}.$$

We set $\mathcal{G} = \mathbb{R}^3 \times [0, \frac{1}{N^2})$. Let $\gamma = (y^*, \zeta^*) \equiv (y^*, s^*, \xi_0^*, \xi_1^*, \xi_2^*) \in \partial\mathcal{G} \times S^3$ be an arbitrary point. To finish the proof, we need the following lemma.

LEMMA 20.5

Let $\gamma = (y^*, \zeta^*) \in \partial\mathcal{G} \times S^3$ be an arbitrary point and supp $\chi_\nu \subset \mathcal{O}(\delta_1)$. Then for all sufficiently small $\delta_1 > 0$, the following estimate holds true:

$$\sqrt{|s|}\|\mathbf{w}_\nu\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_8\left(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right). \tag{20.69}$$

Assume for the moment that Lemma 20.5 holds true. Using Carleman estimate (20.69), we have

$$\begin{aligned}
&\sqrt{|s|}\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}}{\partial y_3}, \mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\
&\le \sum_{\nu=1}^{K}\sqrt{|s|}\|\mathbf{w}_\nu\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\
&\le C_8\left(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right) \quad \forall |s| \ge s_0.
\end{aligned} \tag{20.70}$$

By Proposition 5.1 and the argument similar to that of Eqs. (5.10) and (5.11) in Reference 30, we obtain

$$\sqrt{N}\left(\int_{\mathcal{G}}\sum_{|\alpha|=0}^{2}|s|^{4-2|\alpha|}|D_{y'}^\alpha \mathbf{u}e^{|s|\varphi}|^2 dy\right)^{\frac{1}{2}} \le C_8\left(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right), \quad \forall |s| \ge s_0. \tag{20.71}$$

Directly from (20.4) we can obtain the estimates for $\left(\partial_{y_0}^2\mathbf{u}\right)e^{|s|\varphi}$ and $(\partial_{y_0}\partial_{y_1}\mathbf{u})e^{|s|\varphi}$:

$$N^{\frac{1}{4}}\left(\int_{\mathcal{G}}\sum_{|\alpha|=0}^{2}|s|^{4-2|\alpha|}\left|D^\alpha \mathbf{u}e^{|s|\varphi}\right|^2 dy\right)^{\frac{1}{2}} \le C_8\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right], \quad \forall |s| \ge s_0. \tag{20.72}$$

Because the constant C_8 is independent of N, estimate (20.72) implies

$$\begin{aligned}
&N^{\frac{1}{4}}\left(\int_{\mathcal{G}}\sum_{|\alpha|=0}^{2}|s|^{4-2|\alpha|}\left|D^\alpha \mathbf{u}e^{|s|\varphi}\right|^2 dy\right)^{\frac{1}{2}} + \sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}}{\partial y_3}, \mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\
&\le C_9\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \quad \forall |s| \ge s_0.
\end{aligned} \tag{20.73}$$

This estimate immediately implies (20.59).

Now it suffices to prove Lemma 20.5.

Before starting the proof of Lemma 20.5, we need to recall some facts from the theory of pseudodifferential operators and Carleman estimates.

We set

$$P_{\mu,s}(y,s,D) = P_\mu(y,\mathbf{D}), \quad P_{\lambda+2\mu,s}(y,s,D) = P_{\lambda+2\mu}(y,\mathbf{D}), \quad \mathbf{D} = D + i|s|\nabla\varphi.$$

Denote

$$p_\beta(y,s,\xi') = -(\xi_0 + i|s|\varphi_{y_0})^2 + \beta\big[(\xi_1 + i|s|\varphi_{y_1})^2 + (\xi_2 + i|s|\varphi_{y_2})^2 - 2\ell_{y_1}(\xi_1 + i|s|\varphi_{y_1}) \times(\xi_3 + i|s|\varphi_{y_3}) - 2\ell_{y_2}(\xi_2 + i|s|\varphi_{y_2})(\xi_3 + i|s|\varphi_{y_3}) + (\xi_3 + i|s|\varphi_{y_3})^2|G|^2\big], \tag{20.74}$$

where $\beta \in \{\mu, \lambda+2\mu\}$ and $s \neq 0$ is a parameter. The roots $\Gamma_\beta^\pm(y,s,\xi')$ of the polynomial p_β with respect to the variable ξ_3 are given by

$$\Gamma_\beta^\pm(y,s,\xi') = -i|s|\varphi_{y_3} + \alpha_\beta^\pm(y,s,\xi'), \tag{20.75}$$

where

$$\alpha_\beta^\pm(y,s,\xi') = \frac{(\xi_1 + i|s|\varphi_{y_1})\ell_{y_1} + (\xi_2 + i|s|\varphi_{y_2})\ell_{y_2}}{|G|^2} \pm \sqrt{r_\beta(y,s,\xi')}, \tag{20.76}$$

$$r_\beta(y,\zeta) = \frac{\big\{(\xi_0 + i|s|\varphi_{y_0})^2 - \beta\left[(\xi_1 + i|s|\varphi_{y_1})^2 + (\xi_2 + i|s|\varphi_{y_2})^2\right]\big\}|G|^2 + \beta(\xi + i|s|\nabla\varphi, \nabla\ell)^2}{\beta|G|^4}. \tag{20.77}$$

In some situations we can factorize the operator $P_{\beta,s}$ as a product of two first-order pseudodifferential operators.

PROPOSITION 20.1
Let $\beta \in \{\mu, \lambda+2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta}|\zeta|^2 > 0$ for all $(y,\zeta) \in (B_\delta \cap \mathcal{G}) \times \mathcal{O}(2\delta_1)$. Then we can factorize the operator $P_{\beta,s}$ into the product of two first-order pseudodifferential operators:

$$P_{\beta,s}\chi_\nu(s,D')V = \beta|G|^2[D_{y_3} - \Gamma_\beta^-(y,s,D')][D_{y_3} - \Gamma_\beta^+(y,s,D')]\chi_\nu(s,D')V + T_\beta V, \tag{20.78}$$

where supp $V \subset B_\delta \cap \mathcal{G}$ and T_β is a continuous operator:

$$T_\beta : L^2[0,1; H^{1,s}(\mathbb{R}^3)] \to L^2[0,1; L^2(\mathbb{R}^3)].$$

Let us consider the equation

$$[D_{y_3} - \Gamma_\beta^-(y,s,D')]\chi_\nu(s,D')V = q, \quad V|_{y_3=\frac{1}{N^2}} = 0, \quad \text{supp } V \subset B_\delta \cap \mathcal{G}.$$

For solutions of this problem, similar to Proposition 5.4 in Reference 30, we can prove the following.

PROPOSITION 20.2
Let $\beta \in \{\mu, \lambda+2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta}|\zeta|^2 > 0$ for all $(y,\zeta) \in B_\delta \times \mathcal{O}(2\delta_1)$. Then there exists a constant $C_{10} > 0$ independent of N such that

$$\big\|\sqrt{|s|}\chi_\nu(s,D')V|_{y_3=0}\|_{L^2(\mathbb{R}^3)} \le C_{10}\big\|q\|_{L^2(\mathcal{G})}. \tag{20.79}$$

Let $\widetilde{w}(y)$ be a function that satisfies

$$P_{\beta,s}(y,s,D)\widetilde{w}=\widetilde{q}\quad\text{in}\quad\mathcal{G},\quad \left.\frac{\partial\widetilde{w}}{\partial y_3}\right|_{y_3=1/N^2}=\widetilde{w}|_{y_3=1/N^2}=0,\quad \operatorname{supp}\widetilde{w}\subset B_\delta\cap\mathbb{R}^3\times\left[0,\frac{1}{N^2}\right).$$

Let $P^*_{\beta,s}$ be the formally adjoint operator to $P_{\beta,s}$, where $\beta\in\{\mu,\lambda+2\mu\}$. Set $L_{+,\beta}=\frac{P_{\beta,s}+P^*_{\beta,s}}{2}$ and $L_{-,\beta}=\frac{P_{\beta,s}-P^*_{\beta,s}}{2}$. Obviously $L_{+,\beta}\widetilde{w}+L_{-,\beta}\widetilde{w}=\widetilde{q}$. For almost all $s\in\mathbb{R}^1$, the following equality holds true:

$$\Xi_\beta+\|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\operatorname{Re}\int_{\mathcal{G}}([L_{+,\beta},L_{-,\beta}]\widetilde{w},\overline{\widetilde{w}})dy=\|\widetilde{q}\|^2_{L^2(\mathcal{G})},\tag{20.80}$$

where

$$\begin{aligned}\Xi_\beta\;=&\;\int_{\partial\mathcal{G}}\widetilde{p}_\beta(y,\nabla\varphi,-\vec{e}_4)(|s|\widetilde{p}_\beta(y,\nabla\widetilde{w},\overline{\nabla\widetilde{w}})-|s|^3p_\beta(y,\nabla\varphi)|\widetilde{w}|^2)dy_0dy_1dy_2\\ &+\operatorname{Re}\int_{\partial\mathcal{G}}\widetilde{p}_\beta(y,\nabla\widetilde{w},-\vec{e}_4)\overline{L_{-,\beta}\widetilde{w}}dy_0dy_1dy_2,\end{aligned}\tag{20.81}$$

$\vec{e}_4=(0,0,0,1)$ and

$$\widetilde{p}_\beta(y,\xi,\widetilde{\xi})=\xi_0\widetilde{\xi}_0-\beta(\xi_1\widetilde{\xi}_1+\xi_2\widetilde{\xi}_2-\ell_{y_1}(\xi_1\widetilde{\xi}_3+\xi_3\widetilde{\xi}_1)-\ell_{y_2}(\xi_2\widetilde{\xi}_3+\xi_3\widetilde{\xi}_2)+|G|^2\xi_3\widetilde{\xi}_3).$$

We note that $\phi_{y_k}|_{\partial\mathcal{G}}=\varphi_{y_k}|_{\partial\mathcal{G}}$ for $k\in\{0,1,2\}$. Therefore, on $\partial\mathcal{G}$, the function $\nabla_{y'}\varphi$ is independent of N and $|\nabla\phi(y')-\nabla\varphi(y')|\le C/N^2$ with the constant C independent of N. It is convenient for us to rewrite (20.81) in the form

$$\begin{aligned}\Xi_\beta\;=&\;\Xi^{(1)}_\beta+\Xi^{(2)}_\beta,\\ \Xi^{(1)}_\beta\;=&\;\operatorname{Re}\int_{y_3=0}2|s|\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_3}\overline{\left[\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_1}\varphi_{y_1}(y^*)+\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_2}\varphi_{y_2}(y^*)\right.}\\ &\overline{\left.+\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_3}\varphi_{y_3}(y^*)-\frac{\partial\widetilde{w}}{\partial y_0}\varphi_{y_0}(y^*)\right]}dy_0dy_1dy_2\\ &+\int_{y_3=0}|s|\beta(y^*)\varphi_{y_3}(y^*)\left\{\left|\frac{\partial\widetilde{w}}{\partial y_0}\right|^2-\beta(y^*)\left(\left|\frac{\partial\widetilde{w}}{\partial y_1}\right|^2+\left|\frac{\partial\widetilde{w}}{\partial y_2}\right|^2+\left|\frac{\partial\widetilde{w}}{\partial y_3}\right|^2\right)\right.\\ &\left.-|s|^2\left\{\varphi^2_{y_0}(y^*)-\beta(y^*)\left[\varphi^2_{y_1}(y^*)+\varphi^2_{y_2}(y^*)+\varphi^2_{y_3}(y^*)\right]\right\}|\widetilde{w}|^2\right\}dy_0dy_1dy_2.\end{aligned}$$

Then

$$\left|\Xi^{(2)}_\beta\right|\le\epsilon(\delta)|s|\left\|\left(\frac{\partial\widetilde{w}}{\partial y_3},\widetilde{w}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})},\tag{20.82}$$

where $\epsilon(\delta)\to0$ as $\delta\to+0$. We can prove that there exists a parameter $\widehat{\tau}>1$ such that for any $\tau>\widehat{\tau}$ there exists $s_0(\tau)$ such that

$$\begin{aligned}&\frac{3}{4}\|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\frac{3}{4}\|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\operatorname{Re}([L_{+,\beta},L_{-,\beta}]\widetilde{w},\overline{\widetilde{w}})_{L^2(\mathcal{G})}\\ &\quad+C_{11}|s|\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}\widetilde{w}\|_{L^2(\partial\mathcal{G})}\\ &\quad\ge C_{12}|s|\|\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})},\quad\forall|s|\ge s_0(\tau),\end{aligned}\tag{20.83}$$

where $C_{12} > 0$ is independent of s. The proof of (20.83) is done exactly same as in Appendix II in Reference 30. Combining (20.80) and (20.83), we arrive at

$$\begin{aligned}&\frac{1}{4}\|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\frac{1}{4}\|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+C_{12}|s|\|\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})}+\Xi_\beta\\&\quad\le C_{13}\left(\|q\|^2_{L^2(\mathcal{G})}+|s|\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}\widetilde{w}\|_{L^2(\partial\mathcal{G})}\right),\quad \forall|s|\ge s_0(\tau).\end{aligned}\tag{20.84}$$

By Rot, Div, Nab denote the operators obtained from rot, div, $\nabla_{y'}$ after the change of variables. In that case, on $\partial\mathcal{G}$ we can rewrite (20.66) and identify $\operatorname{div}\operatorname{rot}\mathbf{u}=0$ in the following way:

$$i\mu\mathrm{Rot}\,(y,\mathbf{D})\mathbf{w}'-i(\lambda+2\mu)\mathrm{Nab}\,(y,\mathbf{D})w_4=\mathbf{f}e^{|s|\varphi}+K(y,D,s)(\mathbf{u}e^{|s|\varphi}),\quad \mathrm{Div}(y,\mathbf{D})\mathbf{w}'=0,\tag{20.85}$$

where $K(y,D,s)$ is the first-order differential operator. Applying the operator $\chi_\nu(s,D')$ to (20.85), we have

$$i\mu\mathrm{Rot}(y,\mathbf{D})\mathbf{w}'_\nu-i(\lambda+2\mu)\mathrm{Nab}(y,\mathbf{D})w_{4,\nu}=\mathbf{F}_1,\quad \mathrm{Div}(y,\mathbf{D})\mathbf{w}'_\nu+[\chi_\nu,\mathrm{Div}]\mathbf{w}'=0\ \ y\in\partial\mathcal{G},\tag{20.86}$$

where

$$\mathbf{F}_1=\chi_\nu\mathbf{f}e^{|s|\varphi}-i[\chi_\nu,\mu\mathrm{Rot}(y,\mathbf{D})]\mathbf{w}'+i[\chi_\nu,(\lambda+2\mu)\mathrm{Nab}(y,\mathbf{D})]w_4+\chi_\nu K(y,D,s)\left(\mathbf{u}e^{|s|\varphi}\right).$$

We will prove Lemma 20.5 separately in the following three cases:

1. $r_\mu(\gamma)=0$, $r_{\lambda+2\mu}(\gamma)\ne 0$ (Section 5)
2. $r_\mu(\gamma)\ne 0$, $r_{\lambda+2\mu}(\gamma)=0$ (Section 6)
3. $r_\mu(\gamma)\ne 0$, $r_{\lambda+2\mu}(\gamma)\ne 0$ or $r_\mu(\gamma)=r_{\lambda+2\mu}(\gamma)=0$ (Section 7)

20.5 The Case: $r_\mu(\gamma)=0$ and $r_{\lambda+2\mu}(\gamma)\ne 0$

In this section, we treat the case where $r_\mu(\gamma)=0$ and $r_{\lambda+2\mu}(\gamma)\ne 0$. Taking the parameters δ and δ_1 sufficiently small, we can assume that there exists a constant $\widehat{C}>0$ such that

$$|r_{\lambda+2\mu}(y,\zeta)|\ge\widehat{C}|\zeta|^2,\quad \forall(y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1),\ \ |\zeta|\ge 1.$$

We note by (20.84) that there exist $C_1>0$ and $C_2>0$ such that

$$\begin{aligned}&C_1|s|\|w_{k,\nu}\|^2_{H^{1,s}(\mathcal{G})}+\Xi^{(1)}_{\mu,k}\\&\quad\le C_2\left(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right)+\epsilon(\delta)|s|\left\|\left(\frac{\partial\mathbf{w}'_\nu}{\partial y_2},\mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})},\quad k\in\{1,2,3\}.\end{aligned}\tag{20.87}$$

Here and below by $\epsilon(\delta)$ we denote a function such that $\lim_{\epsilon\to+0}\epsilon(\delta) = 0$. Note that $\Xi^{(1)}_{k,\mu}$ can be written in the form:

$$\begin{aligned}
\Xi^{(1)}_{k,\mu} = &\int_{\partial\mathcal{G}} \left(|s|\big(\mu^2\varphi_{y_3}\big)(y^*)\left|\frac{\partial w_{k,\nu}}{\partial y_3}\right|^2 + |s|^3\left(\mu^2\varphi^3_{y_3}\right)(y^*)|w_{k,\nu}|^2\right) d\Sigma \\
&+ \mathrm{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)\frac{\partial w_{k,\nu}}{\partial y_3}\overline{\left[(\mu\varphi_{y_1})(y^*)\frac{\partial w_{k,\nu}}{\partial y_1} + (\mu\varphi_{y_2})(y^*)\frac{\partial w_{k,\nu}}{\partial y_2} - \varphi_{y_0}(y^*)\frac{\partial w_{k,\nu}}{\partial y_0}\right]} d\Sigma \\
&+ \int_{\partial\mathcal{G}} |s|(\mu\varphi_{y_3})(y^*)\big\{\xi_0^2 - \mu(y^*)\left(\xi_1^2+\xi_2^2\right) \\
&- s^2\varphi^2_{y_0}(y^*) + s^2\mu(y^*)\big[\varphi^2_{y_1}(y^*)+\varphi^2_{y_2}(y^*)\big]\big\}|\widehat{w}_{k,\nu}|^2 d\Sigma \\
\equiv\ & J_1^{(k)} + J_2^{(k)} + J_3^{(k)}.
\end{aligned} \tag{20.88}$$

By (20.75) to (20.77), there exists $C_3 > 0$ such that

$$\begin{aligned}
&\left|\xi_0^2 - s^2\varphi^2_{y_0}(y^*) - \mu(y^*)\left(\xi_1^2+\xi_2^2\right) + \mu(y^*)s^2\left[\varphi^2_{y_1}(y^*)+\varphi^2_{y_2}(y^*)\right]\right| \\
&\quad + |s||\xi_0\varphi_{y_0}(y^*) - \mu(y^*)\xi_1\varphi_{y_1}(y^*) - \mu(y^*)\xi_2\varphi_{y_2}(y^*)| \\
&\quad \le \delta_1 C_3\left(|\xi_0|^2+|\xi_1|^2+|\xi_2|^2+s^2\right), \quad \forall\zeta\in\mathcal{O}(\delta_1).
\end{aligned} \tag{20.89}$$

Next we take the parameter δ_1 sufficiently small such that

$$|\xi_0|^2 \le C_4\left(\xi_1^2+\xi_2^2+s^2\right), \quad \forall\zeta\in\mathcal{O}(\delta_1), \tag{20.90}$$

where the constant $C_4 > 0$ is independent of ζ. Then, by (20.89), we have

$$\sum_{k=1}^{3}\left|J_3^{(k)}\right| \le \delta_1(\mu\varphi_{y_3})(y^*)|s|\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \quad k\in\{1,2,3\}. \tag{20.91}$$

Moreover we claim that there exists $\delta_0 > 0$ such that if $\delta_1 \in (0,\delta_0)$, then there exists $C_5 > 0$ such that

$$|\xi_0| \le C_5(|\xi_1|+|\xi_2|+|s|), \quad \forall\zeta\in\mathcal{O}(\delta_1). \tag{20.92}$$

We set $V^+_{\lambda+2\mu} = [D_{y_3} - \Gamma^+_{\lambda+2\mu}(y,s,D')]w_{4,\nu}$. Then by Proposition 20.1

$$P_{\lambda+2\mu,s}w_{4,\nu} = (\lambda+2\mu)|G|^2[D_{y_3} - \Gamma^-_{\lambda+2\mu}(y,s,D')]V^+_{\lambda+2\mu} + T_{\lambda+2\mu}w_{4,\nu},$$

where $T_{\lambda+2\mu}\in\mathcal{L}[H^{1,s}(\mathcal{G}), L^2(\mathcal{G})]$. This decomposition and Proposition 20.2 immediately imply

$$\left\|\sqrt{|s|}\left[D_{y_3} - \Gamma^+_{\lambda+2\mu}(y,s,D')\right]w_{4,\nu}|_{y_3=0}\right\|_{L^2(\partial\mathcal{G})} \le C_6\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\right]. \tag{20.93}$$

Next we estimate the term $J_2^{(k)}$. First we note that, thanks to the homogeneous Dirichlet boundary conditions, we have the *a priori* estimate

$$\sqrt{|s|}\|w_{3,\nu}\|_{H^{1,s}(\partial\mathcal{G})} \le \epsilon(\delta)\sqrt{|s|}\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{20.94}$$

Using (20.94), we have from (20.84) with $w_{3,\nu}$ instead of $\widetilde{w}$

$$\begin{aligned}
\sqrt{|s|}\left\|\left(\frac{\partial w_{3,\nu}}{\partial y_3}, w_{3,\nu}\right)\right\|_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \le\ & C_7\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\right] \\
&+ \epsilon(\delta)\sqrt{|s|}\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.
\end{aligned} \tag{20.95}$$

Now we consider the following two cases.

CASE A Assume that $s^* \neq 0$. In this case by (20.89)

$$\sum_{k=1}^{3} \left| J_2^{(k)} \right| + \left| J_3^{(k)} \right| \leq \epsilon(\delta)|s| \left\| \left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu \right) \right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

Therefore, for some constant $C_8 > 0$,

$$\sum_{k=1}^{3} \Xi^{(1)}_{k,\mu} \geq |s| C_8 \left\| \left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu \right) \right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{20.96}$$

Combining (20.96) and (20.87), we have

$$\sqrt{|s|} \left\| \left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \leq C_9 \big(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \big). \tag{20.97}$$

Using (20.97) we obtain from (20.86)

$$\begin{aligned} &\sqrt{|s|} \|\mathrm{Nab}(y, \mathbf{D}) w_{4,\nu}\|_{L^2(\partial\mathcal{G})} \\ &\leq C_{10} \big(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \big) + \epsilon(\delta) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \end{aligned} \tag{20.98}$$

On the other hand, thanks to (20.93) we have

$$\begin{aligned} &\sqrt{|s|} \left\| \left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu} \right) \right\|_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \\ &\leq C_{11} \sqrt{|s|} \|\mathrm{Nab}(y, \mathbf{D}) w_{4,\nu}\|_{L^2(\partial\mathcal{G})} + \epsilon(\delta) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \end{aligned}$$

Combining this estimate with (20.97) and (20.98), we obtain

$$\sqrt{|s|} \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \leq C_{12} \left(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \right). \tag{20.99}$$

Inequalities (20.99) and (20.87) imply (20.69).

CASE B Assume that $s^* = 0$. By (20.86) and (20.94) the following equality is true:

$$\begin{aligned} R(y, s, D')(w_{1,\nu}, w_{2,\nu}) &\equiv (\mathbf{D}_1 w_{1,\nu} + \mathbf{D}_2 w_{2,\nu}, -\mathbf{D}_2 w_{1,\nu} + \mathbf{D}_1 w_{2,\nu}) \\ &= \left[F_1, \frac{\lambda + 2\mu}{\mu} \alpha^+_{\lambda+2\mu}(y, s, D') w_{4,\nu} + F_2 \right], \end{aligned}$$

where

$$\sqrt{|s|} \|(F_1, F_2)\|_{\mathbf{L}^2(\partial\mathcal{G})} \leq \epsilon(\delta) \sqrt{|s|} \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{13} \left(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \right).$$

The principal symbol of the operator R is

$$R(y^*, s, \xi_1, \xi_2) = \begin{pmatrix} \xi_1 + i|s|\varphi_{y_1}(y^*) & \xi_2 + i|s|\varphi_{y_2}(y^*) \\ -\xi_2 - i|s|\varphi_{y_2}(y^*) & \xi_1 + i|s|\varphi_{y_1}(y^*) \end{pmatrix}.$$

Because $\det R(y^*, s^*, \xi_1^*, \xi_2^*) \neq 0$, there exists a parametrix of the operator R such that

$$(w_{1,\nu}, w_{2,\nu}) = R(y, s, D')^{-1}\left[0, \frac{\lambda+2\mu}{\mu}\alpha^+_{\lambda+2\mu}(y, s, D')w_{4,\nu}\right] + R(y, s, D')^{-1}(F_1, F_2) + T_{-1}(w_{1,\nu}, w_{2,\nu}). \tag{20.100}$$

By the first and second equations in (20.86), we have

$$(D_3w_{1,\nu}, D_3w_{2,\nu}) = \frac{\lambda+2\mu}{\mu}(D_2w_{4,\nu}, -D_1w_{4,\nu}) + (F_4, F_5), \tag{20.101}$$

where

$$\sqrt{|s|}\|(F_4, F_5)\|_{\mathbf{L}^2(\partial\mathcal{G})} \leq \epsilon(\delta, \delta_1)\sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{14}\left(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\right).$$

Using Eq. (20.100) and (20.101), we can reduce $J_2^{(1)}$, $J_2^{(2)}$ to the form

$$J_2^{(1)} = \mathrm{Re}\int_{\partial\mathcal{G}} 2|s|(\lambda+2\mu)(y^*)\frac{\partial w_{4,\nu}}{\partial y_2}\overline{[i(\mu\varphi_{y_1})(y^*)D_1 + i(\mu\varphi_{y_2})(y^*)D_2 - i\varphi_{y_0}(y^*)D_0]} \overline{(R(y, s, D)^{-1}[0, \alpha^+_{\lambda+2\mu}(y, s, D')w_{4,\nu}]\cdot \vec{j}_1)}d\Sigma + I_1, \tag{20.102}$$

$$J_2^{(2)} = -\mathrm{Re}\int_{\partial\mathcal{G}} 2|s|(\lambda+2\mu)(y^*)\frac{\partial w_{4,\nu}}{\partial y_1}\overline{[i(\mu\varphi_{y_1})(y^*)D_1 + i(\mu\varphi_{y_2})(y^*)D_2 - i\varphi_{y_0}(y^*)D_0]} \overline{(R(y, s, D)^{-1}[0, \alpha^+_{\lambda+2\mu}(y, s, D')w_{4,\nu}]\cdot \vec{j}_2)}d\Sigma + I_2, \tag{20.103}$$

where $\vec{j}_1 = (1, 0)$, $\vec{j}_2 = (0, 1)$, and I_1 and I_2 are terms that are estimated by

$$|I_1| + |I_2| \leq \epsilon(\delta)|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{15}\left[\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right].$$

Because $\mathrm{Re}\,\alpha^+_\mu(\gamma) = 0$ and $\mathrm{Im}\,R(\gamma)^{-1} = 0$, by Gårding's inequality we obtain from (20.102) and (20.103) that

$$\left|J_2^{(1)}\right| + \left|J_2^{(2)}\right| \leq \epsilon(\delta)\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{16}\left[\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right].$$

This inequality and (20.91) imply for $k \in \{1, 2, 3\}$

$$\Xi^{(1)}_{k,\mu} \geq \int_{\partial\mathcal{G}}\left\{|s|\left(\mu^2\varphi_{y_3}\right)(y^*)\left|\frac{\partial w_{k,\nu}}{\partial y_3}\right|^2 + |s|^3\left(\mu^2\varphi^3_{y_3}\right)(y^*)|w_{k,\nu}|^2\right\}d\Sigma - \epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{20.104}$$

In terms of (20.101) and (20.104), we have

$$|s|\|w_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \leq C_{17}\sum_{k=1}^{3}\Xi^{(1)}_{k,\mu} + \epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This inequality and (20.93) imply

$$|s|\left\|\left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \le C_{18}\left[\sum_{k=1}^{3}\Xi^{(1)}_{k,\mu} + \|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right]$$
$$+\epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_3}, \mathbf{w}_{\nu}\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{18}\left[\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right]. \tag{20.105}$$

By (20.100), (20.104), and (20.105), we obtain

$$|s|\left\|\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_3}, \mathbf{w}_{\nu}\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_{19}\left[\sum_{k=1}^{3}\Xi^{(1)}_{k,\mu} + \|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right]$$
$$+\epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_3}, \mathbf{w}_{\nu}\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This estimate and (20.84) imply (20.69).

20.6 The Case: $r_{\lambda+2\mu}(\gamma) = 0$ and $r_\mu(\gamma) \neq 0$

Let $\gamma = (y^*, \zeta^*) = (y^*, s^*, \xi_0^*, \xi_1^*, \xi_2^*)$ be a point on $\partial\mathcal{G}\times S^3$ such that $r_{\lambda+2\mu}(\gamma) = 0$, $r_\mu(\gamma) \neq 0$ and supp$\chi_\nu \subset \mathcal{O}(\delta_1)$. Taking the parameters δ and δ_1 sufficiently small, we can assume that there exists a constant $\widehat{C} > 0$ such that

$$|r_\mu(y,\zeta)| \ge \widehat{C}|\zeta|^2, \quad \forall (y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1), \quad |\zeta|\ge 1.$$

By (20.75) to (20.77), there exist $\delta_0 > 0$ and $C_1 > 0$ such that for all $\delta_1 \in (0,\delta_0)$ we have

$$\xi_0^2 \le C_1\left(\xi_1^2+\xi_2^2+s^2\right), \quad \forall\zeta\in\mathcal{O}(\delta_1). \tag{20.106}$$

We consider the following three cases.

CASE A Assume that $s^* = 0$ and

$$\varphi_{y_3}(y^*) > \frac{\left|\frac{1}{\mu(y^*)}\xi_0^*\varphi_{y_0}(y^*) - \xi_1^*\varphi_{y_1}(y^*) - \xi_2^*\varphi_{y_2}(y^*)\right|}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*,\xi_2^*)|}.$$

In that case, there exists a constant $C_2 > 0$ such that

$$-\mathrm{Im}\,\Gamma^{\pm}_{\mu}(y,\zeta) \ge C_2|s|, \quad \forall (y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1),$$

provided that $|\delta| + |\delta_1|$ is sufficiently small. Because $s^* = 0$, we may assume that

$$|\xi_0|^2 + s^2 \le C_3\left(\xi_1^2+\xi_2^2\right), \quad \forall\zeta\in\mathcal{O}(\delta_1) \tag{20.107}$$

for some constant $C_3 > 0$, taking a sufficiently small δ_1. We set $V^{\pm}_{\mu} = [D_{y_3} - \Gamma^{\pm}_{\mu}(y,s,D')]\mathbf{w}'_\nu$. Then, by Proposition 20.1, we have

$$P_{\mu,s}(y,D)\mathbf{w}'_\nu = |G|^2\mu\left[D_{y_3} - \Gamma^{-}_{\mu}(y,s,D')\right]V^{+}_{\mu} + T^{+}_{\mu}\mathbf{w}'_\nu$$
$$= |G|^2\mu\left[D_{y_3} - \Gamma^{+}_{\mu}(y,s,D')\right]V^{-}_{\mu} + T^{-}_{\mu}\mathbf{w}'_\nu, \tag{20.108}$$

where $T_\mu^\pm \in \mathcal{L}[\mathbf{H}^{1,s}(\mathcal{G}), \mathbf{L}^2(\mathcal{G})]$. This decomposition and Proposition 20.2 imply

$$\left\| \sqrt{|s|}\left[D_{y_3} - \Gamma_\mu^\pm(y,s,D')\right]\mathbf{w}'_\nu|_{y_3=0}\right\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_4\big(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})}\big). \quad (20.109)$$

We have

$$\big(-V_\mu^+ + V_\mu^-\big)\big|_{y_3=0} = \big[\alpha_\mu^+(y,s,D') - \alpha_\mu^-(y,s,D')\big]\mathbf{w}'_\nu \quad \text{on } \partial\mathcal{G}. \quad (20.110)$$

Because $\alpha_\mu^+(y^*,\zeta^*) - \alpha_\mu^-(y^*,\zeta^*) = 2\sqrt{r_\mu(y^*,\zeta^*)} \neq 0$, by (20.109), and (20.110) and Gårding's inequality we have

$$\sqrt{|s|}\|\mathbf{w}'_\nu\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_5\big[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})}\big]. \quad (20.111)$$

By (20.111) and (20.109), we obtain

$$\int_{\partial\mathcal{G}} |s|\left|\frac{\partial \mathbf{w}'_\nu}{\partial y_3}\right|^2 d\Sigma \le C_6\big[\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\big] \quad (20.112)$$

Finally, by (20.111) and (20.112) combined with (20.86), we obtain

$$|s|\left\|\left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \le C_7\big(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\big). \quad (20.113)$$

Inequalities of Eqs. (20.111) to (20.113) and (20.84) imply (20.69).

CASE B Assume that $s^* = 0$ and

$$\varphi_{y_3}(y^*) \le \frac{\left|\frac{1}{\mu(y^*)}\xi_0^*\varphi_{y_0}(y^*) - \xi_1^*\varphi_{y_1}(y^*) - \xi_2^*\varphi_{y_2}(y^*)\right|}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*,\xi_2^*)|}. \quad (20.114)$$

In that case $\lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s| \neq 0$. Because $s^* = 0$, we note that $\operatorname{Re} r_\mu(y^*,\zeta^*) > 0$. Set $I = \operatorname{sign} \lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s|$. Then we have

$$\Gamma_\mu^+(y^*,\zeta^*) = I\sqrt{\operatorname{Re} r_\mu(y^*,\zeta^*)}. \quad (20.115)$$

Therefore,

$$-\operatorname{Re}\Gamma_\mu^+(y^*,\zeta^*)[(\mu\varphi_{y_1})(y^*)\xi_1^* + (\mu\varphi_{y_2})(y^*)\xi_2^* - \varphi_{y_0}(y^*)\xi_0^*] > 0.$$

Taking the parameters $\delta > 0$ and $\delta_1 > 0$ sufficiently small, we obtain

$$-\operatorname{Re}\Gamma_\mu^+(y,\zeta)[\mu\varphi_{y_1}(y)\xi_1 + \mu\varphi_{y_2}(y)\xi_2 - \varphi_{y_0}(y)\xi_0] > 0, \quad \forall (y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1). \quad (20.116)$$

Using the definition of V_μ^+, we have

$$\begin{aligned}
J_2 &= \operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_3}\overline{\left[\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right]}d\Sigma \\
&= \operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)i\Gamma_\mu^+(y,s,D')\mathbf{w}'_\nu\overline{\left[\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right]}d\Sigma \\
&\quad + \operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)iV_\mu^+(\cdot,0)\overline{\left[\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right]}d\Sigma \\
&= \operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)[\mu(y^*)D_{y_1}\varphi_{y_1}(y^*) + \mu(y^*)D_{y_2}\varphi_{y_2}(y^*) - D_{y_0}\varphi_{y_0}(y^*)]\Gamma_\mu^+(y,s,D')\mathbf{w}'_\nu\overline{\mathbf{w}'_\nu}d\Sigma \\
&\quad + \operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)iV_\mu^+(\cdot,0)\overline{\left[\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right]}d\Sigma.
\end{aligned} \quad (20.117)$$

By (20.116) we obtain from Gårding's inequality that the first integral at the right-hand side of (20.117) is negative. Consider two cases. First let

$$[\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^*]\Gamma_\mu^+(y^*, \zeta^*) \ge 0.$$

This inequality and (20.116) yield that $|\xi_0^*\varphi_{y_0}(y^*)| > \mu(y^*)|\xi_1^*\varphi_{y_1}(y^*) + \xi_2^*\varphi_{y_2}(y^*)|$. If $\xi_0^*\varphi_{y_0}(y^*) > 0$, then $\Gamma_\mu^+(y^*, \zeta^*) = |\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(y^*) + \xi_2^*\varphi_{y_2}(y^*) \ge 0$. By the first condition in (20.9), we obtain

$$\frac{\mu(y^*)}{\sqrt{\lambda+2\mu}(y^*)} \frac{[\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^*]}{|(\xi_1^*, \xi_2^*)|} + \frac{\sqrt{\mu}(y^*)\sqrt{\lambda+\mu}(y^*)}{\sqrt{\lambda+2\mu}(y^*)}|\varphi_{y_3}(y^*)| \ge |\varphi_{y_0}(y^*)|.$$

Again by the third condition in (20.9), we note that $|\varphi_{y_3}(y^*)| = \varphi_{y_3}(y^*)$. On the other hand, from $r_{\lambda+2\mu}(y^*, 0, \xi_0^*, \xi_1^*, \xi_2^*) = 0$, we see that $|\xi_0^*| = \sqrt{(\lambda+2\mu)(y^*)}|(\xi_1^*, \xi_2^*)|$. By $\xi_0^*\varphi_{y_0}(y^*) > 0$, we obtain

$$\varphi_{y_3}(y^*) > \frac{-\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_2}(y^*)\xi_2^* + \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*, \xi_2^*)|}.$$

This contradicts (20.114).

If $\xi_0^*\varphi_{y_0}(y^*) < 0$, then $\Gamma_\mu^+(y^*, \zeta^*) = -|\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(y^*) + \xi_2^*\varphi_{y_2}(y^*) < 0$. Therefore,

$$\begin{aligned}\varphi_{y_3}(y^*) &> \frac{\left|\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^* - \frac{\varphi_{y_0}(y^*)\xi_0^*}{\mu(y^*)}\right|}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*, \xi_2^*)|}\\ &= \frac{\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^* - \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*, \xi_2^*)|}.\end{aligned}$$

By (20.60) this again contradicts (20.114).

As the second case, one has to consider $[\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^*]\Gamma_\mu^+(y^*, \zeta^*) < 0$. By Gårding's inequality, for $k \in \{1, 2\}$, we have

$$\mathrm{Re}\int_{\partial\mathcal{G}} 2|s|i\Gamma_\mu^+(y, D')w_{k,\nu}\overline{\left[\mu(y^*)\varphi_{y_1}(y^*)\frac{\partial w_{k,\nu}}{\partial y_1} + \mu(y^*)\varphi_{y_2}(y^*)\frac{\partial w_{k,\nu}}{\partial y_2}\right]}d\Sigma < 0.$$

This inequality, (20.3), and the fact that J_2 is negative implies that

$$\begin{aligned}&-\mathrm{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)i\Gamma_\mu^+(y, D')w_{k,\nu}\\ &\times\overline{\left[(\lambda+2\mu)(y^*)\frac{\partial w_{k,\nu}}{\partial y_1}\varphi_{y_1}(y^*) + (\lambda+2\mu)(y^*)\varphi_{y_2}(y^*)\frac{\partial w_{k,\nu}}{\partial y_2} - \frac{\partial w_{k,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right]}d\Sigma > 0. \quad (20.118)\end{aligned}$$

Note that

$$\begin{aligned}\Xi_{\lambda+2\mu}^{(1)} &= \int_{\partial\mathcal{G}}\left\{|s|\left[(\lambda+2\mu)^2\varphi_{y_3}\right](y^*)\left|\frac{\partial w_{4,\nu}}{\partial y_3}\right|^2 + |s|^3\left[(\lambda+2\mu)^2\varphi_{y_3}^3\right](y^*)|w_{4,\nu}|^2\right\}d\Sigma\\ &\quad + \mathrm{Re}\int_{\partial\mathcal{G}} 2|s|(\lambda+2\mu)(y^*)\frac{\partial w_{4,\nu}}{\partial y_3}\\ &\quad \times\overline{\left\{[(\lambda+2\mu)\varphi_{y_1}](y^*)\frac{\partial w_{4,\nu}}{\partial y_1} + [(\lambda+2\mu)\varphi_{y_2}](y^*)\frac{\partial w_{4,\nu}}{\partial y_2} - \varphi_{y_0}(y^*)\frac{\partial w_{4,\nu}}{\partial y_0}\right\}}d\Sigma\\ &\quad + \int_{\partial\mathcal{G}}|s|[(\lambda+2\mu)\varphi_{y_3}](y^*)\{\xi_0^2 - (\lambda+2\mu)(y^*)\left(\xi_1^2+\xi_2^2\right) - s^2\varphi_{y_0}^2(y^*)\\ &\quad + s^2(\lambda+2\mu)(y^*)\left[\varphi_{y_1}^2(y^*) + \varphi_{y_2}^2(y^*)\right]\}|w_{4,\nu}|^2 d\Sigma\\ &\equiv \widetilde{J}_1 + \widetilde{J}_2 + \widetilde{J}_3.\end{aligned}$$

Using (20.86), we can transform $\widetilde{J}_2$ as

$$\widetilde{J}_2 = -\mathrm{Re}\sum_{k=1}^{2}\int_{\partial\mathcal{G}} 2|s|\frac{\mu^2}{\lambda+2\mu}\frac{\partial w_{k,\nu}}{\partial y_3} \times \left[(\lambda+2\mu)\varphi_{y_1}(y^*)\frac{\partial w_{k,\nu}}{\partial y_1} + (\lambda+2\mu)\varphi_{y_2}(y^*)\frac{\partial w_{k,\nu}}{\partial y_2} - \varphi_{y_0}(y^*)\frac{\partial w_{k,\nu}}{\partial y_0}\right] d\Sigma + I,$$

where

$$|I| \le \epsilon(\delta)|s| \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_8\left(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right).$$

Then by (20.118)

$$\widetilde{J}_2 > C_{10}|s| \left\| \left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} - C_9\left(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right). \tag{20.119}$$

Note

$$|\widetilde{J}_3| \le C_{11}\delta_1|s| \left\| \left(\frac{\partial w_{2,\nu}}{\partial y_3}, w_{2,\nu}\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This inequality and Eq. (20.119) imply

$$\begin{aligned}\Xi^{(1)}_{\lambda+2\mu} \ge\ & C_{12}\left\{\int_{\partial\mathcal{G}}\left(|s|\left|\frac{\partial w_{4,\nu}}{\partial y_3}\right|^2 + |s|^3|w_{4,\nu}|^2\right)d\Sigma + |s|\left\|\left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\right\} \\ & - \epsilon(\delta)|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} - C_9\left(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right).\end{aligned} \tag{20.120}$$

Now we will estimate J_3. By Eqs. (20.75) and (20.77), there exists a constant $C_{13} > 0$ such that

$$\begin{aligned}&\Big|\xi_0^2 - |s|^2\varphi^2_{y_0}(y^*) - (\lambda+2\mu)(y^*)\xi_1^2 + \left[(\lambda+2\mu)\varphi^2_{y_1}\right](y^*)|s|^2 \\ &\quad - (\lambda+2\mu)(y^*)\xi_2^2 + \left[(\lambda+2\mu)\varphi^2_{y_2}\right](y^*)|s|^2\Big| \\ &\quad \le C_{13}\delta_1(|\xi'|^2 + s^2), \quad \forall\zeta \in \mathcal{O}(\delta_1).\end{aligned} \tag{20.121}$$

Using (20.121), we obtain

$$\begin{aligned}&\xi_0^2 - \mu(y^*)\xi_1^2 - \mu(y^*)\xi_2^2 - s^2\varphi^2_{y_0}(y^*) + s^2\left(\mu\varphi^2_{y_1}\right)(y^*) + s^2\left(\mu\varphi^2_{y_2}\right)(y^*) \\ &\quad = (\lambda+\mu)(y^*)\left[\xi_1^2 + \xi_2^2 - s^2\varphi^2_{y_1}(y^*) - s^2\varphi^2_{y_2}(y^*)\right] \\ &\qquad + \xi_0^2 - (\lambda+2\mu)(y^*)\xi_1^2 - (\lambda+2\mu)(y^*)\xi_2^2 - s^2\varphi^2_{y_0}(y^*) \\ &\qquad + s^2\left[(\lambda+2\mu)\varphi^2_{y_1}\right](y^*) + s^2\left[(\lambda+2\mu)\varphi^2_{y_2}\right](y^*) \\ &\quad \ge (\lambda+\mu)(y^*)\left[\xi_1^2 + \xi_2^2 - s^2\varphi^2_{y_1}(y^*) - s^2\varphi^2_{y_2}(y^*)\right] - C_{14}\delta_1(|\xi'|^2 + s^2).\end{aligned}$$

Therefore, for all sufficiently small δ_1, there exists $C_{15} > 0$ such that for all $\zeta \in \mathcal{O}(\delta_1)$

$$\xi_0^2 - \mu(y^*)\xi_1^2 - \mu(y^*)\xi_2^2 - s^2\varphi^2_{y_0}(y^*) + s^2\left(\mu\varphi^2_{y_1}\right)(y^*) + s^2\left(\mu\varphi^2_{y_2}\right)(y^*) \ge C_{15}\delta_1(|\xi|^2 + s^2). \tag{20.122}$$

By (20.122), we see that $J_3 \geq 0$. Hence, by (20.120) and (20.106), there exist constants $C_{16}, C_{17}, C_{18} > 0$ such that

$$\sum_{k=1}^{3} \Xi^{(1)}_{k,\mu} + C_{16}\Xi^{(1)}_{\lambda+2\mu} \geq C_{17}|s| \left\| \left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu \right) \right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})} - C_{18}(\delta,\delta_1)\left[\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right].$$

This inequality and (20.86) imply

$$\sum_{k=1}^{3} \Xi^{(1)}_{k,\mu} \geq C_{19}|s| \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})} - C_{18}(\delta,\delta_1)\left(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right). \tag{20.123}$$

By (20.123), (20.87), and (20.84), we obtain Eq. (20.69).

CASE C Assume that $s^* \neq 0$. If $\delta_1 > 0$ is small enough, then there exists a constant $C_{20} > 0$ such that

$$|\xi_0\varphi_{y_0}(y^*) - (\lambda+2\mu)(y^*)\xi_1\varphi_{y_1}(y^*) - (\lambda+2\mu)(y^*)\xi_2\varphi_{y_2}(y^*)|^2 \leq \delta_1^2 C_{20}\left(\xi_1^2+\xi_2^2+s^2\right). \tag{20.124}$$

By (20.84), there exists $C_{21} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} + C_{21}\|\sqrt{|s|}w_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})} \leq C_{21}\left(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right) + \epsilon(\delta)|s| \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{20.125}$$

By (20.121) and (20.124), we have

$$|\widetilde{J}_2 + \widetilde{J}_3| \leq C_{22}\delta_1^2|s| \left\| \left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}. \tag{20.126}$$

By (20.126) we see from (20.124) that there exists a constant $C_{23} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} \geq -\epsilon|s| \left\| \left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} + C_{23}\int_{\partial\mathcal{G}} \left\{ |s|\left[(\lambda+2\mu)^2\varphi_{y_3}\right](y^*) \left| \frac{\partial w_{4,\nu}}{\partial y_3} \right|^2 + |s|^3\left[(\lambda+2\mu)^2\varphi^3_{y_3}\right](y^*)|w_{4,\nu}|^2 \right\} d\Sigma. \tag{20.127}$$

Because $s^* \neq 0$, without loss of generality, taking δ_1 sufficiently small, we can assume that

$$|\xi'| \leq C_{24}|s|, \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{20.128}$$

By (20.127) and (20.128) for some constants $C_{25} > 0$ and $C_{26} > 0$, we have

$$\Xi^{(1)}_{\lambda+2\mu} \geq C_{25}|s| \left\| \left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} - C_{26}\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}. \tag{20.129}$$

By (20.86) we have

$$\mu \mathrm{Rot}(y,\mathbf{D})\mathbf{w}'_\nu = \mathbf{F}^* \quad \text{on } \partial\mathcal{G}, \tag{20.130}$$

where we set $\mathbf{F}^* = \frac{1}{i}\mathbf{F}_1 + (\lambda+2\mu)\mathrm{Nab}(y, \mathbf{D})w_{4,\nu}$ and $\|\mathbf{F}_1\|_{\mathbf{L}^2(\mathcal{G})} \le C_{27}(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})})$. Next in the operator Rot, we put instead of $\mathbf{D}_3 w_{k,\nu}$ the function $\alpha^+(y, s, D')w_{k,\nu} + V^-_{k,\mu}$ for $k \in \{1, 2, 3\}$. We can represent (20.130) in the form

$$\mu\mathbb{G}(y, s, D')\mathbf{w}'_\nu = \mathbf{F}^* + \widetilde{R}(V^-_\mu). \tag{20.131}$$

By (20.109)

$$\sqrt{|s|}\|\widetilde{R}(V^-_\mu)\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_{28}\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})}\right]. \tag{20.132}$$

The principal symbol of the operator $\mathbb{G}$ at the point γ is given by the matrix

$$\mathbb{G}(\gamma) = \begin{pmatrix} 0 & \alpha^+_\mu(\gamma) & -\xi^*_2 - i|s^*|\varphi_{y_2}(y^*) \\ -\alpha^+_\mu(\gamma) & 0 & \xi^*_1 + i|s^*|\varphi_{y_1}(y^*) \\ -\xi^*_2 - i|s^*|\varphi_{y_2}(y^*) & \xi^*_1 + i|s^*|\varphi_{y_1}(y^*) & 0 \end{pmatrix}. \tag{20.133}$$

Thanks to the Dirichlet boundary condition, we note that

$$\|w_{3,\nu}\|_{H^{1,s}(\partial\mathcal{G})} \le \epsilon(\delta)\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This inequality and (20.109) imply

$$\begin{aligned} |s|^{\frac{1}{2}}\left\|\left(\frac{\partial w_{3,\nu}}{\partial y_3}, w_{3,\nu}\right)\right\|_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} &\le C_{29}\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})}\right] \\ &+\epsilon(\delta)\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \end{aligned} \tag{20.134}$$

By the first two equations of (20.131) and $r_\mu(\gamma) \neq 0$, we obtain

$$\begin{aligned} |s|^{\frac{1}{2}}\|w_{k,\nu}\|_{H^{1,s}(\partial\mathcal{G})} \le\ & C_{29}\Big(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})} + |s|^{\frac{1}{2}}\|\mathbf{F}^*\|_{\mathbf{L}^2(\partial\mathcal{G})}\Big) \\ &+ \epsilon(\delta)\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \qquad k = 1, 2. \end{aligned} \tag{20.135}$$

By (20.109), (20.134), and (20.135)

$$\begin{aligned} |s|^{\frac{1}{2}}\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} &\le C_{29}\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})} + |s|^{\frac{1}{2}}\|\mathbf{F}^*\|_{\mathbf{L}^2(\partial\mathcal{G})}\right] \\ &+ \epsilon(\delta)\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \end{aligned} \tag{20.136}$$

By (20.126) and the definition of the function $\mathbf{F}^*$, we obtain

$$|s|\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \le C_{30}\left[\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \Xi^{(1)}_{\lambda+2\mu}\right]. \tag{20.137}$$

In view of (20.137) and (20.84), we obtain (20.69).

20.7 The Case: $r_\mu(\gamma) \neq 0$ and $r_{\lambda+2\mu}(\gamma) \neq 0$ or $r_\mu(\gamma) = r_{\lambda+2\mu}(\gamma) = 0$

To treat this case, we use the Calderon method. First we introduce the new variables $U = (U_1, \dots, U_6)$, where

$$(U_1, U_2, U_3) = \Lambda(s, D') \left(\mathbf{u}e^{|s|\varphi}\right), \quad (U_4, U_5, U_6) = (D_3 + i|s|\varphi_{y_3}) \left(\mathbf{u}e^{|s|\varphi}\right),$$

and Λ is the pseudodifferential operator with the symbol $(s^2 + |\xi'|^2 + 1)^{\frac{1}{2}}$. In the new notations, the problem (20.66) can be written in the form

$$D_{y_3}U = M(y, s, D')U + \mathbf{F} \quad \text{in } \mathbb{R}^3 \times [0, 1], \quad (U_1, U_2, U_3)(y)|_{y_3=0} = 0, \ U|_{y_3=\frac{1}{N^2}} = 0, \tag{20.138}$$

where $\mathbf{F} = (0, \mathbf{f}e^{|s|\varphi})$. Here $M(y, s, D')$ is the matrix pseudodifferential operator with principal symbol $M_1(y, \zeta)$ given by

$$M_1(y, \zeta) = \begin{pmatrix} 0 & \Lambda_1 E_3 \\ A^{-1}M_{21}\Lambda_1^{-1} & A^{-1}M_{22} \end{pmatrix} - i|s|\varphi_{y_3}E_6$$

(see Reference 58). Here we set $\vec{\theta} = (\xi_1 + i|s|\varphi_{y_1}, \xi_2 + i|s|\varphi_{y_2}, 0)$, $G(y_1, y_2) = (-\partial\ell(y_1, y_2)/\partial y_1, -\partial\ell(y_1, y_2)/\partial y_2, 1)$, $\Lambda_1 = |\zeta|$, $M_{21}[y, \xi' + i|s|\nabla_{y'}\varphi(y)] = ([\xi_0 + i|s|\varphi_{y_0}(y)]^2 - \mu\{[\xi_1 + i|s|\varphi_{y_1}(y)]^2 + [\xi_2 + i|s|\varphi_{y_2}(y)]^2\})E_3 - (\lambda + \mu)(y)\vec{\theta}^T\vec{\theta}$, $M_{22}(y, \xi') = -(\lambda + \mu)(y)(\vec{\theta}^T G + G^T\vec{\theta}) - 2\mu\vec{\theta}G^T E_3$, $A = (\lambda + \mu)(y)G^T G + \mu(y)|G|^2 E_3$. Here $\vec{\theta}^T$ denotes the transpose of the row vector $\vec{\theta}$.

CASE A Suppose that $r_\mu(\gamma) = r_{\lambda+2\mu}(\gamma) = 0$. Then $\operatorname{Im}\Gamma^\pm_\mu(\gamma) < 0$ and $\operatorname{Im}\Gamma^\pm_{\lambda+2\mu}(\gamma) < 0$. Therefore, all the eigenvalues of the matrix $M_1(y, \zeta)$ have negative imaginary parts. There exists $C_1 > 0$ such that

$$\operatorname{Im}\Gamma^\pm_\mu(y, \zeta) < -C_1|\zeta|, \quad \operatorname{Im}\Gamma^\pm_{\lambda+2\mu}(y, \zeta) < -C_1|\zeta|, \quad \forall(y, \zeta) \in B_\delta \times \mathcal{O}(\delta_1).$$

Using the arguments in §4 of Chapter 7 in Reference 43, we obtain

$$\|\chi_\nu U\|_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_2 \left(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\right). \tag{20.139}$$

This estimate implies (20.69).

CASE B Suppose that $r_\mu(\gamma) \neq r_{\lambda+2\mu}(\gamma)$, $r_\mu(\gamma) \neq 0$, $r_{\lambda+2\mu}(\gamma) \neq 0$. In this case, the matrix M_1 has four smooth eigenvalues given by (20.75) to (20.77) and the corresponding six smooth eigenvectors $s_1^\pm, s_2^\pm, s_3^\pm$ given by the following formulae (e.g., see References 30 and 58):

$$s_1^\pm = \left[\left(\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G\right)\Lambda_1^{-1}, \ \alpha^\pm_{\lambda+2\mu}\left(\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G\right)\Lambda_1^{-2}\right], s_2^\pm = \left(w_2^\pm, \alpha^\pm_\mu\Lambda_1^{-1}w_2^\pm\right),$$
$$s_3^\pm = \left(w_3^\pm, \alpha^\pm_\mu\Lambda_1^{-1}w_3^\pm\right),$$

where we set

$$w_2^\pm = \Lambda_1^{-1}\left(-\xi_2 - i|s|\varphi_{y_2} + \alpha^\pm_\mu\ell_{y_2}, \xi_1 + i|s|\varphi_{y_1} - \alpha^\pm_\mu\ell_{y_1}, 0\right),$$
$$w_3^\pm = \left[\alpha^\pm_\mu\left(\xi_1 + i|s|\varphi_{y_1} - \alpha^\pm_\mu\ell_{y_1}\right), \alpha^\pm_\mu\left(\xi_2 + i|s|\varphi_{y_2} - \alpha^\pm_\mu\ell_{y_2}\right), -\sum_{k=1}^{2}\left(\xi_k + i|s|\varphi_{y_k} - \alpha^\pm_\mu\ell_{y_k}\right)^2\right]\Lambda_1^{-2}. \tag{20.140}$$

Now we describe the construction of the pseudodifferential operator S. We take the symbol S in the form $S = (s_1^+, s_2^+, s_3^+, s_1^-, s_2^-, s_3^-)$. Denote

$$S(y, \zeta) = \begin{pmatrix} S_{11}(y, \zeta) & S_{12}(y, \zeta) \\ S_{21}(y, \zeta) & S_{22}(y, \zeta) \end{pmatrix}, \quad |\zeta| = 1. \tag{20.141}$$

Let $S^{-1}(y, \zeta)$ be the inverse matrix to S. We extend the matrices S and S^{-1} within the C^3-class in ζ such that for $|\zeta| \geq 1$, the elements of these matrices are the homogeneous functions of order zero. Following Reference 55 and using the change of variables $W = S^{-1}(y, s, D')U$, which is constructed above, we can reduce system (20.138) to the form

$$D_{y_3} W = \widetilde{M}(y, s, D')W + T(y, s, D')W + \widetilde{\mathbf{F}}, \tag{20.142}$$

where the matrix $\widetilde{M}$ is diagonal and $T \in L^\infty(0, \frac{1}{N^2}; \mathcal{L}[\mathbf{H}^{1,s}(\mathbb{R}^3), \mathbf{H}^{1,s}(\mathbb{R}^3)])$. Now using a standard argument (see Reference 43, p. 241), we can estimate the last three components of W as follows:

$$\sqrt{s}\|(W_4, W_5, W_6)\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \leq C_3\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right], \tag{20.143}$$

where the constant C_3 is independent of N. Because the Lopatinskii determinant $\det S_{11}(\gamma)$ is not equal to zero, by Eq. (20.143) we have

$$\sqrt{|s|}\|(W_1, W_2, W_3)\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \leq C_4\left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right]. \tag{20.144}$$

By (20.143), (20.144), and (20.84), we have (20.69).

CASE C Suppose that $r_\mu(\gamma) = r_{\lambda+2\mu}(\gamma) \neq 0$. Obviously we may assume

$$\operatorname{Im} \frac{\Gamma_\mu^+}{|s|}(\gamma) \geq 0. \tag{20.145}$$

Otherwise (20.69) has been already obtained in Case A. The matrix $M_1(\gamma)$ has only two eigenvalues given by (20.75) to (20.77). Moreover, it is known that the Jordan form of the matrix $M_1(\gamma)$ has two Jordan blocks of the form

$$M^\pm = \begin{pmatrix} \Gamma_\mu^\pm(\gamma) & 1 & 0 \\ 0 & \Gamma_\mu^\pm(\gamma) & 0 \\ 0 & 0 & \Gamma_\mu^\pm(\gamma) \end{pmatrix}.$$

Similar to Case B, following Reference 55 and using the change of variables $W = S^{-1}(y, s, D')U$ where S^{-1} is constructed through S, we can reduce the system to Eq. (20.142) where the matrix $\widetilde{M}(y, \zeta)$ is represented by

$$\widetilde{M}(y, \zeta) = \begin{pmatrix} \widetilde{M}_+(y, \zeta) & 0 \\ 0 & \widetilde{M}_-(y, \zeta) \end{pmatrix}$$

with

$$\widetilde{M}_\pm(y, \zeta) = \begin{bmatrix} \Gamma_{\lambda+2\mu}^\pm(y, \zeta) & 0 & m_{13}^\pm(y, \zeta) \\ 0 & \Gamma_\mu^\pm(y, \zeta) & m_{23}^\pm(y, \zeta) \\ 0 & 0 & \Gamma_\mu^\pm(y, \zeta) \end{bmatrix},$$

and the operator T is in $L^\infty(0, \frac{1}{N^2}; \mathcal{L}[\mathbf{H}^{1,s}(\mathbb{R}^3), \mathbf{H}^{1,s}(\mathbb{R}^3)])$, $m_{13}^\pm(y, s, D')$, $m_{23}^\pm(y, s, D')$ are first-order operators, and

$$\|\widetilde{\mathbf{F}}\|_{L^2(\mathbb{R}_+^1; \mathbf{H}^{1,s}(\mathbb{R}^3))} \leq C_5\left\{\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|U\|_{L^2[\mathbb{R}_+^1; \mathbf{H}^{1,s}(\mathbb{R}^3)]}\right\}.$$

Now we describe the construction of the pseudodifferential operator S. We take the symbol S in the form $S = (s_1^+, s_2^+, s_3^+, s_1^-, s_2^-, s_3^-)$. Here

$$s_1^{\pm} = \left[(\vec{\theta} + \alpha^{\pm}_{\lambda+2\mu} G)\Lambda_1^{-1},\ \alpha^{\pm}_{\lambda+2\mu}(\vec{\theta} + \alpha^{\pm}_{\lambda+2\mu} G)\Lambda_1^{-2}\right], \quad s_2^{\pm} = \left(w_2^{\pm}, \alpha_\mu \Lambda_1^{-1} w_2^{\pm}\right)$$

are the eigenvectors of the matrix $M_1(y, \zeta)$ on the sphere $\zeta \in S^3$, which corresponds to the eigenvalue $\Gamma^{\pm}_{\lambda+2\mu}$ (with $w_2^{\pm}$ given by (20.140)) and the vector $s_3^{\pm}$ is given by the formula

$$s_3^{\pm} = E_{\pm} s^{\pm}, \quad E_{\pm} = \frac{1}{2\pi i} \int_{C^{\pm}} [z - M_1(y, \zeta)]^{-1} dz,$$

where $C^{\pm}$ are small circles, oriented counterclockwise, centered at $\Gamma^{\pm}_{\mu}(\gamma)$, and $s^{\pm}$ solves the equation $M_1(\gamma)s^{\pm} - \Gamma^{\pm}_{\mu}(\gamma)s^{\pm} = s_1^{\pm}(\gamma)$. For the explicit formula for the vector $s^{\pm}$, see Reference 29. By (20.145) the circles $C^{\pm}$ may be taken such that the disks bounded by these circles do not intersect, provided that δ_1, δ are taken sufficiently small. Note that the vectors $s_j^{\pm} \in C^2(B_\delta \times \mathcal{O}_{\delta_1})$ are homogeneous functions of the order zero in ζ. Now using a standard argument (see Reference 43, p.241), we can estimate the last three components of W as follows:

$$\|(W_4, W_5, W_6)\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} \le C_6 \left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right], \tag{20.146}$$

where the constant C_6 is independent of N. Now we need to estimate the first three components of the vector function W on $\partial\mathcal{G}$. Thanks to the homogeneous boundary conditions for U_4, U_5, and U_6, we have

$$S_{11}(y', 0, s, D')(W_1, W_2, W_3) = -S_{12}(y', 0, s, D')(W_4, W_5, W_6) + T_{-1}(y', 0, s, D')U, \tag{20.147}$$

where $T_{-1} \in \mathcal{L}[\mathbf{H}^{1,s}(\mathbb{R}^3), \mathbf{H}^{2,s}(\mathbb{R}^3)]$ and we set

$$S(y, \zeta) = \begin{bmatrix} S_{11}(y, \zeta) & S_{12}(y, \zeta) \\ S_{21}(y, \zeta) & S_{22}(y, \zeta) \end{bmatrix}.$$

The principal symbol of the pseudodifferential operator S_{11} is a 3×3 matrix such that the j-th column equals the last three coordinates of the vector s_j^+. Therefore, $\det S_{11}(\gamma) \neq 0$. From (20.146) and (20.147) and Gårding's inequality, we obtain

$$\left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G}) \times \mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_7 \left[\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^2(\mathcal{G})}\right], \tag{20.148}$$

where the constant C_7 is independent of N. By (20.148) and (20.84), we obtain (20.69).

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Chapter 21

Forced Oscillations of a Damped Benjamin-Bona-Mahony Equation in a Quarter Plane

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Abstract This chapter studies an infinite-dimensional dynamic system described by a damped Benjamin-Bona-Mahony equation posed in a quarter plane. It shows that if the boundary forcing of the system is time periodic with small amplitude, then the system admits a unique time-periodic solution that, as a limit cycle, is globally exponentially stable. In other words, it comprises an exponentially stable global attractor for the system.

21.1 Introduction

In this chapter we consider an initial-boundary value problem (IBVP) for a damped Benjamin-Bona-Mahony (BBM) equation posed in a quarter plane, namely,

$$u_t + u_x + uu_x - u_{xxt} - \alpha_1 u_{xx} + \alpha_2 u = 0 \quad \text{for } x,\ t \geq 0, \tag{21.1}$$

$$u(x, 0) = \phi(x), \quad u(0, t) = h(t) \tag{21.2}$$

where α_1 and α_2 are nonnegative constants that are proportional to the strength of the damping effect.

The BBM equation, known also as the alternative Korteweg-de Vries (KdV) equation, is commonly used as a mathematical model for unidirectional propagation of nonlinear dispersive waves. In a well-cited paper of Bona et al. [4] aimed at evaluating the effectiveness of the BBM equation as a model equation for water waves, a numerical scheme was developed to solve Eqs. (21.1) and (21.2) with the initial condition $u(x, 0) = 0$. The specified function h corresponds to a given displacement of the free surface at one end of the channel. A comparison was made between the numerical simulations and the results of laboratory experiments. The model was found to give quite a good description of spatial and temporal development of periodic generated waves. It is noteworthy that, as was pointed out in Reference 4, "one feature to emerge from the study was that, in all cases, it was important to make allowance for dissipative effects to obtain reasonable agreement between the empirical results and the theoretical model. But, with this proviso, the model appeared to give a good description of the experimental results at small-wave amplitudes. At larger amplitudes the agreement was not so good..." Moreover, it was observed from both numerical simulations and laboratory experiments that, at each fixed station down the channel (for example, at a spatial point represented by x_0), the wave motion $u(x_0, t)$ rapidly became periodic of the same period as the boundary forcing. This leads to the following conjecture for the IBVP of Eqs. (21.1) and (21.2):

CONJECTURE
If the boundary forcing h is a periodic function of period θ with small amplitude, then for any given initial data ϕ in an appropriate space, the resulting solution $u(x, t)$ of Eqs. (21.1) *and* (21.2) *is, for each fixed x, asymptotically periodic of period θ.*

It is well known in the literature that the IBVP of Eqs. (21.1) and (21.2) is well-posed in the space $H^1(R^+)$. More precisely, *for any $\phi \in H^1(R^+)$ and $h \in C[0, \infty)$ satisfying the compatibility condition*

$$h(0) = \phi(0),$$

the IBVP of Eqs. (21.1) *and* (21.2) *admits a unique solution $u \in C\{(0, \infty); H^1(R^+)\}$. The associated solution map is continuous from the space $H^1(R^+) \times C[0, T]$ to the space $C([0, T]; H^1(R^+))$ for any given $T > 0$.* Thus, Eqs. (21.1) and (21.2) can be viewed as an infinite-dimensional dynamic system in the space $H^1(R^+)$. The following questions are standard from the point of view of dynamics:

1. What is longtime behavior of solutions for Eqs. (21.1) and (21.2)?
2. If the boundary forcing h is a time-periodic function of period $\theta > 0$, does Eq. (21.1) possess a time-periodic solution $u(x, t)$ satisfying the boundary condition $u(0, t) = h(t)$? If such a time-periodic solution exists, what is its stability?

In this paper, assuming that both α_1 and α_2 in Eq. (21.1) are positive, we will demonstrate that if the boundary forcing is small and periodic of period θ, then there is what we will call a unique forced oscillation solution $u^*(x, t)$ of Eq. (21.1) that is temporally periodic of period θ and such that $u^*(0, t) = h(t)$ for $t \geq 0$. Furthermore, it will be shown that this unique time-periodic solution $u^*(x, t)$, as a *limit cycle* for the system of Eqs. (21.1) and (21.2), is also globally exponentially stable: for any $\phi \in H^1(0, \infty)$ satisfying $\phi(0) = h(0)$, the corresponding solution $u(x, t)$ of Eqs. (21.1) and (21.2) satisfies

$$\|u(\cdot, t) - u^*(\cdot, t)\|_{H^1(R^+)} \leq Ce^{-\mu t} \tag{21.3}$$

for any $t \geq 0$ where $\mu > 0$ is independent of ϕ. In other words, this unique time-periodic solution, as a limit cycle, comprises an *inertial manifold* for the infinite-dimensional system described by Eqs. (21.1) and (21.2).

Damped dispersive wave equations have been studied in the past from the point of view of a dynamic system. Ghidaglia [12, 13] and Sell and You [16] considered the damped forced KdV equation

$$\left.\begin{aligned} &u_t + uu_x + u_{xxx} - \alpha_1 u_{xx} + \alpha_2 u = f, \quad x \in (0, 1),\ t \geq 0 \\ &u(0, t) = u(1, t), \quad u_x(1, t) = u_x(0, t), \quad u_{xx}(1, t) = u_{xx}(0, t), \end{aligned}\right\} \tag{21.4}$$

posed on the finite interval (0, 1) with periodic boundary conditions, where the forcing $f = f(x, t)$ is a function of x and t. Assuming that $\alpha_1 = 0, \alpha_2 > 0$ and that the external excitation f is either time independent or time periodic, Ghidaglia [12, 13] proved the existence of a global attractor of finite fractal dimension for the infinite-dimensional system described by Eq. (21.4). Assuming $\alpha_1 > 0$, Sell and You [16] showed that Eq. (21.4) possesses an *inertial manifold* in the case that the external excitation f is time independent.

Zhang [20] studied a damped forced KdV equation posed on a finite interval with homogeneous Dirichlet boundary conditions, viz.,

$$\left.\begin{aligned} &u_t + uu_x + u_{xxx} - \eta u_{xx} + u_x = f, \quad x \in (0,1),\ t \geq 0,\\ &u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0. \end{aligned}\right\} \tag{21.5}$$

It was shown that if the external excitation f is time periodic with small amplitude, then the system admits a *unique time-periodic solution* that, as a limit cycle, forms an *inertial manifold* for the infinite-dimensional system described by Eq. (21.5). Similar results have also been established by Zhang [21] for the following damped BBM equation posed on a finite interval:

$$\left.\begin{aligned} &u_t + u_x + uu_x - u_{xxt} - \eta u_{xx} = f, \quad x \in (0,1),\ t \geq 0,\\ &u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0. \end{aligned}\right\} \tag{21.6}$$

In a recent paper, Bona et al. [5] studied a damped KdV equation posed in a quarter plane:

$$u_t + u_x + uu_x + u_{xxx} - \alpha_1 u_{xx} + \alpha_2 u = 0 \quad \text{for } x,\ t \geq 0, \tag{21.7}$$

$$u(x,0) = \phi(x), \quad u(0,t) = h(t). \tag{21.8}$$

Assuming $\alpha_1 = 0$ and $\alpha_2 > 0$, Bona et al. [5] showed that if the boundary forcing h is small and periodic of period θ, Eq. (21.7) admits a unique time-periodic solution $u^*(x,t)$ of period θ and such that $u^*(0,t) = h(t)$ for $t > 0$. They also showed that this unique time-periodic solution, as a limit cycle, is globally exponentially stable and thus comprises an inertial manifold for the infinite-dimensional dynamic system described by Eqs. (21.7) and (21.8).

There have been many studies concerned with time-periodic solutions of partial differential equations in the literature. Early works on this subject include, for example, Brézis [8], Vejvoda [17], Keller and Ting [10], and Rabinowitz [14, 15]. For recent theory, see Bourgain [7], Craig and Wayne [9], and Wayne [19]. Most of the theory in the literature thus far has been in the context of parabolic or hyperbolic equations. Consideration of this issue for nonlinear dispersive equations is sparse, and the important question of stability of periodic solutions has received little attention.

This chapter is organized as follows. In Section 21.2 we discuss longtime behavior of the system of Eqs. (21.1) and (21.2) without assuming time periodicity of the boundary forcing. In particular, asymptotic bounds on solutions are established, which are important for the main analysis that is developed in Section 21.3. The existence of the forced oscillations and their stability analysis will be discussed in Section 21.3.

21.2 Asymptotic Behavior

Our main concern in this section for the IBVP of Eqs. (21.1) and (21.2) is longtime behavior of its solutions. We have the following stability results, which will also play important roles in establishing the existence of time-periodic solutions of Eqs. (21.1) and (21.2) in the next section.

THEOREM 21.1
Assume $\alpha = \min\{\alpha_1,\ \alpha_2\} > 0$. There exist positive constants β, γ, and T such that if $\phi \in H^1(R^+)$ and $h \in C_b[0,+\infty)$ satisfy the compatibility condition $h(0) = \phi(0)$ and

$$\overline{\lim_{t\to\infty}} |h(t)| < \beta,$$

then the corresponding solution u of Eqs. (21.1) *and* (21.2) *satisfies*

$$\sup_{t\geq 0} \|u(\cdot,t)\|_{H^1(R^+)} \leq C_1 \left(\|\phi\|_{H^1(R^+)} + \|h\|_{C_b(0,\infty)} \right) \tag{21.9}$$

and

$$\|u(\cdot,t)\|_{H^1(R^+)} \leq C_2 e^{-\gamma t} + C_3 \|h\|_{C_b(T,t)} \tag{21.10}$$

for any $t \geq T$ where C_j, $j = 1, 2, 3$ are positive constants depending continuously on $\|\phi\|_{H^1(R^+)}$ and $\|h\|_{C_b(0,\infty)}$. In particular, if

$$\lim_{t\to\infty} |h(t)| = 0,$$

then

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{H^1(R^+)} = 0.$$

In addition, if there exists a $\beta^* > 0$ such that

$$|h(t)| \leq C e^{-\beta^* t}$$

for some constant $C > 0$ and any $t \geq 0$, then there exists $\gamma > 0$ depending on α and β such that

$$\|u(\cdot,t)\|_{H^1(R^+)} \leq C^* e^{-\gamma t}$$

for any $t \geq 0$ where $C^* > 0$ depending on $\|\phi\|_{H^1(R^+)}$ and $\|h\|_{C_b(R^+)}$.

To prove Theorem 21.1, we consider first the linear problem

$$\begin{cases} u_t + u_x - u_{xxt} - \alpha_1 u_{xx} + \alpha_2 u = 0, & x \in (0,\infty), \quad t \geq 0, \\ u(x,0) = \phi(x) \quad u(0,t) = 0. \end{cases} \tag{21.11}$$

For any $\phi \in H_0^1(R^+)$, Eq. (21.11) admits a unique solution $u = W(t)\phi \in C_b[R^+; H_0^1(R^+)]$ where $\{W(t)\}_{t=0}^{\infty}$ is an analytic semigroup generated by the bounded linear operator A in the space $H_0^1(R^+)$ defined by

$$Af = \left(I - \partial_x^2\right)^{-1} \left(\partial_x - \alpha_1 \partial_x^2 + \alpha_2\right) f$$

for any $f \in H_0^1(R^+)$. Here $(I - \partial_x^2)^{-1}$ is the inverse of the elliptic operator $I - \partial_x^2$ with the domain $\{g \in H^2(R^+), g(0) = 0\}$. A direct computation shows that

$$\max\{\text{Re}\,\lambda, \quad \lambda \in \sigma(A)\} \leq -\alpha.$$

As a result, we have the following.

PROPOSITION 21.1
There exists a constant $C > 0$ such that

$$\|W(t)\phi\|_{H^1(R^+)} \leq \|\phi\|_{H^1(R^+)} e^{-\alpha t} \tag{21.12}$$

for any $t \geq 0$ and $\phi \in H_0^1(R^+)$.

Next we consider the nonhomogeneous problem

$$\begin{cases} u_t + u_x - u_{xxt} - \alpha_1 u_{xx} + \alpha_2 u = 0, \\ u(x,0) = \phi(x), \quad u(0,t) = h(t) \end{cases} \tag{21.13}$$

for $x \in (0, \infty)$ and $t \geq 0$. Let $q(x,t) = e^{-x}h(t)$ and $w(x,t) = u(x,t) - q(x,t)$. Then w solves

$$\begin{cases} w_t + w_x - w_{xxt} - \alpha_1 w_{xx} + \alpha_2 w = (1 + \alpha_1 - \alpha_2)q, \\ w(x,0) = \phi(x) - q(x,0) \equiv \phi^*(x), \quad w(0,t) = 0 \end{cases}$$

for $x \in (0, \infty)$ and $t \geq 0$. If $\phi \in H^1(R^+)$ and $h \in C[0, +\infty)$ satisfying the compatibility conditions $\phi(0) = h(0)$, then Eq. (21.13) admits a unique solution $u \in C[R^+; H^1(R^+)]$ with

$$u(x,t) = q(x,t) + W(t)\phi^* + (1 + \alpha_1 - \alpha_2) \int_0^t W(t-\tau)\left(I - \partial_x^2\right)^{-1} q(\cdot, \tau)\, d\tau.$$

Applying the estimate of Eq. (21.12) yields the following result.

PROPOSITION 21.2

Let $\phi \in H^1(R^+)$ and $h \in C[0, +\infty)$ satisfying the compatibility condition $\phi(0) = h(0)$. Then there exists a constant μ_α depending only on α such that the corresponding solution u of Eq. (21.13) *satisfies*

$$\|u(\cdot,t)\|_{H^1(R^+)} \leq e^{-\alpha t}\|\phi\|_{H^1(R^+)} + \mu_\alpha \|h\|_{C_b(0,t)} \tag{21.14}$$

for any $t \geq 0$. If, in addition, there exists a $\beta^* > 0$ such that

$$|h(t)| \leq C_1 e^{-\beta^* t}$$

for any $t \geq 0$, then

$$\|u(\cdot,t)\|_{H^1(R^+)} \leq C e^{-\alpha t}\|\phi\|_{H^1(R^+)} + C_\alpha e^{-\gamma t} \tag{21.15}$$

for any $t \geq 0$ with $\gamma = min\{\alpha, \beta^*\}$.

Now we present the proof of Theorem 21.1.

PROOF OF THEOREM 21.1 The solution u of Eqs. (21.1) and (21.2) may be written as

$$u(x,t) = v(x,t) + w(x,t)$$

where v solves the linear problem of Eq. (21.13), and w solves the nonlinear problem

$$\begin{cases} w_t + w_x + ww_x + (vw)_x - w_{xxt} - \alpha_1 w_{xx} + \alpha_2 w = -vv_x, \\ w(x,0) = 0, \quad w(0,t) = 0. \end{cases} \tag{21.16}$$

Multiplying both sides of the equation in Eq. (21.16) by $2w$ and integrating with respect to x over $(0, \infty)$, then integration by parts leads to

$$\frac{d}{dt}\int_0^\infty \left(w^2 + w_x^2\right)dx + \int_0^\infty \left(2\alpha_1 w_x^2 + 2\alpha_2 w^2\right)dx + \int_0^\infty v_x w^2 dx = -2\int_0^\infty vv_x w dx.$$

Because

$$\int_0^\infty |v_x w^2| dx \leq \|w(\cdot,t)\|^2_{L^\infty(R^+)}\|v(\cdot,t)\|_{H^1(R^+)} \leq \|w(\cdot,t)\|^2_{H^1(R^+)}\|v(\cdot,t)\|_{H^1(R^+)}$$

and

$$2\int_0^\infty |vv_x w|dx \le \|w(\cdot,t)\|_{H^1(R^+)}\|v(\cdot,t)\|^2_{H^1(R^+)},$$

one arrives at

$$\frac{d}{dt}\|w(\cdot,t)\|_{H^1(R^+)} + [\alpha - \|v(\cdot,t)\|_{H^1(R^+)}]\|w(\cdot,t)\|_{H^1(R^+)} \le \|v(\cdot,t)\|^2_{H^1(R^+)}. \tag{21.17}$$

According to Proposition 21.2,

$$\|v(\cdot,t)\|_{H^1(R^+)} \le e^{-\alpha t}\|v(\cdot,\tau)\|_{H^1(R^+)} + \mu_\alpha \|h\|_{C_b(\tau,t)}$$

for any $t \ge \tau \ge 0$. For given $\delta < \alpha$, first choose $\tau > 0$ large enough that

$$\mu_\alpha \|h\|_{C_b(\tau,t)} \le \delta/2.$$

Then choose $T \ge \tau$ such that

$$e^{-\alpha T}\|v(\cdot,\tau)\|_{H^1(R^+)} \le \delta/2.$$

As a result,

$$\sup_{T \le t < +\infty} \|v(\cdot,t)\|_{H^1(R^+)} \le \delta.$$

It follows then from Eq. (21.17) that

$$\|w(\cdot,t)\|_{H^1(R^+)} \le \|w(\cdot,T)\|_{H^1(R^+)} e^{-\gamma_1 t} + \int_T^t e^{-\gamma_1(t-s)}\|v(\cdot,s)\|^2_{H^1(R^+)} ds \tag{21.18}$$

with $\gamma_1 = \alpha - \delta$ for any $t \ge T$. Then the proof can be completed easily using the assumptions and the above estimates.

21.3 Forced Oscillations and Their Stability

In this section, we first consider the pure boundary-value problem

$$\begin{cases} u_t + uu_x + u_x - u_{xxt} - \alpha_1 u_{xx} + \alpha_2 u = 0, & x,\ t \ge 0, \\ u(0,t) = h(t). \end{cases} \tag{21.19}$$

It is assumed that the boundary input $h(t)$ is periodic with period θ so that

$$h(t+\theta) = h(t)$$

for all $t \ge 0$. We are concerned with whether or not this periodic forcing generates a time-periodic solution of Eq. (21.19). Because of the dissipative terms, it is not expected that the initial data will play any role in this question of longtime asymptotics.

THEOREM 21.2
Assume $\alpha = \min\{\alpha_1, \alpha_2\} > 0$. *There exists a constant* $\beta > 0$ *such that if* $h \in C_b(R^+)$ *satisfying*

$$\|h\|_{C_b(0,\theta)} < \beta, \tag{21.20}$$

then Eq. (21.19) *admits a solution* $u^* \in C_b[R^+; H^1(R^+)]$*, which is a time-periodic function of period* θ*. Moreover, there exists a* $\beta_1 > 0$ *such that if* $u_1^* \in C_b[R^+; H^1(R^+)]$ *is also a time-periodic solution of Eq.* (21.19) *with*

$$\|u_1^*(\cdot, 0)\|_{H^1(R^+)} \leq \beta_1,$$

then

$$u_1^*(x, t) = u^*(x, t)$$

for all $x,\ t \in R^+$.

PROOF Choose $\phi \in H^1(R^+)$, which is compatible with h, and consider the IBVP

$$\begin{cases} u_t + uu_x + u_x - u_{xxt} - \alpha_1 u_{xx} + \alpha_2 u = 0, & x,\ t \geq 0, \\ u(0, t) = h(t), \quad u(x, 0) = \phi(x). \end{cases} \tag{21.21}$$

For the solution u of Eq. (21.21), let

$$w(x, t) = u(x, t + \theta) - u(x, t).$$

Then, the new function w solves the IBVP

$$\begin{cases} w_t + (vw)_x + w_x - w_{xxt} - \alpha_1 w_{xx} + \alpha_2 w = 0, & x,\ t \geq 0, \\ w(0, t) = 0, \quad w(x, 0) = \phi^*(x). \end{cases} \tag{21.22}$$

where

$$\phi^*(x) = u(x, \theta) - \phi(x), \quad v(x, t) = \frac{1}{2}[u(x, t + \theta) + u(x, t)].$$

For this linearized system, the solution w decays exponentially in time.

CLAIM *There exist* $r > 0$ *and* $\beta > 0$ *such that if*

$$\|\phi\|_{H^1(R^+)} + \|h\|_{C_b(0,\theta)} \leq \beta,$$

then the solution w *of Eq. (21.22) satisfies*

$$\|w(\cdot, t)\|_{H^1(R^+)} \leq Ce^{-rt}\|\phi^*\|_{H^1(R^+)}$$

for all $t \geq 0$*, where* C *is independent of* ϕ^* *and* t.

Indeed, multiplying the both sides of Eq. (21.22) by $2w$ and integrating over R^+ with respect to x, integration by parts leads to

$$\frac{d}{dt}\int_0^\infty \left(w^2 + w_x^2\right)dx + 2\alpha\int_0^\infty \left(w^2 + w_x^2\right)dx + \int_0^\infty v_x w^2 dx \leq 0,$$

or

$$\frac{d}{dt}\int_0^\infty \left(w^2 + w_x^2\right)dx + (2\alpha - \|v(\cdot, t)\|_{H^1(R^+)})\int_0^\infty \left(w^2 + w_x^2\right)dx < 0$$

for any $t \geq 0$, which yields that

$$\|w(\cdot, t)\|_{H^1(R^+)} \leq \|\phi^*\|_{H^1(R^+)} e^{\int_0^t -\left[\alpha - \frac{1}{2}\|v(\cdot,s)\|_{H^1(R^+)}\right] ds}$$

for any $t \geq 0$. By Theorem 21.1, for a given $\alpha^* < 2\alpha$, there exists a $\beta > 0$ such that if

$$\|\phi\|_{H^1(R^+)} + \|h\|_{C_b(0,\theta)} < \beta,$$

then

$$\sup_{t \geq 0} \|v(\cdot, t)\|_{H^1(R^+)} \leq \alpha^*.$$

The claim is therefore true.

With this fact in hand, we show that Eq. (21.22) possesses a time-periodic solution of the same period θ as the boundary forcing. Denote by

$$u_n \equiv u(x, n\theta)$$

for $n \geq 1$. Write the difference $u_{n+m} - u_n$ in telescoping fashion to deduce

$$\begin{aligned}
\|u_{n+m} - u_n\|_{H^1(R^+)} &= \left\| \sum_{k=1}^{m} u_{n+k} - u_{n+k-1} \right\|_{H^1(R^+)} \\
&\leq \sum_{k=1}^{m} \|u_{n+k} - u_{n+k-1}\|_{H^1(R^+)} \\
&\leq \sum_{k=1}^{m} \|w[\cdot, (n+k-1)\theta]\|_{H^1(R^+)} \\
&\leq \sum_{k=1}^{m} Ce^{-(n+k-1)\theta} \|\phi^*\|_{H^1(R^+)} \qquad \text{(by the claim)} \\
&\leq \frac{e^{-n\theta}}{1 - e^{-\theta}} C\|\phi^*\|_{H^1(R^+)}
\end{aligned}$$

for any $m \geq 1$. Thus $\|u_{n+m} - u_n\|_{H^1(R^+)}$ tends to zero uniformly in m as $n \to \infty$, which is equivalent to saying that the sequence $\{u_n\}$ is a Cauchy sequence in $H^1(R^+)$. Let $\psi \in H^1(R^+)$ be the limit of the sequence $\{u_n\}$, viz.,

$$\lim_{n \to \infty} u_n = \psi \quad \text{in } H^1(R^+).$$

By Theorem 21.1,

$$\|\psi\|_{H^1(R^+)} \leq C\|h\|_{C_b(0,\theta)}. \tag{21.23}$$

Taking ψ as an initial datum together with the boundary input h for Eq. (21.21), it is asserted that the corresponding solution u^* is a time-periodic solution of period θ. To see this, note that whereas $u_n(\cdot) = u(\cdot, n\theta)$ converges to ψ strongly in $H^1(R^+)$, it is also the case that $u(\cdot, n\theta + \theta)$ converges to $u^*(\cdot, \theta)$ strongly in the space $H^1(R^+)$ as $n \to \infty$ because of the continuity of the associated solution map. Observing that

$$\begin{aligned}
\|u^*(\cdot, \theta) - u^*(\cdot, 0)\|_{H^1(R^+)} \leq\ & \|u^*(\cdot, \theta) - u(\cdot, n\theta + \theta)\|_{H^1(R^+)} \\
&+ \|u(\cdot, n\theta + \theta) - u(\cdot, n\theta)\|_{H^1(R^+)} + \|u(\cdot, n\theta) - u^*(\cdot, 0)\|_{H^1(R^+)}
\end{aligned}$$

for any $n \geq 1$, it is concluded that

$$u^*(\cdot, \theta) = u^*(\cdot, 0)$$

and therefore that $u^*(x, t)$ is a time-periodic function of period θ.

To demonstrate uniqueness, let u_1 be another time-periodic solution with the same boundary forcing h. Let $z(x, t) = u_1(x, t) - u^*(x, t)$. Then z solves the linear problem of Eq. (21.22) with $v = u_1 + u^*$ and $\phi^*(x, t) = u_1(x, 0) - \psi(x)$. As in the proof of the above claim, there exists a $\beta^* > 0$ such that

$$\sup_{t \geq 0} \|v(\cdot, t)\|_{H^1(R^+)} \leq \alpha^*,$$

if

$$\|u_1(\cdot, 0)\|_{H^1(R^+)} + \|h\|_{C_b(0,\theta)} \leq \beta^*.$$

Then z decays to zero exponentially in the space $H^1(R^+)$, which in turn implies

$$u_1(x, t) = u^*(x, t)$$

for all $x \in R^+$ and $t \geq 0$ because both of them are time-periodic functions. The proof is complete. ∎

For a given periodic boundary forcing h of period θ, the IBVP of Eqs. (21.1) and (21.2) may be considered as a dynamical system in the infinite-dimensional space $H^1(R^+)$. It has just been shown that if the amplitude of h is small, Eq. (21.1) admits a unique time-periodic solution $u^*(x, t)$ of period θ satisfying the boundary condition $u^*(0, t) = h(t)$. Maintaining the dynamical system's perspective, one may view u^* as a limit cycle of the dynamical system. A natural further inquiry is then to study the stability of this limit cycle.

THEOREM 21.3

Under the assumptions of Theorem 21.2, the time-periodic solution u^* is locally exponentially stable in the space $H^1(R^+)$, which is to say there exist $\delta > 0$, $r > 0$, and $C > 0$ such that for any given $\phi \in H^1(R^+)$, which satisfies

$$\|\phi(\cdot) - u^*(\cdot, 0)\|_{H^1(R^+)} \leq \delta$$

and is compatible with h, the corresponding solution u of Eqs. (21.1) *and* (21.2) *satisfies*

$$\|u(\cdot, t) - u^*(\cdot, t)\|_{H^1(R^+)} \leq Ce^{-rt}$$

for any $t \geq 0$.

PROOF Let $w(x, t) = u(x, t) - u^*(x, t)$. Then w solves

$$\begin{cases} w_t + (vw)_x + w_x - w_{xxt} - \alpha_1 w_{xx} + \alpha_2 w = 0, & x,\ t \geq 0, \\ w(0, t) = 0, \quad w(x, 0) = \phi_1(x) \end{cases}$$

where $v(x, t) = \frac{1}{2}[u(x, t) + u^*(x, t)]$ and $\phi_1(x) = \phi(x) - u^*(x, 0)$. As in the proof of Theorem 21.4,

$$\begin{aligned} \sup_{t \geq 0} \|v(\cdot, t)\|_{H^1(R^+)} &\leq \sup_{t \geq 0} \|u(\cdot, t)\|_{H^1(R^+)} + \sup_{t \geq 0} \|u^*(\cdot, t)\|_{H^1(R^+)} \\ &\leq C\big(\|\phi\|_{H^1(R^+)} + \|\psi\|_{H^1(R^+)} + \|h\|_{C_b(0,\theta)}\big) \\ &\leq C\big(\|\phi - \psi\|_{H^1(R^+)} + 2\|\psi\|_{H^1(R^+)} + \|h\|_{C_b(0,\theta)}\big) \\ &\leq C(\delta + \beta). \end{aligned}$$

If δ and β are small enough, it follows that

$$\|w(\cdot,t)\|_{H^1(R^+)} \le Ce^{-rt}\|\phi_1\|_{H^1(R^+)}$$

for any $t \ge 0$. The proof is complete. ∎

The results presented in Theorem 21.2 and Theorem 21.3 are local: (i) uniqueness of time-periodic solutions holds under the assumption that the initial value is small; (ii) the time-periodic solution, considered as a limit cycle, is locally exponential stable. The next result shows that uniqueness holds without the smallness assumption and that the unique limit cycle is, in fact, globally exponentially stable.

THEOREM 21.4
Then there exists a $\beta > 0$ such that if $h \in C_b(R^+)$ is a periodic function of period θ satisfying

$$\|h\|_{C_b(0,\theta)} \le \beta,$$

then Eq. (21.19) *admits a unique time-periodic solution u^* of period θ. Moreover, there exist $r > 0$ and $C > 0$ such that for any given $\phi \in H^1(R^+)$, which is compatible with h, the corresponding solution u of Eqs.* (21.1) *and* (21.2) *satisfies*

$$\|u(\cdot,t) - u^*(\cdot,t)\|_{H^1(R^+)} \le Ce^{-rt}$$

for any $t \ge 0$.

PROOF By Theorem 21.1, there exists β_2 such that if

$$\|h\|_{C_b(0,\theta)} \le \beta_2,$$

then

$$\overline{\lim_{t\to\infty}}\|u(\cdot,t)\|_{H^1(R^+)} \le C\|h\|_{C_b(0,\theta)}.$$

One may choose β_2 small enough such that $C\beta_2$ is smaller than β and β_1 determined in Theorem 21.2. Thus, there is a $T_0 > 0$ such that for any $t_0 \ge T_0$, $\|u(\cdot,t_0)\|_{H^1(R^+)} \le \beta_1$. Moreover, let u^* be the time-periodic solution given in Theorem 21.2. We may choose β_1 and β in Theorem 21.2 even smaller so that

$$\|u(\cdot,t_0) - u^*(\cdot,0)\|_{H^1(R^+)} \le \delta$$

where δ is given in Theorem 21.3. Then by Theorem 21.3

$$\|u(\cdot,t+t_0) - u^*(\cdot,t)\|_{H^1(R^+)} \le Ce^{-rt}$$

for all $t \ge 0$, which yields

$$\|u(\cdot,t) - u^*(\cdot,t)\|_{H^1(R^+)} \le \tilde{C}e^{-rt}$$

for all $t \ge 0$ because u^* is a time-periodic function. This estimate implies both that the time-periodic solution of Eq. (21.19) is unique and that it is globally exponentially stable. The proof is complete. ∎

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Chapter 22

Exact Controllability of the Heat Equation with Hyperbolic Memory Kernel

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Abstract The exact controllability for the heat equation with hyperbolic memory kernel is considered. Because of the appearance of such kind of memory, the speed of propagation for heat pulses is finite, and the corresponding controllability property has a certain nature similar to the wave equation and is significantly different from that of the usual heat equation.

22.1 Introduction

Let $T > 0$ and Ω be a bounded domain of $\mathbb{R}^n$ with a C^2 boundary $\partial\Omega$, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$. Let ω be an open nonempty subset of Ω and denote by χ_ω the characteristic function of ω.

To begin with, let us recall the controllability theory on the classical heat equation:

$$\begin{cases} y_t - \nabla \cdot (A(t,x)\nabla y) = u\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{22.1}$$

where $A(\cdot,\cdot)$ is a uniformly positive definite matrix-valued function, which represents the *thermal conductivity* of the material occupying Ω, $y = y(t,x)$ is the *state*, and $u = u(t,x)$ is the *control*. In the system in Eq. (22.1), the *state space* is chosen as $L^2(\Omega)$, and the *control space* $L^2(\omega)$.

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It is well known (see for example Reference 4) that for any given $T > 0$ and any given non-empty open subset ω of Ω, Eq. (22.1) is null exactly controllable (*resp.* approximately controllable) in $L^2(\Omega)$, that is, for any given $y_0 \in L^2(\Omega)$ (*resp.* for any given $\varepsilon > 0$, $y_0, y_1 \in L^2(\Omega)$), one can find a control $u \in L^2[(0,T)\times\omega]$ such that the weak solution $y(\cdot) \in C\{[0,T]; L^2(\Omega)\} \cap C\{(0,T]; H_0^1(\Omega)\}$ of Eq. (22.1) satisfies $y(T) = 0$ (*resp.* $|y(T) - y_1|_{L^2(\Omega)} \le \varepsilon$). However, because of the smoothing effect of solutions for heat equations, exact controllability for Eq. (22.1) is impossible (i.e., the above ε may not be taken to be zero).

It is notable that in the above discussion, the controllability time T and the controller ω can be chosen as small as one likes. This is because the classical heat equation admits an infinite speed of propagation for a finite heat pulse.

However, it has been known (see References 2 and 3) for quite a long time that the property of instantaneous propagation for the heat equation is not really physical! To eliminate this paradox, a modified Fourier law was introduced [5], which results in a heat equation with memory. We refer to our recent work [10] for an updated analysis on the well posedness and the propagation speed of the heat equation with memory from a more general modified Fourier law. In particular, we show in Reference 10 that the heat equation with hyperbolic memory kernel admits a finite speed of propagation for heat pulses. Hence, this is a more realistic model for heat conduction. Especially, under suitable conditions on $a(\cdot,\cdot)$ (see Eq. (22.5)), this is the case for the following controlled system:

$$\begin{cases} y_t - \nabla \cdot \displaystyle\int_0^t a(t-s,x)\nabla y(s,x)\,ds = u\chi_\omega \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0) = y_0 \quad \text{in } \Omega. \end{cases} \tag{22.2}$$

The same as in the system in Eq. (22.1), the *state* and *control spaces* for Eq. (22.2) will be chosen as $L^2(\Omega)$ and $L^2(\omega)$, respectively. We refer to Reference 10 for the well posedness of Eq. (22.2).

This chapter is devoted to the study of the (instantaneous) exact controllability of Eq. (22.2), which means that, for any given $y_0, y_1 \in L^2(\Omega)$, there is a control $u \in L^2[(0,T)\times\omega]$ such that the solution $y \in C\{[0,T]; L^2(\Omega)\}$ of Eq. (22.2) satisfies

$$y(T) = y_1 \quad \text{in } \Omega. \tag{22.3}$$

Because of its hyperbolic nature, we will show the exact controllability of Eq. (22.2) under suitable conditions on the waiting time T and the controller ω. On the other hand, because of its finite propagation speed, it is clear that the exact controllability of Eq. (22.2) is impossible unless T is large enough. Also, a Gaussian beam construction of highly localized solutions (in ω) for the dual system of Eq. (22.2) shows that the exact controllability of Eq. (22.2) is impossible without geometric conditions on the controller ω (the detailed analysis will be given elsewhere). Consequently, our results show that the controllability property of Eq. (22.2) is very similar to that of the usual wave equation and significantly different from that of Eq. (22.1).

22.2 Main Results

For any $\varepsilon > 0$ and $S \subseteq \mathbb{R}^n$, we set $\mathcal{O}_\varepsilon(S) \stackrel{\triangle}{=} \{x \in \mathbb{R}^n \mid |x-y| < \varepsilon \text{ for some } y \in S\}$. Fix a function $d = d(\cdot) \in C^2(\overline{\Omega})$. Denote

$$\begin{cases} \Gamma_0 = \{x \in \partial\Omega \mid [\nabla d(x)] \cdot \nu(x) > 0\}, \\ \omega = \mathcal{O}_{\varepsilon_0}(\Gamma_0) \cap \Omega, \end{cases} \tag{22.4}$$

where $\nu(x)$ is the unit out normal vector of Ω at $x \in \partial\Omega$, and $\varepsilon_0 > 0$ is a given (small) real number.

Our exact controllability result for Eq. (22.2) is stated as follows.

THEOREM 22.1

Let ω be given by Eq. (22.4). *Assume that the kernel $a(\cdot,\cdot)$ satisfies*

$$\begin{cases} a(t,x) \in C^3([0,+\infty)\times\overline{\Omega}), \\ m_0 \le a(0,x) \le m_1, \quad \forall\, x \in \Omega \end{cases} \tag{22.5}$$

for some positive constants $m_1 \ge m_0 > 0$. Assume $d = d(\cdot) \in C^2(\overline{\Omega})$ satisfies for some constant $\delta_0 > 0$ that

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} d_{x_i x_j}(x)\xi_i\xi_j \ge \delta_0 \sum_{i=1}^{n} \xi_i^2, \quad \forall\,(x,\xi_1,\dots,\xi_n) \in \Omega\times\mathbb{R}^n, \\ R_0 \stackrel{\triangle}{=} \min_{x\in\Omega} d(x) > 0, \\ r_0 \stackrel{\triangle}{=} \min_{x\in\Omega} |\nabla d(x)| > 0, \\ [\nabla a(0,x)]\cdot[\nabla d(x)] \le 0, \quad \forall\, x\in\Omega. \end{cases} \tag{22.6}$$

Then there exists a $T_0 > 0$, depending on Ω, $a(0,\cdot)$ and $d(\cdot)$, such that for all $T \ge T_0$, Eq. (22.2) *is exactly controllable in $L^2(\Omega)$ on $[0,T]$ by means of control $u \in L^2[(0,T)\times\omega]$.*

REMARK 22.1 For the particular case that $a(0,x)$ is a positive constant α in Ω, one can simply choose $d(x) = \frac{1}{2}|x-x_0|^2$ for any given $x_0 \in \mathbb{R}^n\setminus\overline{\Omega}$. Then Eq. (22.6) holds. In this case, from the proof of Theorems 22.1 and 22.2 below, we see that T_0 in Theorem 22.1 can be chosen as $\alpha^{-1/2}\max_{x\in\Omega}|x-x_0|$.

To prove Theorem 22.1, we need to derive an observability estimate for the following (dual) system of Eq. (22.2):

$$\begin{cases} p_t(t,x) + \nabla\cdot \displaystyle\int_t^T a(s-t,x)\nabla p(s,x)\,ds = 0 \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(T) = p_0 \quad \text{in } \Omega. \end{cases} \tag{22.7}$$

We have the following result, which has an independent interest.

THEOREM 22.2

Let the assumptions in Theorem 22.1 *hold. Then there exist two constants $T_0 > 0$ and $C > 0$, depending on Ω, $a(0,\cdot)$ and $d(\cdot)$, such that for all $T \ge T_0$, it holds that*

$$\int_\Omega p(T,x)^2 dx \le C\int_0^T\int_\omega p(t,x)^2 dt dx \tag{22.8}$$

for any solution p to Eq. (22.7).

REMARK 22.2 The same controllability (*resp.* observability) problem for Eq. (22.2) (*resp.* Eq. (22.7)) was analyzed in Reference 1 for the case of one space dimension (i.e., $n = 1$) under some technical conditions on the memory kernel $a(t,x) \equiv a(t)$ by means of Laplace transform. In this Chapter, we need to develop a different method based on Carleman-type estimate, for the argument in Reference 1 does not apply here because of the spatial dependence of the leading coefficient $a(t,x)$.

22.3 Some Preliminaries

In this section, we present some preliminary results. Let us begin with the following lemma.

LEMMA 22.1

Let $\gamma = \gamma(x) \in C^1(\mathbb{R}^n)$, $w = w(t,x) \in C^2(\mathbb{R}^{1+n})$ and $\ell = \ell(t,x) \in C^2(\mathbb{R}^{1+n})$. Let $\Psi = \Psi(x) \in C^2(\mathbb{R}^n)$, $\theta = e^\ell$, and $v = \theta w$. Then the following point-wise inequality holds:

$$\begin{aligned}
&\theta^2|\gamma w_{tt} - \Delta w|^2 \\
&\geq \left[-2\gamma\ell_t\left(\gamma v_t^2 + \sum_{j=1}^n v_{x_j}^2\right) + 4\gamma\sum_{j=1}^n \ell_{x_j}v_{x_j}v_t + 2\gamma\Psi v_t v - 2\gamma\ell_t(A+\Psi)v^2\right]_t \\
&\quad -2\sum_{j=1}^n\left\{2\sum_{i=1}^n \ell_{x_i}v_{x_i}v_{x_j} - \ell_{x_j}\sum_{i=1}^n v_{x_i}^2 - 2\gamma\ell_t v_{x_j}v_t + \Psi v_{x_j}v + \gamma\ell_{x_j}v_t^2\right. \\
&\quad \left. - \left[(A+\Psi)\ell_{x_j} + \frac{1}{2}\Psi_{x_j}\right]v^2\right\}_{x_j} \\
&\quad +2\left[-\gamma\Psi + \sum_{i=1}^n\left(\gamma\ell_{x_i}\right)_{x_i} + \gamma^2\ell_{tt}\right]v_t^2 - 4\sum_{j=1}^n\left[\gamma\ell_{tx_j} + (\gamma\ell_t)_{x_j}\right]v_{x_j}v_t \\
&\quad +4\sum_{i,j=1}^n \ell_{x_ix_j}v_{x_i}v_{x_j} + 2\left(\Psi - \sum_{i=1}^n \ell_{x_ix_i} + \gamma\ell_{tt}\right)\sum_{j=1}^n v_{x_j}^2 + Bv^2,
\end{aligned} \tag{22.9}$$

where

$$A \triangleq \gamma\left(\ell_t^2 - \ell_{tt}\right) - \sum_{j=1}^n\left(\ell_{x_j}^2 - \ell_{x_jx_j}\right) - \Psi \tag{22.10}$$

and

$$B \triangleq 2[\gamma\ell_t(A+\Psi)]_t - 2\sum_{j=1}^n\left[\ell_{x_j}(A+\Psi)\right]_{x_j} + 2A\Psi - \Delta\Psi + \Psi^2. \tag{22.11}$$

PROOF We borrow some ideas from Reference 7 (p. 124). Recall that

$$v(t,x) = \theta(t,x)w(t,x), \quad (t,x) \in \mathbb{R}^{1+n}. \tag{22.12}$$

Hence,

$$\begin{cases} w_t = \theta^{-1}(v_t - \ell_t v), \quad w_{tt} = \theta^{-1}\left[v_{tt} - 2\ell_t v_t + \left(\ell_t^2 - \ell_{tt}\right)v\right], \\ w_{x_jx_j} = \theta^{-1}\left[v_{x_jx_j} - 2\ell_{x_j}v_{x_j} + \left(\ell_{x_j}^2 - \ell_{x_jx_j}\right)v\right], \quad j = 1, \ldots, n, \end{cases} \tag{22.13}$$

leads to

$$\begin{aligned}
&\theta^2|\gamma w_{tt} - \Delta w|^2 \\
&= \left|\gamma\left[v_{tt} - 2\ell_t v_t + \left(\ell_t^2 - \ell_{tt}\right)v\right] - \sum_{j=1}^n\left[v_{x_jx_j} - 2\ell_{x_j}v_{x_j} + \left(\ell_{x_j}^2 - \ell_{x_jx_j}\right)v\right]\right|^2 \\
&= |I_1 + I_2 + I_3|^2,
\end{aligned} \tag{22.14}$$

where (recall A is defined in Eq. (22.10))

$$I_1 \stackrel{\triangle}{=} \gamma v_{tt} - \sum_{j=1}^{n} v_{x_j x_j} + Av, \quad I_2 \stackrel{\triangle}{=} -2\gamma \ell_t v_t + 2\sum_{j=1}^{n} \ell_{x_j} v_{x_j}, \quad I_3 \stackrel{\triangle}{=} \Psi v. \tag{22.15}$$

Then

$$\begin{aligned} \theta^2 |\gamma u_{tt} - \Delta u|^2 &= I_1^2 + I_2^2 + I_3^2 + 2(I_1 I_2 + I_2 I_3 + I_1 I_3) \\ &\geq I_3^2 + 2(I_1 I_2 + I_2 I_3 + I_1 I_3). \end{aligned} \tag{22.16}$$

Recalling that γ is independent of t, we have

$$\begin{aligned} I_1 I_2 &= 2\left(\gamma v_{tt} - \sum_{j=1}^{n} v_{x_j x_j} + Av\right)\left(-\gamma \ell_t v_t + \sum_{j=1}^{n} \ell_{x_j} v_{x_j}\right) \\ &= -\gamma^2 \ell_t \left(v_t^2\right)_t + 2\sum_{j=1}^{n} (\gamma \ell_t v_{x_j} v_t)_{x_j} - \gamma \ell_t \sum_{j=1}^{n} \left(v_{x_j}^2\right)_t - 2\sum_{j=1}^{n} (\gamma \ell_t)_{x_j} v_{x_j} v_t \\ &\quad - \gamma \ell_t A (v^2)_t + 2\sum_{j=1}^{n} \left(\gamma \ell_{x_j} v_{x_j} v_t\right)_t - \gamma \sum_{j=1}^{n} \ell_{x_j} \left(v_t^2\right)_{x_j} - 2\gamma \sum_{j=1}^{n} \ell_{t x_j} v_{x_j} v_t \\ &\quad - 2\sum_{i,j=1}^{n} \left[\left(\ell_{x_i} v_{x_i} v_{x_j}\right)_{x_j} - \ell_{x_i x_j} v_{x_i} v_{x_j}\right] + \sum_{i,j=1}^{n} \ell_{x_i} \left(v_{x_j}^2\right)_{x_i} + A\sum_{j=1}^{n} \ell_{x_j} (v^2)_{x_j} \\ &= \left[-\gamma \ell_t \left(\gamma v_t^2 + \sum_{j=1}^{n} v_{x_j}^2 + Av^2\right) + 2\gamma \sum_{j=1}^{n} \ell_{x_j} v_{x_j} v_t\right]_t \\ &\quad - \sum_{j=1}^{n} \left[2\sum_{i=1}^{n} \ell_{x_i} v_{x_i} v_{x_j} - \ell_{x_j} \sum_{i=1}^{n} v_{x_i}^2 - 2\gamma \ell_t v_{x_j} v_t + \gamma \ell_{x_j} v_t^2 - A\ell_{x_j} v^2\right]_{x_j} \\ &\quad - 2\sum_{j=1}^{n} \left[\gamma \ell_{t x_j} + (\gamma \ell_t)_{x_j}\right] v_{x_j} v_t + \left(\sum_{i=1}^{n} \left(\gamma \ell_{x_i}\right)_{x_i} + \gamma^2 \ell_{tt}\right) v_t^2 \\ &\quad + 2\sum_{i,j=1}^{n} \ell_{x_i x_j} v_{x_i} v_{x_j} - \left(\sum_{i=1}^{n} \ell_{x_i x_i} - \gamma \ell_{tt}\right) \sum_{j=1}^{n} v_{x_j}^2 \\ &\quad - \left[\sum_{i=1}^{n} \left(A\ell_{x_i}\right)_{x_i} - (\gamma \ell_t A)_t\right] v^2. \end{aligned} \tag{22.17}$$

On the other hand, because Ψ and γ are independent of t, one has

$$\begin{aligned} I_1 I_3 &= \left(\gamma v_{tt} - \sum_{j=1}^{n} v_{x_j x_j} + Av\right) \Psi v \\ &= (\gamma \Psi v_t v)_t - \gamma \Psi v_t^2 - \sum_{j=1}^{n} \left(\Psi v_{x_j} v - \frac{1}{2} \Psi_{x_j} v^2\right)_{x_j} \\ &\quad + \Psi \sum_{j=1}^{n} v_{x_j}^2 + \left(A\Psi - \frac{\Delta \Psi}{2}\right) v^2. \end{aligned} \tag{22.18}$$

Furthermore,

$$I_2 I_3 = 2\left[-\gamma\ell_t v_t + \sum_{j=1}^n \ell_{x_j} v_{x_j}\right]\Psi v = -\gamma\ell_t\Psi(v^2)_t + \Psi\sum_{j=1}^n \ell_{x_j}(v^2)_{x_j}$$
$$= \left(-\gamma\ell_t\Psi v^2\right)_t + \sum_{j=1}^n\left(\Psi\ell_{x_j}v^2\right)_{x_j} + \left[\gamma\ell_{tt}\Psi - \sum_{j=1}^n(\ell_{x_j}\Psi)_{x_j}\right]v^2. \tag{22.19}$$

Combining Eqs. (22.16)–(22.19), we obtain Eq. (22.9). This completes the proof of Lemma 22.1. ∎

COROLLARY 22.1
Let $\lambda > 0, \delta > 0$ and $\varepsilon > 0$. Let $\gamma = \gamma(x) \in C^1(\mathbb{R}^n)$ and $d = d(x) \in C^2(\mathbb{R}^n)$ satisfy $(\nabla\gamma)\cdot(\nabla d) \geq 0$. Put

$$\begin{cases} \ell = \ell(t,x) = \lambda\left[d(x) - \dfrac{\delta}{2}t^2\right] \\ \Psi = (\Delta d - \gamma\delta - \varepsilon)\lambda. \end{cases} \tag{22.20}$$

Let $w \in C^2(\mathbb{R}^{1+n})$, and $v = \theta w$ with $\theta = e^{\ell}$. Then

$$\theta^2|\gamma w_{tt} - \Delta w|^2$$
$$\geq \left[-2\gamma\ell_t\left(\gamma v_t^2 + \sum_{j=1}^n v_{x_j}^2\right) + 4\gamma\sum_{j=1}^n \ell_{x_j}v_{x_j}v_t + 2\gamma\Psi v_t v - 2\gamma\ell_t(A+\Psi)v^2\right]_t$$
$$-2\sum_{j=1}^n\left[2\sum_{i=1}^n \ell_{x_i}v_{x_i}v_{x_j} - \ell_{x_j}\sum_{i=1}^n v_{x_i}^2 - 2\gamma\ell_t v_{x_j}v_t + \Psi v_{x_j}v + \gamma\ell_{x_j}v_t^2 - (A+\Psi)\ell_{x_j}v^2\right]_{x_j}$$
$$+2\lambda\left[\gamma\varepsilon v_t^2 + 2\delta t\sum_{j=1}^n \gamma_{x_j}v_{x_j}v_t + 2\sum_{i,j=1}^n d_{x_ix_j}v_{x_i}v_{x_j} - (2\delta\gamma+\varepsilon)\sum_{j=1}^n v_{x_j}^2\right] + Bv^2, \tag{22.21}$$

where

$$A \overset{\triangle}{=} \lambda^2(\gamma\delta^2t^2 - |\nabla d|^2) + \lambda(\delta\gamma + \Delta d) - \Psi \tag{22.22}$$

and

$$B \overset{\triangle}{=} 2\lambda^3\left\{2\sum_{i,j=1}^n d_{x_ix_j}d_{x_i}d_{x_j} - [2\delta\gamma^2 + (\nabla\gamma)\cdot(\nabla d)]\delta^2t^2 + (|\nabla d|^2 - \gamma\delta^2t^2)(2\delta\gamma+\varepsilon)\right\} + O(\lambda^2). \tag{22.23}$$

PROOF Using Lemma 22.1 with ℓ and Ψ given by (22.20), one can obtain Corollary 22.1 immediately. ∎

Next, we need the following lemma.

LEMMA 22.2
Let $b(\cdot\,,\cdot) \in C^1(\overline{Q})$. Then for any $w(\cdot\,,\cdot) \in C\{[0,T]; L^2(\Omega)\}$, there is a unique solution $z(\cdot\,,\cdot) \in C\{[0,T]; L^2(\Omega)\}$

$$z(t,x) = w(t,x) + \int_0^t b(t-s,x)z(s,x)\,ds, \quad \text{a.e. } (t,x) \in Q. \tag{22.24}$$

Moreover, there exists a $\beta(\cdot\,,\cdot) \in C^1(\overline{Q})$, such that the solution $z(\cdot\,,\cdot)$ of Eq. (22.24) admits the following representation:

$$z(t,x) = w(t,x) + \int_0^t \beta(t-s,x)w(s,x)\,ds, \quad \forall (t,x) \in (0,T)\times\Omega. \tag{22.25}$$

Furthermore, for any $\eta \in L^\infty(Q)$ satisfying

$$|\eta(t,x)| \le |\eta(s,x)|, \quad \forall\, 0 \le s \le t \le T, x \in \Omega, \tag{22.26}$$

it holds that

$$|\eta z|_{L^2(Q)} \le C|\eta w|_{L^2(Q)}, \tag{22.27}$$

where $C = C(b)$ is a constant, independent of η.

PROOF First, Eq. (22.24) is a Volterra integral equation with $x \in \Omega$ being a parameter. Thus, by a standard theory, we obtain the first two conclusions.

We now show Eq. (22.27). For any $r \in [0,T]$, by Eqs. (22.24) and (22.26), we have

$$\begin{aligned}
|\eta z|^2_{L^2[\Omega\times(0,r)]} &\le |\eta w|^2_{L^2[\Omega\times(0,r)]} + \int_{\Omega\times(0,r)} \eta(t,x)^2 \left| \int_0^t b(t-s,x)z(s,x)ds \right|^2 dtdx \\
&\le |\eta w|^2_{L^2[\Omega\times(0,r)]} + \int_{\Omega\times(0,r)} \int_0^t |b(t-s,x)|^2 ds \int_0^t |\eta(s,x)z(s,x)|^2 dsdtdx \\
&\le |\eta w|^2_{L^2[\Omega\times(0,r)]} + T|b|^2_{L^\infty(Q)} \int_0^r |\eta z|^2_{L^2[\Omega\times(0,t)]}\,dt.
\end{aligned} \tag{22.28}$$

Then Eq. (22.27) follows from Gronwall's inequality. ∎

Finally, similar to the proof of Lemma 3.3 in Reference 11, we have the following identity.

LEMMA 22.3

Let $h \triangleq (h^1, \ldots, h^n) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^1, $\beta = \beta(x) \in C^1(\mathbb{R}^n)$. Then for any $q \in C^2(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\begin{aligned}
&\nabla \cdot \left\{ 2(h\cdot\nabla q)(\nabla q) + h\left[\beta q_t^2 - \sum_{i=1}^n q_{x_i}^2\right]\right\} \\
&\quad = -2(\beta q_{tt} - \Delta q)h\cdot\nabla q + (2\beta q_t h\cdot\nabla q)_t - 2\beta q_t h_t \cdot \nabla q \\
&\qquad + \left\{[\nabla\cdot(\beta h)]q_t^2 - (\nabla\cdot h)\sum_{i=1}^n q_{x_i}^2\right\} + 2\sum_{i,j=1}^n \left(\frac{\partial h^j}{\partial x_i} q_{x_i} q_{x_j}\right).
\end{aligned} \tag{22.29}$$

22.4 Proof of Theorem 22.2

This section is devoted to a proof of Theorem 22.2. Some ideas from References 6, 9, and 11 are adopted. The proof is split into several steps.

1. We shall reduce Eq. (22.7) to the following forward equation on the variable $z(t,x) \stackrel{\triangle}{=} p(T-t,x)$:

$$\begin{cases} z_t(t,x) - \nabla \cdot \int_0^t a(t-s,x)\nabla z(s,x)\,ds = 0 \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(0) = z_0 \stackrel{\triangle}{=} p_0 \quad \text{in } \Omega. \end{cases} \tag{22.30}$$

Thus, the inequality of Eq. (22.8) reduces to

$$\int_\Omega z_0^2(x)\,dx \le C \int_0^T \int_\omega z^2(t,x)\,dtdx. \tag{22.31}$$

Denote

$$q(t,x) \stackrel{\triangle}{=} \int_0^t a(t-s,x)z(s,x)\,ds, \quad (t,x) \in Q. \tag{22.32}$$

Then, we get

$$z_t - \Delta q + \nabla \cdot \int_0^t z(s,x)\nabla a(t-s,x)\,ds = 0 \quad \text{in } Q. \tag{22.33}$$

However, by Eq. (22.32), we have

$$q_t(t,x) = \gamma^{-1} z(t,x) + \int_0^t a_t(t-s,x)z(s,x)\,ds, \quad (t,x) \in Q, \tag{22.34}$$

where $\gamma = \gamma(x) \stackrel{\triangle}{=} \frac{1}{a(0,x)}$. By the second conclusion in Lemma 22.2 and noting $q(0) = 0$, one can find a $\beta \in C^1(\overline{Q})$ such that

$$\begin{aligned} z(t,x) &= \gamma q_t(t,x) + \int_0^t \beta(t-s,x)q_t(s,x)\,ds \\ &= \gamma q_t(t,x) + \beta(0,x)q(t,x) + \int_0^t \beta_t(t-s,x)q(s,x)\,ds, (t,x) \in Q. \end{aligned} \tag{22.35}$$

From Eq. (22.34), one also gets

$$z(t,x) = \gamma \left[q_t(t,x) - \int_0^t a_t(t-s,x)z(s,x)\,ds \right], \quad (t,x) \in Q. \tag{22.36}$$

By Eqs. (22.30), (22.33), (22.35), and (22.36), and noting again $q(0) = 0$, we obtain

$$\begin{cases} \gamma q_{tt} - \Delta q = H^z(t,x) \quad \text{in } Q, \\ q = 0 \quad \text{on } \Sigma, \\ q(0) = 0, \ q_t(0) = \gamma^{-1} z(0) \quad \text{in } \Omega, \end{cases} \tag{22.37}$$

where

$$\begin{aligned} H^z(t,x) \stackrel{\triangle}{=} \ & \gamma a_t(0,x)z + \gamma \int_0^t a_{tt}(t-s,x)z(s,x)\,ds - \nabla \cdot [\gamma q(t,x)\nabla a(0,x)] \\ & -\nabla \cdot \int_0^t q(s,x) \Big[\gamma \nabla a_t(t-s,x) + \beta(0,x)\nabla a(t-s,x) \\ & + \int_s^t \beta_t(\tau - s)\nabla a(t-\tau,x)\,d\tau \Big] ds. \end{aligned} \tag{22.38}$$

2. Denote $R_1 \overset{\triangle}{=} \max_{x\in\Omega} d(x)$. By our assumption in Eq. (22.6), for any $c \in (0, R_0/4)$, we can find two small constants $\varepsilon > 0$ and $\delta > 0$ and a big constant $T > 0$ such that (recall Eq. [22.6] for R_0, r_0, and δ_0)

$$R_1 < c + \frac{\delta}{2}T^2, \tag{22.39}$$

$$\gamma\varepsilon\xi_0^2 + 2\delta t\sum_{j=1}^n \gamma_{x_j}\xi_j\xi_0 + 2\sum_{i,j=1}^n d_{x_ix_j}\xi_i\xi_j - (2\delta\gamma+\varepsilon)\sum_{j=1}^n \xi_j^2 \geq \frac{\varepsilon}{2}\left(\gamma\xi_0^2 + \sum_{j=1}^n \xi_j^2\right), \quad \forall\ (t, x, \xi_0, \ldots, \xi_n) \in (0, T)\times\Omega\times\mathbb{R}^{1+n}, \tag{22.40}$$

and

$$2\sum_{i,j=1}^n d_{x_ix_j}d_{x_i}d_{x_j} - [2\delta\gamma^2 + (\nabla\gamma)\cdot(\nabla d)]\delta^2t^2 + (|\nabla d|^2 - \gamma\delta^2t^2)(2\delta\gamma+\varepsilon) \geq \delta_0 r_0^2, \quad \forall\ (t, x) \in (0, T)\times\Omega. \tag{22.41}$$

For any $b > 0$, set

$$\begin{cases} \varphi(t, x) = d(x) - \dfrac{\delta}{2}t^2, & (t, x) \in \mathbb{R}^{n+1} \\ Q(b) = \{(t, x) \in (0, \infty)\times\Omega \mid \varphi(t, x) > b\}. \end{cases} \tag{22.42}$$

Clearly, $Q(b)$ is decreasing in b, and there is a $T_1 \in (0, T)$, which is close to T, such that

$$Q(c) \subset (0, T_1)\times\Omega. \tag{22.43}$$

Noting that because $\{0\}\times\Omega \subset Q(c)$, by $c \in (0, R_0/4)$, thus there exists a $T_0 \in (0, T_1)$ such that

$$(0, T_0)\times\Omega \subset Q(3c) \subset Q(2c) \subset Q(c). \tag{22.44}$$

Now, we choose a smooth $\zeta(\cdot) \in C^\infty(\overline{Q}; [0, 1])$ such that

$$\zeta(t, x) = \begin{cases} 1, & \text{for } (t, x) \in Q(3c), \\ 0, & \text{for } (t, x) \in Q/Q(2c). \end{cases} \tag{22.45}$$

Put

$$w(t, x) = \zeta(t, x)q(t, x), \quad (t, x) \in Q, \tag{22.46}$$

where $q(\cdot)$ is defined in Eq. (22.32). Then by Eq. (22.37), we see that $u(\cdot)$ satisfies

$$\begin{cases} \gamma w_{tt} - \Delta w = F^z(t, x), & \text{in } Q, \\ w = 0, & \text{on } \Sigma, \\ w(0) = 0,\ w_t(0) = \gamma^{-1}z(0), & \text{in } \Omega, \end{cases} \tag{22.47}$$

where (recall Eq. (22.38) for $H^z(t, x)$)

$$F^z(t, x) = \zeta(t, x)H^z(t, x) + \left[\gamma\zeta_{tt}(t, x) - \sum_{i=1}^n \zeta_{x_ix_i}(t, x)\right]q(t, x) + 2\left[\gamma\zeta_t(t, x)q_t(t, x) - \sum_{i=1}^n \zeta_{x_i}(t, x)q_{x_i}(t, x)\right]. \tag{22.48}$$

3. Let us use Corollary 22.1 with $\gamma \overset{\triangle}{=} \frac{1}{a(0,x)}$, δ given in Eqs. (22.39) to (22.41), w defined by (22.46), and $v = \theta w = e^{\ell} w$ (recall Eq. (22.41) for ℓ). Integrating Eq. (22.21) on Q, by Eqs. (22.43)–(22.47) and using integration by parts, noting Eqs. (22.40) and (22.41), we conclude that there is a constant $\lambda_1 > 0$ such that for any $\lambda \geq \lambda_1$, it holds

$$\begin{aligned} &\varepsilon\lambda \int_{Q(c)} \left(\gamma v_t^2 + \sum_{i=1}^n v_{x_i}^2\right) dtdx + \delta_0 r_0^2 \lambda^3 \int_{Q(c)} v^2 dtdx \\ &\leq C\lambda \int_{\Sigma_0^1} \theta^2 \left|\frac{\partial v}{\partial \nu}\right|^2 d\Sigma_0^1 + \int_{Q(c)} \theta^2 |F^z|^2 dtdx \\ &\leq C\lambda e^{C\lambda} \int_{\Sigma_0^1} \left|\frac{\partial q}{\partial \nu}\right|^2 d\Sigma_0^1 + \int_{Q(c)} |F^z|^2 dtdx, \end{aligned} \tag{22.49}$$

where $\Sigma_0^1 \overset{\triangle}{=} (0, T_1) \times \Gamma_0$, F^z is defined by Eq. (22.48). However, by Eqs. (22.48) and (22.38), we have

$$\begin{aligned} &\int_{Q(c)} \theta^2 |F^z|^2 dtdx \\ &\leq C \left[\int_{Q(c)} \theta^2 z^2 dtdx + \int_{Q(c)} \theta^2 \int_0^t (|z(s,x)|^2 + |p(s,x)|^2 + |\nabla p(s,x)|^2)\, dsdtdx \right. \\ &\quad \left. + \int_{Q(c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx\right]. \end{aligned} \tag{22.50}$$

Note that

$$\begin{aligned} &\int_{Q(c)} \theta^2 \int_0^t [|z(s,x)|^2 + |p(s,x)|^2 + |\nabla p(s,x)|^2]\, dsdtdx \\ &\leq C \int_{Q(c)} \theta^2 (z^2 + p^2 + |\nabla p|^2)\, dtdx. \end{aligned} \tag{22.51}$$

Thus, by Eqs. (22.50) and (22.51), we get

$$\int_{Q(c)} \theta^2 |F^z|^2 dtdx \leq C \left[\int_{Q(c)} \theta^2 z^2 dtdx + \int_{Q(c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx\right]. \tag{22.52}$$

Now, denote

$$\eta(t,x) = \chi_{Q(c)}(t,x)\theta(t,x), \quad (t,x) \in Q, \tag{22.53}$$

where $\chi_{Q(c)}$ is the characteristic function of the set $Q(c)$. Then, by Eq. (22.42), it is easy to check that

$$\eta^2(t,x) \leq \eta^2(s,x), \quad \forall\, 0 \leq s \leq t \leq T, x \in \Omega. \tag{22.54}$$

Thus, by Eq. (22.27) in Lemma 22.2 and Eq. (22.36), we get

$$\int_{Q(c)} \theta^2 z^2 dtdx = \int_Q \eta^2 z^2 dtdx \leq C \int_Q \eta^2 |q_t|^2 dtdx = C \int_{Q(c)} \theta^2 |q_t|^2 dtdx. \tag{22.55}$$

Thus, combining Eqs. (22.49), (22.52), and (22.55), we arrive at

$$\begin{aligned} &\lambda \int_{Q(c)} \left(\gamma v_t^2 + \sum_{i=1}^n v_{x_i}^2\right) dtdx + \lambda^3 \int_{Q(c)} v^2 dtdx \\ &\leq C \left[\lambda e^{C\lambda} \int_{\Sigma_0^1} \left|\frac{\partial q}{\partial \nu}\right|^2 d\Sigma_0^1 + \int_{Q(c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx\right]. \end{aligned} \tag{22.56}$$

Consequently, by $w = \theta^{-1}v$ and Eq. (22.41), and by Eq. (22.56), we get

$$\begin{aligned}
&\lambda \int_{Q(c)} \theta^2 \left(\gamma w_t^2 + \sum_{i=1}^n w_{x_i}^2\right) dtdx + \lambda^3 \int_{Q(c)} \theta^2 w^2 dtdx \\
&\le C \left[\lambda \int_{Q(c)} \left(\gamma v_t^2 + \sum_{i=1}^n v_{x_i}^2\right) dtdx + \lambda^3 \int_{Q(c)} v^2 dtdx\right] \\
&\le C \left[\lambda e^{C\lambda} \int_{\Sigma_0^1} \left|\frac{\partial q}{\partial \nu}\right|^2 d\Sigma_0^1 + \int_{Q(c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx\right], \quad \forall\, \lambda \ge \lambda_1. \quad (22.57)
\end{aligned}$$

Thus, by Eqs. (22.45) and (22.46), we conclude that

$$\begin{aligned}
&\lambda \int_{Q(3c)} \theta^2 \left(\gamma |q_t|^2 + \sum_{i=1}^n |q_{x_i}|^2\right) dtdx + \lambda^3 \int_{Q(3c)} \theta^2 q^2 dtdx \\
&\le C \left[\lambda e^{C\lambda} \int_{\Sigma_0^1} \left|\frac{\partial q}{\partial \nu}\right|^2 d\Sigma_0^1 + \int_{Q(c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx\right], \quad \forall\, \lambda \ge \lambda_1. \quad (22.58)
\end{aligned}$$

Now, note that by Eq. (22.42), we have

$$\begin{aligned}
&\int_{Q(c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx \\
&= \left[\int_{Q(3c)} + \int_{Q(c)\setminus Q(3c)}\right] \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx \\
&\le \int_{Q(3c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx \\
&\quad + e^{6c\lambda} \int_{Q(c)\setminus Q(3c)} \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx \\
&\le \int_{Q(3c)} \theta^2 \left(q^2 + q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx + Ce^{6c\lambda} \int_Q \left(q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx, \quad (22.59)
\end{aligned}$$

where we have used Poincaré's inequality in the last inequality of Eq. (22.59). Thus, by Eqs. (22.58) and (22.59), one can find a $\lambda_2 \ge \lambda_1$, such that for any $\lambda \ge \lambda_2$, it holds that

$$\begin{aligned}
&\lambda \int_{Q(3c)} \theta^2 \left(\gamma q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx \\
&\le C \left[\lambda e^{C\lambda} \int_{\Sigma_0^1} \left|\frac{\partial q}{\partial \nu}\right|^2 d\Sigma_0^1 + e^{6c\lambda} \int_Q \left(q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx\right], \quad \forall\, \lambda \ge \lambda_1. \quad (22.60)
\end{aligned}$$

However, by Eqs. (22.42) and (22.44), we have

$$\begin{aligned}
\int_{Q(3c)} \theta^2 \left(\gamma q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx &\ge e^{6c\lambda} \int_{Q(3c)} \left(\gamma q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx \\
&\ge e^{6c\lambda} \int_0^{T_0} \int_\Omega \left(\gamma q_t^2 + \sum_{i=1}^n q_{x_i}^2\right) dtdx. \quad (22.61)
\end{aligned}$$

Thus, by Eqs. (22.60) and (22.61), we conclude that

$$\lambda \int_0^{T_0} E(t)\,dt \le C \left[\lambda e^{C\lambda} \int_{\Sigma_0^1} \left| \frac{\partial q}{\partial \nu} \right|^2 d\Sigma_0^1 + \int_0^T E(t)\,dt \right], \quad \forall\, \lambda \ge \lambda_2, \tag{22.62}$$

where

$$E(t) = \frac{1}{2} \int_\Omega \left(\gamma q_t^2 + \sum_{i=1}^n q_{x_i}^2 \right) dx. \tag{22.63}$$

4. We now estimate "$\int_{\Sigma_0^1} |\frac{\partial q}{\partial \nu}|^2 d\Sigma_0^1$". For this purpose, let us use Lemma 22.3. Choose $h_0 = h_0(x) \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that $h_0 = \nu$ on Γ (cf. Reference 8), and choose $h_i = h_i(x) \in C^\infty(\overline{\Omega}; [0, 1])$ $(i = 1, 2)$ such that

$$\begin{cases} 0 \le h_i(x) \le 1, & x \in \overline{\Omega}, \\ h_i(x) \equiv 1, & x \in \omega_i, \\ h_i(x) \equiv 0, & x \in \Omega \backslash \omega_{i+1}, \end{cases} \tag{22.64}$$

where $\omega_1 \subset \omega_2 \subset \omega_3 \equiv \omega (\subset \Omega)$ are all intersection of some neighborhood of Γ_0 and Ω. Fix $T_2 \in (T_1, T)$. Now, we apply identity of Eq. (22.50) (in Lemma 22.3) with q defined by Eq. (22.32), $\beta = \gamma$ and

$$h = h(t, x) = g_1(t) h_0(x) h_1(x), \quad (t, x) \in Q, \tag{22.65}$$

where $g_1(\cdot) \in C^\infty([0, T]; \mathbb{R})$ satisfies

$$\begin{cases} g_1(t) \equiv 1, & t \in [0, T_1], \\ g_1(t) > 0, & t \in (T_1, T_2), \\ g_1(T_2) = 0. \end{cases} \tag{22.66}$$

Integrating Eq. (22.29) on $(0, T_2) \times \Omega$, using Eq. (22.37) and Eqs. (22.64) to (22.66), and by Green's formula, we obtain

$$\int_0^{T_2} \int_{\Gamma_0} g_1 \left| \frac{\partial q}{\partial \nu} \right|^2 d\Gamma_0\, dt \le C \int_0^{T_2} \int_{\omega_2} z^2 dt dx + \int_0^{T_2} \int_{\omega_2} \left(q_t^2 + \sum_{i=1}^n q_{x_i}^2 \right) dt dx. \tag{22.67}$$

On the other hand, denote (recall Eq. (22.64) for h_2)

$$\eta = \eta(t, x) \stackrel{\triangle}{=} g_2(t) h_2(x), \tag{22.68}$$

where $g_2(\cdot) \in C^\infty([0, T]; \mathbb{R})$ satisfies

$$\begin{cases} g_2(t) \equiv 1, & t \in [0, T_2], \\ g_2(t) > 0, & t \in (T_2, T), \\ g_2(T) = 0. \end{cases} \tag{22.69}$$

By Eqs. (22.37), (22.64), and (22.68) to (22.69), we get

$$\begin{aligned} \int_Q \eta q H^z(t, x) dt dx &= \int_Q \eta q (\gamma q_{tt} - \Delta q) dt dx \\ &= -\int_Q \gamma q_t (\eta_t q + \eta q_t) dt dx + \int_Q \eta |\nabla q|^2 dt dx + \int_Q (\nabla q) \cdot (\nabla \eta) q dt dx \\ &\ge -\int_Q \gamma q_t (\eta_t q + \eta q_t) dt dx + \frac{1}{2} \int_Q \eta |\nabla q|^2 dt dx - C \int_0^T \int_\omega q^2 dt dx. \end{aligned} \tag{22.70}$$

Thus, by Eqs. (22.70) and (22.64), we get

$$\begin{aligned}\int_0^{T_2}\int_{\omega_2}\sum_{i=1}^{n} q_{x_i}^2 dtdx &\le \int_Q \eta|\nabla q|^2 dtdx \\ &\le C\left[\int_0^T\int_\omega \left(q_t^2+q^2\right) dtdx + \int_0^T\int_\omega z^2 dtdx\right]. \end{aligned}\tag{22.71}$$

Combining Eqs. (22.67) and (22.71), and noting Eqs. (22.32) and (22.34), we end up with

$$\begin{aligned}&\int_0^{T_2}\int_{\Gamma_0} g_1\left|\frac{\partial q}{\partial\nu}\right|^2 d\Gamma_0 dt \\ &\le C\left[\int_0^T\int_\omega z^2 dtdx + \int_0^T\int_\omega \left(q_t^2+q^2\right) dtdx\right] \le C\int_0^T\int_\omega z^2 dtdx. \end{aligned}\tag{22.72}$$

Thus, by Eq. (22.67), we conclude that

$$\int_{\Sigma_0^1}\left|\frac{\partial q}{\partial\nu}\right|^2 d\Sigma_0^1 \le C\int_0^T\int_\omega z^2 dtdx. \tag{22.73}$$

Combining Eqs. (22.62) and (22.73), we arrive at

$$\lambda\int_0^{T_0} E(t)\,dt \le C\left[\lambda e^{C\lambda}\int_0^T\int_\omega z^2 dtdx + \int_0^T E(t)\,dt\right], \quad \forall\,\lambda\ge\lambda_2. \tag{22.74}$$

5. Let us complete the proof of Theorem 22.2. First of all, multiplying the first equation of (22.37) by q_t, integrating it on $(0,t)\times\Omega$, by Eqs. (22.63), (22.38), and (22.35), one gets

$$E(t)-E(0)\le C\int_0^t E(s)\,ds, \quad \forall\, t\in[0,T]. \tag{22.75}$$

Thus, by Gronwall's inequality, we have

$$E(t)\le CE(0), \quad \forall\, t\in[0,T]. \tag{22.76}$$

Next, applying the usual energy estimate to Eq. (22.37) and noting its time reversibility, one gets

$$E(0)\le C\left[E(t)+\int_0^t\int_\Omega |H^z(t,x)|^2 dxdt\right], \quad \forall\, t\in[0,T]. \tag{22.77}$$

Now, by Eqs. (22.38) and (22.35), recalling Eq. (22.63), and by Eq. (22.27) in Lemma 22.2, we get

$$\int_0^t\int_\Omega |H^z(s,x)|^2 dxds \le C\int_0^t E(s)\,ds, \quad \forall\, t\in[0,T]. \tag{22.78}$$

Combining Eqs. (22.76) and (22.78), we have

$$\int_0^t\int_\Omega |H^z(s,x)|^2 dsdx \le C_1 tE(0), \quad \forall\, t\in[0,T]. \tag{22.79}$$

Thus, by Eqs. (22.77) and (22.79), there exists a sufficiently small $t_0\in(0,T_0)$ such that

$$E(0)\le CE(t), \quad \forall\, t\in[0,t_0]. \tag{22.80}$$

Finally, by Eqs. (22.74), (22.76), and (22.80), we see that

$$\lambda E(0) \le C_2 \left[\lambda e^{C\lambda} \int_0^T \int_\omega z^2 dt dx + E(0)\right], \quad \forall\, \lambda \ge \lambda_2. \tag{22.81}$$

Thus, if we take $\lambda_3 = \max(\lambda_2, C_2 + 1)$, we get

$$E(0) \le C \int_0^T \int_\omega z^2 dt dx. \tag{22.82}$$

Recalling Eqs. (22.63) and (22.37), we see that Eq. (22.82) implies Eq. (22.31) immediately, which completes the proof of Theorem 22.2.

22.5 Proof of Theorem 22.1

We use the duality argument. Fix an initial datum $y_0 \in L^2(\Omega)$ and a final datum $y_1 \in L^2(\Omega)$. First, we solve

$$\begin{cases} v_t - \nabla \cdot \int_0^t a(t-s,x)\nabla v(s,x)\,ds = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(0) = y_0 & \text{in } \Omega. \end{cases} \tag{22.83}$$

For any $p_0 \in L^2(\Omega)$, we solve

$$\begin{cases} p_t + \nabla \cdot \int_t^T a(s-t,x)\nabla p(s,x)\,ds = 0 & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(T) = p_0 & \text{in } \Omega \end{cases} \tag{22.84}$$

and

$$\begin{cases} \eta_t - \nabla \cdot \int_0^t a(t-s,x)\nabla \eta(s,x)\,ds = \chi_\omega(x) p(t,x) & \text{in } Q, \\ \eta = 0 & \text{on } \Sigma, \\ \eta(0) = 0 & \text{in } \Omega. \end{cases} \tag{22.85}$$

Then, we define a linear and continuous operator $\Lambda : L^2(\Omega) \to L^2(\Omega)$ by

$$\Lambda(p_0) \stackrel{\triangle}{=} \eta(T), \tag{22.86}$$

where $\eta \in C\{[0,T]; L^2(\Omega)\}$ is the weak solution of Eq. (22.85). It suffices to show the existence of some $p_0 \in L^2(\Omega)$ such that

$$\Lambda(p_0) = y_1 - v(T), \tag{22.87}$$

where $v \in C\{[0,T]; L^2(\Omega)\}$ is the weak solution of Eq. (22.83). To solve Eq. (22.87), we observe that (by Eqs. (22.84) to (22.86))

$$[\Lambda(p_0), p_0]_{L^2(\Omega)} = \int_0^T \int_\omega p^2(t,x)\,dt dx. \tag{22.88}$$

However, by Theorem 22.2 and Eq. (22.88), we have

$$[\Lambda(p_0), p_0]_{L^2(\Omega)} \geq \frac{1}{C}|p_0|^2_{L^2(\Omega)}, \quad \forall\, p_0 \in L^2(\Omega). \tag{22.89}$$

Therefore, $\Lambda : L^2(\Omega) \to L^2(\Omega)$ is an isomorphism. Thus, Eq. (22.86) admits a unique solution $p_0 \in L^2(\Omega)$ and

$$u = p \tag{22.90}$$

is the desired control such that the weak solution of Eq. (22.2) satisfies $y(T) = y_1$. This completes the proof of Theorem 22.1.

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